1. Stereo projection spin-offs

St 1. Using the homogeneous version of inverse stereographic projection to find five Pythagorean triples [1] a, b, c with $50 < a < b < c < 100$ and with the differences $b - a, c - b$ greater than 2.

2. Spherical Trig.

Sph 1. a) Derive the Spherical Pythagorean theorem. $\cos(c/R) = \cos(a/R)\cos(b/R)$.

In this formula a, b, c are the side lengths of a right triangle drawn on a sphere of radius R and c is the length of the triangle's hypotenuse.

Hints: Start by taking $R = 1$. Argue that without loss of generality you may put vertex C on the positive x-axis at $(1, 0, 0)$, and vertices A and B on the coordinate planes $z = 0$ and $y = 0$. Use spherical coordinates to express the Cartesian coordinates (x, y, z) of A, B in terms of a, b. Use the dot product formula to find distance c.

b) Taylor expand both sides of this equation in powers of $1/R$ for $R >> 1$ up to order $1/R^4$ and equate terms. Show that the order $1/R^2$ term in the expansion reproduces the usual Euclidean Pythagorean formula.

Sph 2. The sphere of radius R has constant curvature $1/R^2$. The hyperbolic plane has curvature -1 so formally has radius $\sqrt{-1} = i$. Just like there are a family of spheres, there are a family of hyperbolic planes having curvatures $-1/R^2$. Replace R by iR in the formula you derived in 1 a) to formally derive the hyperbolic Pythagorean theorem on a hyperbolic plane of curvature $-1/R^2$.

Sph 3. Approximate GPS coordinates of Portland OR are 45^0 N, 122^0 W and the approximate GPS coordinates of Bordeaux, France are 45^0 N, 2^0 W.

What is the distance between Portland and Bordeaux as travelled on the surface of the Earth when measured in units of R = radius of the earth,

What is this same distance measured in miles if we take for R its approximate value 4,000 miles?

[1] A Pythagorean triple is a triple of positive integers a, b, c such as $3, 4, 5$ which are relatively prime and form side lengths of a right triangle.

3. Three reflections theorem.

Use the notation R_ℓ for reflection about the line ℓ.

R 1. [Flat.] Consider the three lines as indicated in the figure 1 so that ℓ_1 and ℓ_2 are parallel, 1 unit apart and ℓ_3 is perpindicular to them. Let $R_i = R_{\ell_i}$. Approximately compute with sketches, the result of applying R_1, R_2R_1 and finally the isometry $R_3R_2R_1$ to the little smiley face there.

