Disclaimer:

I have not verified much of this work. ~~B.~~ However, I know Paul thought hard about this & much of it is on the money

-R.M.

128 HW

Paul Sintetos

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1.) The map which takes a point on the Cartesian x-axis to a point upper-half of the unit circle is given by $f(x) = (\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1})$. Making the substitution $x = \frac{m}{n}$ yields that the point $(\frac{2mn}{m^2+n^2}, \frac{m^2-n^2}{m^2+n^2})$ is on the unit circle when $n \neq 0$. The Pythagorean theorem tells us then that $(\frac{2mn}{m^2+n^2})^2 + (\frac{m^2-n^2}{m^2+n^2})^2 = 1 \implies (2mn)^2 + (m^2-n^2)^2 = (m^2+n^2)^2$. Therefore, allowing integer values for m and n yields Pythagorean triples.

If we seek Pythagorean triples a, b, c such that $50 \leq a, b, c \leq 100$, then we seek natural numbers m and n such that $c = (m^2+n^2) \leq 100$, and at the same time, $50 \leq (m^2+n^2)$. Therefore we need only look at integers whose squares are under 100, that is 1 through 10. We require, secondly, that $=m^2 - n^2 \geq 50 \implies m^2 \geq 50 + n^2$ and thirdly $50 < 2mn < 100 \implies 25 < mn < 50$. This second condition, since $n \neq 0$ implies the weaker condition that $m^2 \geq 50$, therefore we must consider only $8 \leq m \leq 10$. In view of the third condition, we have the following possible pairs $(m, n) : (8,4), (8,5), (8,6), (9,3), (9,4), (9,5)$ (any pair containing 10 is impossible, for then $c > 100$). Under the second condition, all these pairs are ruled out except $(9,3), (9,4)$, and $(9,5)$. Testing these under the first condition, we see that $(9,5)$ is ruled because $81 + 25 > 100$. Therefore there are two solutions: $m = 9, n = 3$ and $m = 9, n = 4$ which gives the triples $a = 54, b = 72, c = 90$ and $a = 72, b = 65, c = 97$. However, making the additional condition that a, b, and c be relatively prime to one another, we see that the only possible solution to our problem is the second pair: $72^2+65^2 = 97^2$.

(see diagram)

2.) We want to find the formula which relates sides of a right spherical triangle on a sphere of radius R. Using the conventions of spherical coordinates in $\mathbb{R}^3$, we assume without loss of generality that $A = (0,0,1)$ and $B = (R\sin\phi, 0, R\cos\phi)$, and $C = (0, R\sin\psi, R\cos\psi)$ so B and C are on the Cartesian x-axis and $y-axis$, respectively, therefore the angle at $A = \frac{\pi}{2}$. Let the length of side $BC = a$, the length of $AC = b$ and the length of $AB = c$. By the arc length formula, we have $b = R\phi \implies \frac{b}{R} = \phi$ and $c = R\psi \implies \frac{c}{R} = \psi$. We now use the following dot product formula to find the length of a: for vectors b and c, $b \cdot c = |b||c|\cos\alpha$, where α is the angle between the vectors b and c. In our case, since b and c lie on the sphere, $|b| = |c| = R$ and since a is the distance between b and c, $a = R\alpha \implies \frac{a}{R} = \alpha$, therefore computing the dot product yields:

$$\begin{aligned} R^2 \cos \frac{a}{R} &= R^2 \cos \alpha \\ &= (R \sin \phi, 0, R \cos \phi) \cdot (0, R \sin \psi, R \cos \psi) \\ &= 0 + 0 + R^2 \cos \phi \cos \psi \\ &= R^2 \cos \frac{b}{R} \cos \frac{c}{R} \end{aligned}$$

Finally, dividing both sides of the preceding equation by R^2 yields the Pythagorean formula for spherical triangles:

$$\cos(\frac{a}{R}) = \cos(\frac{b}{R}) \cos(\frac{c}{R})$$

b.) Making the substitution $R = iR$ in the above Pythagorean formula yields $\cos(\frac{a}{iR}) = \cos(\frac{b}{iR}) \cos(\frac{c}{iR})$. It thus follows from the definition of the complex continuation of cosine that $\frac{e^{i\frac{a}{iR}} + e^{-i\frac{a}{iR}}}{2} = \frac{(e^{i\frac{b}{iR}} + e^{-i\frac{b}{iR}})(e^{i\frac{c}{iR}} + e^{-i\frac{c}{iR}})}{4} \implies \frac{e^{\frac{a}{R}} + e^{-\frac{a}{R}}}{2} = \frac{(e^{\frac{b}{R}} + e^{-\frac{b}{R}})(e^{\frac{c}{R}} + e^{-\frac{c}{R}})}{4}$ and so by definition of $\cosh z$, we have:

$$\cosh \frac{a}{R} = \cosh \frac{b}{R} \cosh \frac{c}{R}$$

.

