Problems to help prepare for the 2013, Winter, Chaos (Nonlinear Dynamics) Final

0. [A Maps Problem] Let N be an integer. Consider the map $F(x) = \frac{1}{2}(x + \frac{N}{x})$

a) Find the fixed point of F.

b) Show that this fixed point is attracting. What is its basin of attraction?

c) Take $N = 2$ and $x_0 = 1$. Write down the first part of the orbit of x_0: that is x_0, x_1, x_2, x_3 as rational numbers, not decimals. Convert to decimals. Compare to what you know of $\sqrt{2}$.

d) Define what it means for the rate of convergence of an iteration scheme $x_i \mapsto x_{i+1} = F(x_i)$ to converges exponentially (to a desired answer x_*). Define what it means to be super-exponential. Show the rate of our scheme for approximating $\sqrt{N}$ is superexponential.

1. [An ODE PROBLEM. Type 1] A match-the-picture with the linear algebra, algebraic formulae, and topology type question. [See SCAN IN.]

2. [AN ODE PROBLEM, TYPE 2; piecing together fixed point and linearization analysis]

Follow the example which makes up section 6.4 of Strogatz, regarding the Lotka-Voltera type system. Here is a variant of that example...

Consider the following system of differential equations

$$\begin{aligned} \dot{x} &= x(1 - 2x - y) \\ \dot{y} &= y(1 - x - 2y) \end{aligned} \tag{1}$$

(i) Work out the equilibrium points of the system (1).

(ii) Classify them and compute their index.

(iii) Draw the phase portrait of (1). Do we get any periodic orbits? Why, or why not?

(iv) A *nullcline* is a curve on the plane such that either $\dot{x}$ or $\dot{y}$ is 0 (hence the prefix "null"). In general, what is the relationship between nullclines and equilibrium points?

(v) What happens if we change the coefficients of x and y in parenthesis? Work out a few examples; find the nullclines, equilibrium points, their stability, (non-)existence of periodic orbits.

3. [SHIFT MAP]. Consider the point

$$p = \ldots\overline{01}01010101.0001100010010010010010\overline{100}$$

Is p in the stable manifold of $\overline{01} = 010101.0101010\ldots$

Is p in the unstable manifold of $\overline{01} = 010101.0101010\ldots$

Is p in the stable manifold of $\overline{100} = \ldots 1001001.00100100100\ldots$?

Is p in the unstable manifold of $\overline{100} = \ldots 1001001.00100100100\ldots$?

Is p in the stable manifold of $\overline{0} = \ldots 0000.0000\ldots$?

Is p in the unstable manifold of $\overline{0} = \ldots 0000.0000\ldots$?

ADDITIONAL PRACTICE PROBLEMS.

4. [ANOTHER MAP Q.] Newton iteration for finding zeros as a dynamical system: SEE: Strogatz problems : 10.1.12, to 10.1.13, and 10.1.14.

The Newton iteration scheme for finding zeros of a differentiable function is depicted below. In it, we make an initial guess for a zero x_0. Then we define x_1 to be the zero of the Taylor approximate to $f(x)$ at x_1. Thus $f(x_0)+f'(x_0)(x_1-x_0)=0$ so that $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. We can think of this as the result of interating a map $F(x) = x - f(x)/f'(x)$.

a) Suppose that f has exactly one zero inside the unit interval $0 < x < 1$. Describe conditions on f which will guarantee that the Newton scheme converges to this zero.

b) Describe problems that may arise in the convergence.

c) Give examples in which f has exactly one zero in the unit interval, but such that the Newton iteration scheme does not converge to this zero.

5. [MAPS] Prove or disprove. A continuous invertible map of the real line to itself, viewed as defining a dynamical system, has no periodic orbits besides fixed points.

6. [A LITTLE KNOWLEDGE CAN BE A DANGEROUS THING.] Understand and work with the relation between flows [ODEs] and maps, and more specifically with the relations between the spectrum of their linearizations and the corresponding stability relations. In a nutshell: the exponential relates linear ODEs to linear maps. Corresponding eigenvalues are related by $\lambda \to e^{\lambda}$.

Example: The time 1 map for the ODE $dx/dt = kx$ is the map takes an initial condition $x \mapsto e^k x$. If $k > 0$ then $e^k > 1$ and the map is unstable. If $k, 0$ then $e^k < 1$ and 0 is a stable fixed point for the map. What about the time 3 flow?

The above example carries over to n-dimensional linear systems: the time t map for the linear system $\frac{d}{dt}\vec{v} = A\vec{v}$ is the map $\vec{v} \to e^{tA}\vec{v}$. Write $\sigma(A)$ for the spectrum of the matrix A: the list of its eigenvalues. Then $\sigma(e^{tA}) = e^{t\sigma(A)}$, meaning that the eigenvalues of e^{tA} are precisely the numbers $e^{t\lambda}$ as λ varies over the eigenvalues of A.

Example: $A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$ One computes that $e^{tA} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}$ The spectrum of A is $\pm i\omega$. The spectrum of e^{tA} is $e^{\pm i\omega t}$ corresponding to the rotating solutions.

If f is a vector field on a 3 dimensional space and $f(p) = 0$, what is $Df(p)$? How do you determine the stability of p from the matrix $Df(p)$? If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a map with a fixed point p, what is $DF(p)$? How do you determine the stability of p from the matrix $DF(p)$? Etc . Etc.