Solution to HW 7 problem 3:

Find a linear fractional transformation $F(z) = (az+b)/(cz+d)$ which takes the upper half plane to itself and maps the hyberbolic line $x = 0$ $y > 0$ to the hyperbolic line $x^2 + y^2 = 1, y > 0$). [I dropped part of the problem.It is added in the solution below.]

SOLUTION.

a) Strategy: Since the transformation takes the upper half plane to itself, we can assume that a, b, c, d are real. Both hyperbolic lines are circles in the extended plane. Circles are determined by three points on them. If we can find a linear fractional transformation that takes three points on the first circle to three points on the second we will be done

The points $0, i, \infty$ lie on the first circle. The points -1, i, and 1 lie on the second circle. Set $F(z) = (az+b)/(cz+d)$. And insist that

$$F(0) = -1, \quad fFi) = i, \quad F(\infty) = +1.$$

Plugging in, these equations read

$$b/d = -1, \quad (ai+b)/(ci+d) = i, \quad a/c = 1$$

Thus

$$b = -d, \quad (ai+b) = i(ci+d) \quad a = c.$$

Plugging the first and last results into the middle we get $(ai+b) = -a - bi$ which is equivalent to the single equation

$$a = -b,$$

(We have used that a, b are real here.) Thus

$$F(z) = (az-a)/(az+a) = (z-1)/(z+1),$$

the desired result.

b) Write $z = iy$ with y real. Later we will substitute in $y = e^t$ as asked. Then $F(iy)$ should run over the upper half-circle as $y > 0$ variess over the positive real axis. We have:

$$F(iy) = (iy-1)/(iy+1)$$

Clearing denominators by multiplying through by the conjugate of the denominator, $-iy + 1 = -(iy-1)$, yields

$$F(iy)) = -(iy-1)^2/(y^2+1) = -((1-y^2) - 2iy)/(y^2+1)$$

Or: if we write $u + iv = F(iy)$ then

$$u = y^2 - 1/(y^2+1); v = 2y/(y^2+1)$$

Observe these are precisely the formulas for 1 dimensional stereo projection $S^1 \setminus \{N\} \to I\!R$, with the variable subsitutions (u, v) for (x, y) and y for $t \in I\!R$.

These formulae parameterize the circle,as t varies over the ENTIRE real line. To see how the semicircle is traversed, move from $t = 0$ up to $t = +\infty$: $t = 0$ is the point $x = -1, y = 0$, $t = 1$ is the point $x = 0, y = 1$ and $t = \infty$ is the point $x = 1, y = 0$, as it should be.

To get the hyperbolic formulae multiply top and bottom of our displayed expression for u, v by y^{-1}:

$$u = (y - y^{-1})/(y + y^{-1}), v = 2/(y + y^{-1}).$$

Observe that if $y = e^t$ then $y + y^{-1} = 2\cosh(t)$ and $y - y^{-1} = 2\sinh(t)$ so that

$$(u, v) = (\tanh(t), 1/cosh(t))$$