PORTLAND TO BORDEAUX: I now derive the formula the for distance a in diagram 2. $B =< R \sin \phi, 0, R \cos \phi >$, $C =< R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi >$.

$R^2 \cos \frac{a}{R} = R^2 B \cdot C = (\sin \phi \sin \phi \cos \theta + 0 + \cos \phi \cos \phi)$. Plugging in the given coordinates to this equation yields $\cos \frac{a}{R} = \frac{\pi}{6} + \frac{1}{2} \implies \frac{a}{R} = \cos^{-1}(\frac{\pi}{6} + \frac{1}{2})$. Plugging this into an algorithm yields $\frac{a}{R} \approx 1.305 \implies a \approx 5223$ miles. Steven Wolfram's computational engine gives the distance as about 5275 miles, so this was a pretty reasonable approximation, unless you have to walk the remaining 50 miles.

Reflections:

1.) See attached sketches.

2.) I also have provided a sketch, but it is less clear so I will explain. The dot will be first flipped about the equator (ℓ_3), then about the prime meridian (ℓ_2), and finally about ℓ_1, which is perpindicular to ℓ_2, therefore placing the dot near the north pole. This is tough to draw, since we cannot look at the sphere from all sides at once.

3.) The map $z \mapsto \frac{-1}{z}$ inverts the unit cirlce and flips it about the x-axis, before flipping the sign and sending it directly to the 'other side' of the unit circle. This is equivalent to flipping about the 'line' which is the unit circle, and then the y-axis (which is indeed a line in this geometry). Since the intersection of these two lines is the point i, we conclude that this is rotation about i. To find the angle of rotation, we look at the Moebius transformation induced by the rotation matrix (about an angle θ) and find which angle we must put in to obtain our original map ($z \mapsto \frac{-1}{z}$): $F(i) = \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$. We can see that $\theta = \frac{\pi}{2}$ is in fact our desired angle: $\frac{\cos\frac{\pi}{2}z - \sin\frac{\pi}{2}}{\sin\frac{\pi}{2}z + \cos\frac{\pi}{2}} = \frac{(0i)-1}{1z+0} = \frac{-1}{z}$. So the map $z \mapsto \frac{-1}{z}$ is a rotation of $\frac{\pi}{2}$ about i.

Gauss-Bonet Problems

1.) a.) The angle deficit formula is $(A + B + C - \pi) = \alpha$, where α stands for area, so in our case we have $(\frac{\pi}{2} + \frac{\pi}{2} + \theta_0 - \pi = \alpha \Longrightarrow \alpha = \theta_0$.

b.) Let us recall Green's theorem: for a vector field $< P, Q >$ and a region R with boundary ∂R,

$$\int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \int_{\partial R} P\,dx\ + Q\,dy$$

Therefore, since the desired area of a section of the sphere can be found by computing the surface integral $\int\int_R \frac{\partial Q}{\partial \phi} - \frac{\partial P}{\partial \theta} d\phi d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\theta_0} \sin\phi\ d\theta d\phi$, we seek a field $< P, Q >$ such that $\frac{\partial Q}{\partial \phi} - \frac{\partial P}{\partial \theta} = \sin\phi$. We see that $< 0, -\cos\phi >$ will suffice for this purpose. Therefore, Green's gives us

$$\int_0^{\frac{\pi}{2}} \int_0^{\theta_0} \sin\phi\ d\theta d\phi = \int_C 0\ d\phi\ - \cos\phi\ d\theta\ = \int_C -\cos\phi\ d\theta$$

where C is the curve which first runs from A to B, then from B to C, and finally from C back to A. Consider the map from spherical to cylindrical space: $(r, \theta, \phi) \mapsto (r, \theta, \cos\phi)$, that is, $r = r, \theta = \theta, z = \cos\phi$. This map is area preserving, since $\sin\phi d\phi d\theta = -dz d\theta$, so we have that the area for the triangle will be given by the integral

$$\int_C -z\ d\theta$$

, which now involves four curves: The path from $z = 1$ to $z = 0$, where $\theta = 0$, the path along $z = 0$ to the desired angle θ_0, the path back up from $z = 0$ to $z = 1$ when $\theta = \theta_0$, and finally the path along $z = 1$ where θ runs from θ_0 to 0. The paths along what were formerly longitutude lines (and what are now sides of a cylinder) do not involve a change in theta, and the integral for the path along the former equator (now the radius) has z = 0, so is given by $\int_0^{\theta_0} 0 d\theta = 0$. So the only nonzero component of the integral is given by the final path, where $z = 1 : \int_{\theta_0}^0 -1 d\theta = \int_0^{\theta_0} 1 d\theta = \theta_0$.

