HW Number 7. Due Fri. Nov 15

The (*)s in front of a problem mean that these are given priority in grading.

1. [Review of Complex ARithmetic. (Based on Levi problem 40)

a) Compute $(1+i)^{20}$.

b) Compute $(1+i)^{100}$.

Suggestion: Please do your arithmetic in polar coordinates $re^{i\theta}$ not Cartesian $x+iy$.

Transformation Problems

(*) 2 .(Based on Levi problem 41) Rotate (R) the plane by 30 degrees *about the point* $(3,6)$. Then dilate (D) the plane by a factor of 3 with dilation center being the origin.

a) Where does $(1,2) = 1+2i$ end up?

d) Express R , D and their composition DR in complex arithmetic notation.

b) Reverse order: dilate, then rotate. Now where does (1,2) end up?

c) Express R, D, DR and RD in matrix-vector notation.

3. [Inversions through Circles] Let C_1 and C_2 be two circles in the plane with centers O_1 and O_2. Let R_1 and R_2 be the reflection about these circles. Let $T = R_2 \circ R_1$.

a) Show that if C_1 and C_2 are concentric, i.e if $O_1 = O_2$ then T is a dilation of the plane about the common center O_1.

For the remainder of the problem assume that

(*) $O_1 \neq O_2$ and that the circles intersect in two distinct points A and B.

b) Where does T send O_1?

c) Where does T send the line joining O_1 to O_2?

d) Where does T send the oriented line segment O_1O_2.

e) Where does T send the line segment AB?

(*) 4. [Inversions through Circles] ct'd . Continue with the notation and assumptions of problem 3, including the assumption (*) on the two circles. Suppose moreover that C_1 passes throug O_2 and C_2 passes through O_1.

Draw the figure. Observe that something is special about triangle $\Delta = O_1AO_2$. What?

a) Sketch the image of Δ under R_1.

b) Sketch the image of Δ under T. Include locations of vertices and angle measures of the resulting curvilinear triangle.

For the remainder of this problem suppose that $O_1 = O \in \mathbb{C}$ and that $O_2 = 1 \in \mathbb{C}$ so that both C_1 and C_2 have radii equal to 1.

c) Find the algebraic formulae, using complex notation, for R_1, for R_2 and T. (The formulae for the R_i will involve complex conjugation. The formula for T will not, and will be linear fractional.)

d) Express A and B as complex numbers. Verify that A and B are fixed by T, according to your formula.

e) Verify algebraically that $T^3 = Id$.

5. Extra Credit. Continuing with the notation of problem 3 and the assumption (*). True of False/ Suppose that the angle between the two circles is π/N in radians. ($N > 1$ a positive integer) Then $T^N = Id$. [Prove or give a counterexample.]

(*) 6. a)Find a linear fractional transformation $F(z) = (az+b)/(cz+d)$ which takes the y-axis to the unit circle $x^2 + y^2 = 1$ by sending the points $0, i, \infty$ on the y-axis to the points $-1, i, +1$, in that order. In other words: insist that $F(0) = -1, F(i) = i, F(\infty) = +1$. Use thes equations to solve for a, b, c, d.

b) Now parametrize the positive y-axis by its hyperbolic arclenght: $x + iy = 0 + ie^t$. Use part (a) to work out from this the resulting parameterization $x(t) + iy(t)$ of the upper half of the unit circle. Your answer should be all in terms of hyperbolic functions $\cosh(t), \sinh(t), \tanh(t)$.

[Commentary: Why are we doing this? The y-axis and the upper half of the unit circle are lines in the upper half plane model. We are finding an isometry which takes the first line to the second.