1. HW for the week of Feb 19-21; due Feb 26

Exercise 1. *The harmonic oscillator is the system* $\ddot{x} = -\omega^2 x$ *with* $\omega \neq 0$ *a constant called the frequency. Verify that* $H = \frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2$ *is constant along solutions by*

a) plugging the general solution into H *and differentiating.*

b) differentiating H *along the corresponding 1st orderized system.*

c) Draw the phase portrait of the flow in the $(x, \dot{x})$ *plane, indicating several solutions and the fixed point.*

Exercise 2. *Repeat Exer. 1 (b) above for the general 'conservative force' 1 degree of freedom system:* $\ddot{x} = -V'(x)$, *the prime denoting* d/dx. *That is: prove that the energy* $H = \frac{1}{2}\dot{x}^2 + V(x) =$ *(kinetic) + (potential). is constant.*

b) Relate the critical points of the potential V *to the critical points (zeros) of the corresponding vector field in the* $(x, \dot{x})$ *plane.*

c). Draw the phase portrait for the 'Mexican hat ' potential $V(x) = (1 - x^2)^2$ *making sure to indicate the fixed points and the homoclinic orbits.*

Exercise 3. *(a) Draw the phase portrait for the pendulum* $\ddot{\theta} = -\sin(\theta)$ *by finding the potential* $V(\theta)$ *and hence the energy. Here* θ *is the angle that the pendulum bob makes with the vertical. Take* θ *to be an angle: so view it mod* 2π.

(b) Is $\theta = 0$ *straight up, or straight down? What is the meaning of the homoclinic orbit?*

Exercise 4. *Add some damping to the previous exercise: consider the damped pendulum* $\ddot{\theta} = -\sin(\theta) - \mu\dot{\theta}$, μ *small, positive.*

a) Draw the phase portrait.

b) What happened to the critical points of the previous exercise? How did their linearization change? What happened to the homoclinic orbit?

Exercise 5. *Add forcing to the pendulum of the previous exercise: consider the damped driven pendulum is the* $\ddot{\theta} = -\sin(\theta) - \mu\dot{\theta} + \epsilon f(t)$ *Here* $f(t)$ *is a periodic function of time called the forcing function, for example* $f(t) = \cos(\omega t)$. *This is now a non-autonomous system since time explicitly occurs on the r.h.s of the differential equation.*

a) turn the system into an autonomous vector field in 3 dimensions by the following trick. Introduce a variable τ *to play the role of* t. *(Set* $v = \dot{\theta}$ *as per usual.) Write out the expression* G *explicitly:*

$$\dot{\tau} = 1$$
$$\dot{\theta} = v$$
$$\dot{v} = G(\theta, y, \tau) = ?$$

Exercise 6. *Do the n-dimensional version of exer. 2 above: for* $\vec{x} \in \mathbb{R}^n, V : \mathbb{R}^n \to \mathbb{R}$ *smooth, consider the 2nd order ODE [Newton's eq.]:* $\ddot{\vec{x}} = -\nabla V(\vec{x})$. *Define the associated energy* H. *Show your* H *is constant along solutions.*

Exercise 7 (Gradient flows). *Again:* $x \in \mathbb{R}^n, V : \mathbb{R}^n \to \mathbb{R}$ *smooth. But now look at the 1st order gradient system:* $\dot{\vec{x}} = -\nabla V(\vec{x})$ *Prove that if* $\vec{x}(t)$ *is a solution then* $V(\vec{x}(t))$ *is strictly monotonically decreasing, unless* $\vec{x}(t)$ *is constant.*