2.) a.) The angle deficit formula predicts Area $= \pi - (\pi - \theta_2) - \theta_1 \Longrightarrow Area = \theta_2 - \theta_1$.

b.) Since the area of the triangle is given by this double integral (where R is the hyperbolic triangle): $\int\int_R \frac{dxdy}{y^2}$ We have that in terms of Green's theorem, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{y^2} \Longrightarrow \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = -\frac{1}{y^2} + 0$, so, integrating, we can see that a field which satisfies this conditions is obtained by setting $Q = 0$ and P$= \frac{1}{y}$. Therefore, Green tells us that the area will be attained by the following line integral: $\int_{\partial R} \frac{dx}{y}$

where ∂R is clearly the path around R which first moves straight down from ∞ to $x = \cos\theta_2$, then from θ_2 to θ_1 along the unit circle ($= C$), and finally goes straight up:

$$\int_{x=\cos\theta_2} \frac{dx}{y} + \int_C \frac{dx}{y} + \int_{x=\cos\theta_2} \frac{dx}{y}$$

Since we are integrating with respect to dx, the first and third paths will both be zero, since x is not changing on the lines $x = \cos\theta_2$ and $x = \cos\theta_1$. Parameterizing the second path by $x = \cos\theta, y = \sin\theta$ yields $\int_C \frac{dx}{y} = \int_{\theta_2}^{\theta_1} \frac{-\sin\theta d\theta}{sin\theta} = \int_{\theta_2}^{\theta_1} -1 d\theta = -(\theta_1 - \theta_2 = \theta_2 - \theta_1$.

CL 1:

It is not incredibly hard to find an invertible linear map that does not preserve angles. Take for example $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 3 \end{smallmatrix}\right)$. This takes the lines $y = x$ and $y = 0$ (which have an angle between them of $\frac{\pi}{4}$ to the lines $y = \frac{x}{5}$ and $y = \frac{x}{2}$ respectively, which do not form an angle of $\frac{\pi}{2}$.

Isometries:

1.) a.) Isometries having one fixed point are the rotations, whose fixed points are their center of rotation. One can rotate about any point (x_0, y_0) by first translating the plane by the map $\tau : (x, y) \mapsto (x - x_0, y - y_0)$, rotating by the desired angle (as long as the angle is not a multiple of 2π, in which case the whole plane would be fixed), and then translating back with $\tau^{-1} : (x, y) \mapsto (x + x_0, y + y_0)$.

b.) The transformations which fix lines are the reflections. The two easy reflectionsare $f : (x, y) \mapsto (-x, y)$ and $g : (x, y) \mapsto (x, -y)$, which fix the y-axis and the x-axis, respectively. We can obtain a map which fixes and line by first rotating by $-\theta$, then reflecting about the x-axis, then finally rotating by θ, where $\tan\theta$ is the slope of the desired line, as shown in the diagram.

c.) Maps with no fixed points are all the translations (or compositions of translations, since translations of the plane from a group) τ such that $\tau : (x, y) \to (x + a, y + b)$, as long as τ is not the identity transformation.

2.) a.) There are no isometries which have a single fixed point (although rotations about a diametric axis have TWO fixed points).

b.) Maps which fix lines are the reflections which flip every point to the other side of a particular great circle. The great circle will be fixed, but no other point will be.

c.) Translations (which slide every point over by a fixed angle from a particular axis) and compositions of translations will fix no points, as long as they do not translate all the way around the sphere.

3.) a.) The isometries of the hyperbolic plane that have one fixed point are those Moebius transformations whose matrix has exactly one distinct eigenvalue. λ will be an eigenvalue of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ if the characteristic equation $(a - \lambda)(d - lambda) - bc = 0$ has exactly one solution. Setting the discriminant of this quadratic equation equal to zero obtains the condition: $(a-d)^2 \neq -4bc$. Therefore any transformation whose matrix satisfies this condition will have exactly one fixed point.

