We can certainly extend a given rectangle to a square and hence reconstruct the square on the hypotenuse. The main problem is to reconstruct the right-angled triangle, from the hypotenuse, so that the other vertex lies on the dashed line. See whether you can think of a way to do this; a really elegant solution is given in Section 2.7. Once we have the right-angled triangle, we can certainly construct the squares on its other two sides—in particular, the gray square equal in area to the gray rectangle.

Exercises

It follows from the Pythagorean theorem that a right-angled triangle with sides 3 and 4 has hypotenuse $\sqrt{3^2+4^2} = \sqrt{25} = 5$. But there is *only one* triangle with sides 3, 4, and 5 (by the SSS criterion mentioned in Exercise 2.2.2), so putting together lengths 3, 4, and 5 always makes a right-angled triangle. This triangle is known as the $(3,4,5)$ triangle.

2.5.1 Verify that the $(5,12,13)$, $(8,15,17)$, and $(7,24,25)$ triangles are right-angled.

2.5.2 Prove the converse Pythagorean theorem: If $a,b,c > 0$ and $a^2+b^2 = c^2$, then the triangle with sides a,b,c is right-angled.

2.5.3 How can we be sure that lengths $a,b,c > 0$ with $a^2+b^2 = c^2$ actually fit together to make a triangle? (*Hint:* Show that $a+b > c$.)

Right-angled triangles can be used to construct certain irrational lengths. For example, we saw in Section 1.5 that the right-angled triangle with sides 1, 1 has hypotenuse $\sqrt{2}$.

2.5.4 Starting from the triangle with sides 1, 1, and $\sqrt{2}$, find a straightedge and compass construction of $\sqrt{3}$.

2.5.5 Hence, obtain constructions of $\sqrt{n}$ for $n = 2,3,4,5,6,\ldots$.

2.6 Proof of the Thales theorem

We mentioned this theorem in Chapter 1 as a fact with many interesting consequences, such as the proportionality of similar triangles. We are now in a position to prove the theorem as Euclid did in his Proposition 2 of Book VI. Here again is a statement of the theorem.

The Thales theorem. *A line drawn parallel to one side of a triangle cuts the other two sides proportionally.*

The proof begins by considering triangle ABC, with its sides AB anc cut by the parallel PQ to side BC (Figure 2.15). Because PQ is paralle BC, the triangles PQB and PQC on base PQ have the same height, nan the distance between the parallels. They therefore have the same area.

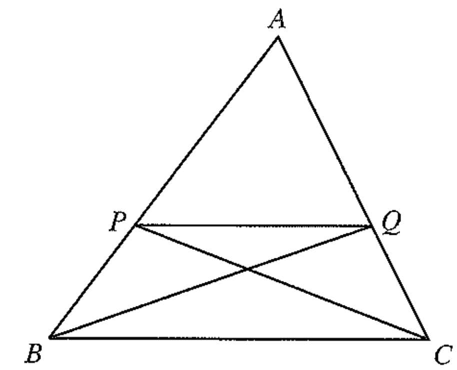

Figure 2.15: Triangle sides cut by a parallel

If we add triangle APQ to each of the equal-area triangles PQB PQC, we get the triangles AQB and APC, respectively. Hence, the la triangles are also equal in area.

Now consider the two triangles—APQ and PQB—that make up tr gle AQB as triangles with bases on the line AB. They have the same he relative to this base (namely, the perpendicular distance of Q from A Hence, their bases are in the ratio of their areas:

$$\frac{|AP|}{|PB|} = \frac{\text{area } APQ}{\text{area } PQB}.$$

Similarly, considering the triangles APQ and PQC that make up the trian APC, we find that

$$\frac{|AQ|}{|QC|} = \frac{\text{area } APQ}{\text{area } PQC}.$$

Because area PQB equals area PQC, the right sides of these two equati are equal, and so are their left sides. That is,

$$\frac{|AP|}{|PB|} = \frac{|AQ|}{|QC|}.$$

In other words, the line PQ cuts the sides AB and AC proportionally.