MATH 145 - HW1 SOLUTIONS

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(A) Since the function has positive slope m for $x < \frac{1}{2}$ and $f(0) = 0$, we get $T_m(x) = mx$ for $x \leqslant \frac{1}{2}$. On the other hand, T_m has slope $-m$ for $x > \frac{1}{2}$ and hence $T_m(x) = -mx + b$ for some $b \in \mathbb{R}$. Since T_m is continuous at $x = \frac{1}{2}$, $\frac{m}{2} = T_m(\frac{1}{2}) = -\frac{m}{2} + b$, which implies that $b = m$. Therefore

$$T_m(x) = \begin{cases} mx, & x \leqslant \dfrac{1}{2} \\ -mx + m, & x \geqslant \dfrac{1}{2} \end{cases}$$

(B) Assume that $0 < m < 1$. We shall prove the claim in two cases

(i) In this case, we investigate seeds $x_0 \leqslant \frac{1}{2}$.

Claim 1. *For $x_0 \leqslant \frac{1}{2}$, $T_m^n(x_0) = m^n x_0$ and $T_m^n(x_0) < \frac{1}{2}$ for all $n \in \mathbb{N}$.*

Proof. For $n = 1$, we have $T_m(x) = mx_0$. If $x_0 < 0$, $mx_0 < 0$ since $m > 0$ and therefore $mx_0 < \frac{1}{2}$. If $x_0 \geqslant 0$, $mx_0 < x_0 \leqslant \frac{1}{2}$ since $m < 1$ and $x_0 \leqslant \frac{1}{2}$. In both cases, we conclude that $T_m^n(x_0) < \frac{1}{2}$ and that the statement holds for $n = 1$. Assume the statement is true for n. Now

$$T_m^{n+1}(x_0) = T_m(T_m^n(x_0)) = T_m(m^n x_0) = m \cdot m^n x_0 = m^{n+1} x_0,$$

where the penultimate equality follows from assuming that the claim holds for n. Now, since $m^n x_0 < \frac{1}{2}$, the same argument we had for case $n = 1$ holds and hence $m^{n+1} x_0 < \frac{1}{2}$. Therefore, by induction, the claim holds for all $n \in \mathbb{N}$. □

Since $m < 1$, $\lim_{n\to\infty} T_m^n(x_0) = \lim_{n\to\infty} m^n x_0 = 0$. Hence all $x_0 \leqslant \frac{1}{2}$ is in the basin of attraction of 0.

(ii) Now, assume that we have the seed $x_0 \geqslant \frac{1}{2}$. Then $x_1 = T_m(x_0) = -mx_0 + m = m(1 - x_0)$. Since $x_0 \geqslant \frac{1}{2}$ and $m < 1$, $m(1 - x_0) < \frac{1}{2}$ and therefore, by (i), $x_{n+1} = T_m^n(x_1) = m^n x_1 = m^{n+1}(1 - x_0)$. Similarly, since $m < 1$, $\lim_{n\to\infty} T_m^n(x_0) = \lim_{n\to\infty} m^n(1 - x_0) = 0$. Hence all $x_0 \geqslant \frac{1}{2}$ is also in the basin of attraction of 0.

Culminating the results in both cases, we conclude that the basin of attraction of 0 is $\mathbb{R}$.

(C) Assume that $m > 1$. Again, we examine two cases.

(i) We start by investigating orbits with seeds $x_0 < 0$.

Claim 2. *For $x_0 < 1$, $T_m^n(x_0) = m^n x_0$ and $T_m^n(x_0) < 0$ for all $n \in \mathbb{N}$.*

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Proof. For $n = 1$, we have $T_m(x) = mx_0$. Since $x_0 < 0$ and $m > 1$, $mx_0 < x_0 < 0$ and thus the statement holds for $n = 1$. Assume the statement is true for n. Now

$$T_m^{n+1}(x_0) = T_m(T_m^n(x_0)) = T_m(m^n x_0) = m \cdot m^n x_0 = m^{n+1} x_0,$$

where the penultimate equality follows from assuming that the claim holds for n. Now, since $m^n x_0 < 0$, the same argument we had for case $n = 1$ holds and hence $m^{n+1} x_0 < 0$. Therefore, by induction, the claim holds for all $n \in \mathbb{N}$. □

Following the claim, we observe that $\lim\limits_{n\to\infty} T_m^n(x_0) = \lim\limits_{n\to\infty} m^n x_0 = -\infty$ since $m > 1$ and $x_0 < 0$.

(ii) We continue by investigating orbits with seeds $x_0 > 1$. In this case $x_1 = T_m(x_0) = m(1 - x_0)$. Since $x_0 > 1$ and $m > 0$, $x_1 < 0$. Therefore, by case (i), $x_n = T_m^{n-1}(x_1) = m^{n-1} x_1 = m^n(1 - x_0)$. We again observe that $\lim\limits_{n\to\infty} T_m^n(x_0) = \lim\limits_{n\to\infty} m^n(1 - x_0) = -\infty$ since $m > 1$ and $1 - x_0 < 0$.

As we saw in (i) and (ii), any orbit having a seed outside of the closed unit interval $[0, 1]$ diverges to $-\infty$.

(D) For this part, we assume that $m = 3$, so we consider

$$T(x) = \begin{cases} 3x, & x \leqslant \dfrac{1}{2} \\ -3x + 3, & x \geqslant \dfrac{1}{2} \end{cases}$$

Let $I = [0, 1]$ and $I^C = \mathbb{R} - I$. It is easy to see that $T([0, \frac{1}{3}]) = T([\frac{2}{3}, 1]) = I$ and $T((\frac{1}{3}, \frac{2}{3})) = (1, \frac{3}{2}) \subset I^C$. So, iterations of T will map first and third thirds of I back to itself and middle third outside of I, which will be eventually mapped to $-\infty$ by part (C). In order to achieve a precise proof of the fact in question, let B_n be the set of real numbers x_0 such that all iterations of length n stay in I, i.e. $B_n = \{x_0 \in \mathbb{R} \mid T^k(x_0) \in I \ \forall k = 0, 1, \ldots, n\}$. Notice that $B_n \subset B_m$ when $n > m$. To obtain $B = \bigcap\limits_{n=1}^{\infty} B_n$, we need to remove the middle thirds iteratively. As shown above, $B_1 = I - (\frac{1}{3}, \frac{2}{3})$. In the next iteration, the middle third of the interval $[0, \frac{1}{3}]$ (which is $(\frac{1}{9}, \frac{2}{9})$) and the middle of the interval $[\frac{2}{3}, 1]$ (which is $(\frac{7}{9}, \frac{8}{9})$) are removed from B_1 since they map to $(\frac{1}{3}, \frac{2}{3})$ after one iteration. Hence $B_2 = B_1 - (\frac{1}{9}, \frac{2}{9}) - (\frac{7}{9}, \frac{8}{9})$. Iteratively, if we encounter any interval J in B_n, the middle third of J will be mapped to an interval outside of B_n in one iteration and thus will not be contained in B_{n+1}. Hence B will be the Cantor set.