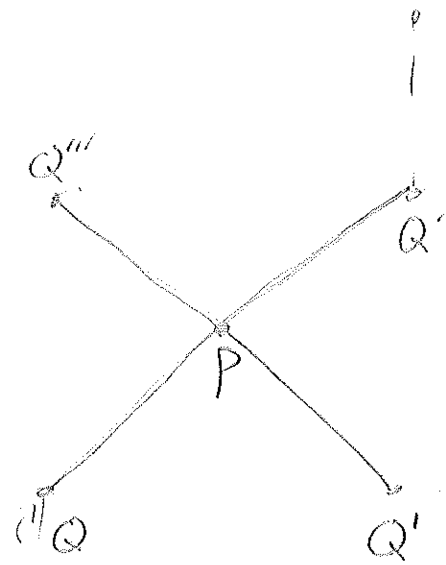
b.) A line of fixed points corresponds to an infinite amount of fixed points, therefore an infinite amount of eigenvectors, which is clearly impossible for a 2 by 2 matrix.

Euclitean dilatons ← Are hyperbolic Isometries

c.) The dilations maps $z \mapsto cz$ where $c \in \mathbb{R}$ fix no points in the (strict) upper plane, but if we extend this map to the ideal, we see that 0 and ∞ are fixed points.

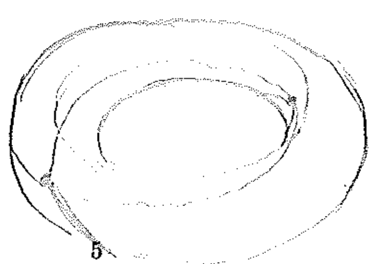
d.) The map $z \mapsto ze^{i\theta}$ where $0 < \theta < 2\pi$ fixes only the point 0, which is on the ideal.

OS 1: Torus:



$Q, Q', Q'', Q''' \in [Q]$

$(x, y) \mapsto \left(\frac{1+\cos 2\pi y}{2}, 2\pi x\right)$ (r, θ)
in polar coordinates,



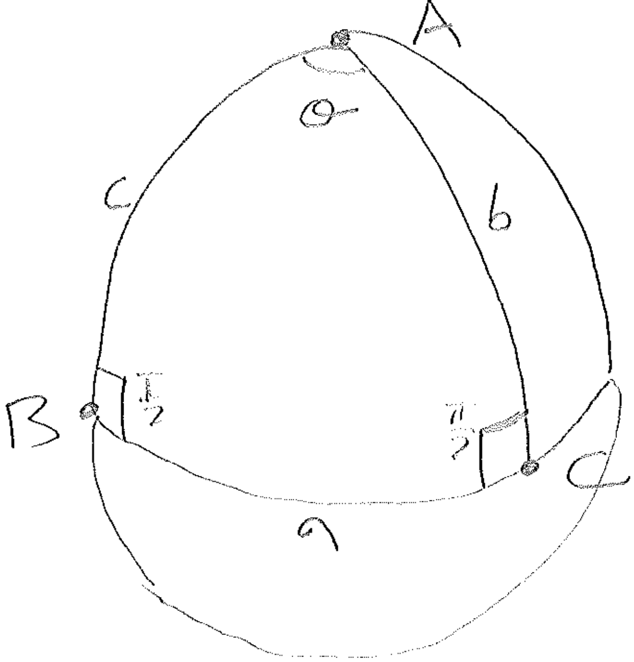


Diagram 1

(problem 7)

