A Snell's law derivation of the circular shape of upper-half plane geodesics.

Step 0. Recall Snell: If a light ray crosses from an interface from a medium with index of refraction n_1 into a medium with index of refraction n_2, then the incident angle θ_1 of the ray with the normal to the interface and the incident angle θ_2 of the outgoing ray to that normal are related by $\sin(\theta_1)/\sin(\theta_2) = n_2/n_1$ or

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

Step 1. Many layers. Now, suppose there are many layers of medium whose bounding interfaces are parallel lines. Then our light ray crossses through a number of different media with indices of refraction $n_1, n_2, \ldots, n_k$. At each step we get $n_j \sin(\theta_j) = n_{j+1} \sin(\theta_{j+1})$ where θ_i is the angle which the ray segment makes with the normal direction to the parallel lines upon crossing out of region j into region $j+1$. But then $n_1 \sin(\theta_1) = n_2 \sin(\theta_2) = \ldots n_k \sin(\theta_k)$ or:

$$n_j \sin(\theta_j) = const.$$

independent of what medium j we are in.

Step 2. Continuum limit.

Let j become a continuous index.

Suppose that index of refraction varies continuously in such a way as to only depend on y. Thus $n = n(y)$. We can view this as a continuum limit of the situation of the previous Step, with finer and finer slabs, interfaces consisting of lines $y = const$ parallel to the x axis. (Imagine approximating $n(y)$ by step functions.) We let $\theta(y)$ denote the angle a ray (allowable light ray; solution of Fermat's least time principle) makes with the y axis at the interface labelled by y. Then Snell's law yields

$$n(y) \sin(\theta(y)) = const.$$

Step 3. Deriving the upper half plane geodesics. In the upper half plane model $n(y) = 1/y$, so we get the law

$$\frac{1}{y} \sin(\theta) = const.$$

for our rays. Label that constant 'const.' as $1/R$. Then

$$\sin(\theta) = \frac{y}{R}.$$

Now look at the figure below. Through a point $P = (x, y)$ we've drawn a trajectory (ray) and its angle of incidence θ with the vertical. We've also drawn the circle through P tangent to the ray and with center O on the x-axis. From the figure we see that the first θ equals the the angle between OP and the axis. Thus R above, the const. is the radius of this circle! We have proved the ray is the arc of a circle.

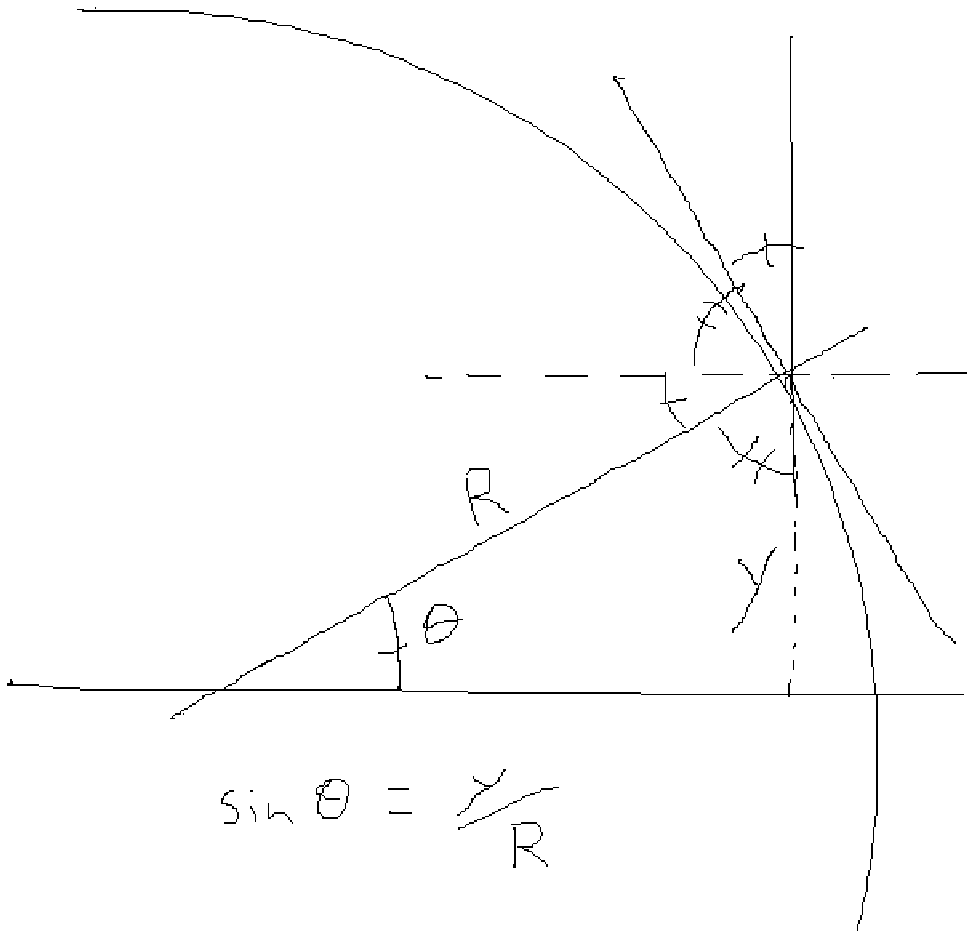

Figure 1. Snell law derivation of hyperbolic geodesics .