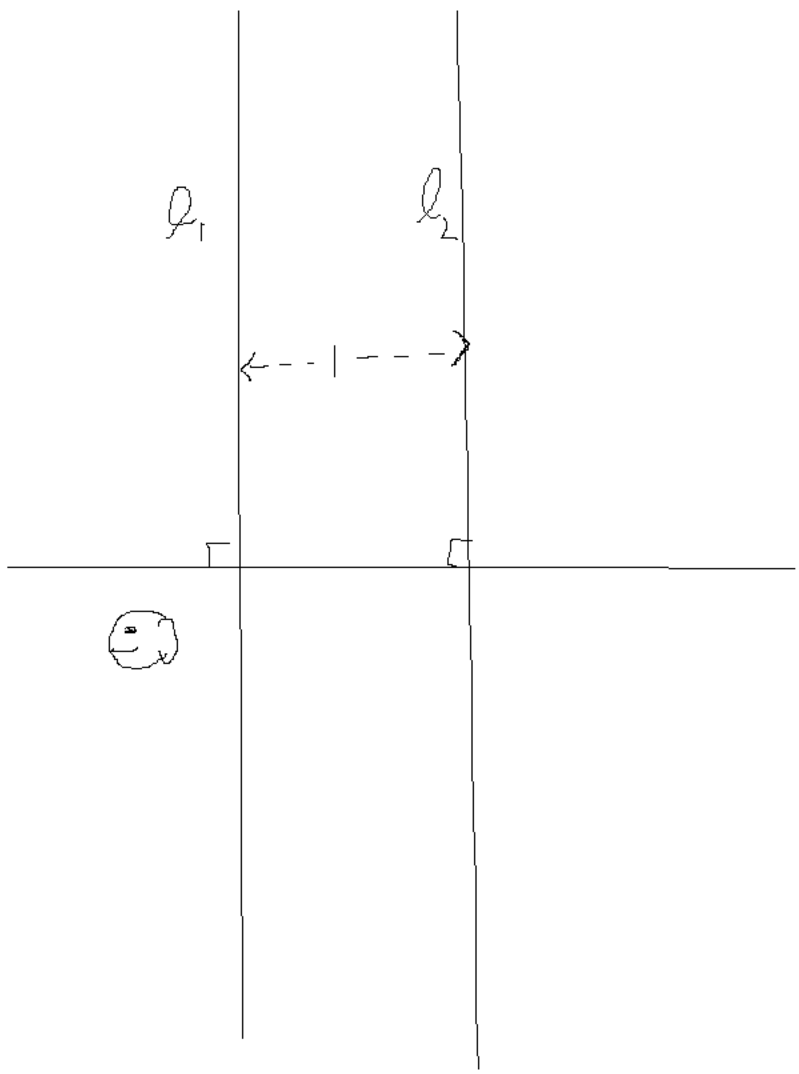

Figure 1. 3 reflections.

R 2. [Spherical.] The three coordinate planes define three lines ℓ_1, ℓ_2, ℓ_3 on the sphere by intersection. Thus for example line ℓ_1 is the great circle defined by $x = 0$ Let $R_i = R_{\ell_i}$. Compute the action of $R_1 R_2 R_3$ on a point (x, y, z) of the sphere.

R 3. [Hyperbolic.] Express the hyperbolic isometry $z \mapsto \frac{-1}{z}$ as the product of two hyperbolic reflections and draw the corresponding hyperbolic lines. Where do they intersect?

b) Conclude that $z \mapsto \frac{-1}{z}$ is a rotation about this point of intersection and find the angle of rotation.

4. Gauss-Bonnet Problems.

Verify the angle deficit theorem in the following special cases by completing steps (a), (b), and (c) in each case.

GB 1. On the sphere: consider the spherical triangle R with one vertex at $N = (0, 0, 1)$ and the other two vertices on the equator ($z = 0$), separated by an angle (= spherical distance) θ.

a) What is the area of R according to the angle deficit formula?

b)The spherical element of area is $d^2A = \sin(\phi) d\phi d\theta$ in terms of standard spherical coordinates. *Apply* Green's theorem[2] (= Stoke's theorem in the plane) to show that

$$\int_R d^2A = \int_{\partial R} (-\cos(\phi) d\theta).$$

where in our case ∂R is the curve formed by the three edges of the triangle.

c) Perform the line integral at the end of (b) and verify that this result agrees with the result from (a).

GB 2. On the hyperbolic plane, upper half-plane model: take R to be the interior of the (ideal) triangle with two sides formed by the lines $x = x_1$, and $x = x_2$ and whose third side is formed by the arc of line $x^2 + y^2 = 1$ between these lines. Parameterize that circle as per usual by $x = \cos(\theta), y = \sin(\theta)$ and suppose that the two finite vertices of the triangle correspond to $\theta = \theta_1$ and $\theta = \theta_2$.

a) What is the hyperbolic area of R according to the angle deficit formula?

b)The hyperbolic element of area is $d^2A = \frac{dxdy}{y^2}$. *Apply* Green's theorem in the plane to show that

$$\int_R d^2A = \int_{\partial R} \frac{dx}{y}.$$

c) Compute the line integral at the end of (b) and verify that it agrees with the result in (a).

5. Conformal vs Linear

CL 1. Find a Linear invertible map of the plane to itself which is *not* angle preserving.

[2]asserting that the line integral of $P(u, v)du + Q(u, v)dv$ around a closed curve surrounding a region R is a certain area integral $(**)dudv$ over R

6. Dynamics and Isometries

DI 1. Describe all isometries of the plane which have
a) exactly one fixed point
b) a line of fixed points
c) no fixed points.

DI 2. Describe all isometries of the sphere which have
a) exactly one fixed point
b) a line of fixed points
c) no fixed points

DI 3. Describe all isometries of the hyperbolic plane which have
a) exactly one fixed point
b) a line of fixed points
c) no fixed points but, when extended to the ideal, exactly two fixed points.
d) no fixed points but when extended to the ideal, exactly one fixed point.

7. Other surfaces

OS 1. Consider the flat torus, $\mathbb{R}^2/\mathbb{Z}^2$ with points written as equivalence classes $[x, y]$ so that $[x + k, y + m] = [x, y]$ for k, m integers.

a) Draw five geodesics connecting $P = [1/2, 1/2]$ to $Q = [0, 0]$.

b) What is the distance between P and Q? Find and sketch four distinct geodesics which realize this distance.