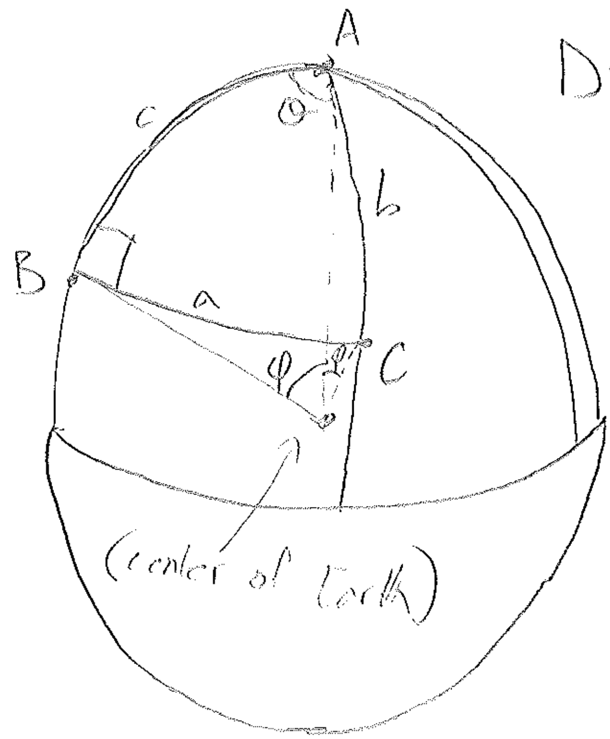
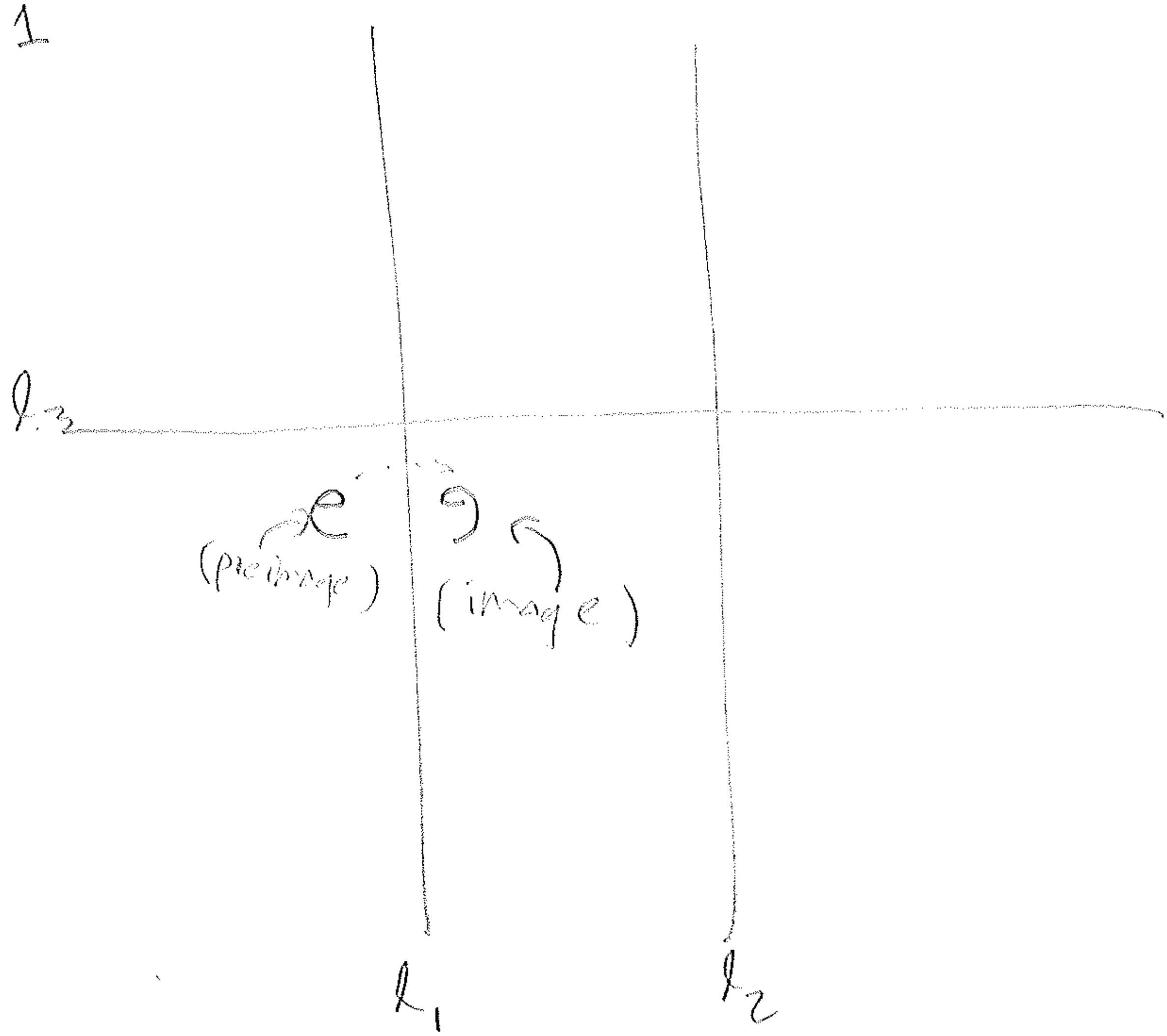


Diagram 2.

(what terribly narrow spheres! ugh)

REFLECTIONS

R_1



$R_2 R_1$