1. HW for the week of Feb 19-21; due Feb 26

Exercise 1. *The harmonic oscillator is the system* $\ddot{x} = -\omega^2 x$ *with* $\omega \neq 0$ *a constant called the frequency. Verify that* $H = \frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2$ *is constant along solutions by*

a) plugging the general solution ($x(t) = a\cos(\omega t - \phi_0)$ *with* a, ϕ_0 *constants) into* H *and verifying using trig identities that the result is independent of* t.

b) B differentiating H *along the corresponding 1st orderized system, using the chain rule.*

c) Draw the phase portrait of the flow in the $(x, \dot{x})$ *plane, indicating several solutions and the fixed point.*

Exercise 2. *Repeat Exer. 1 (b) above for the general 'conservative force' 1 degree of freedom system:* $\ddot{x} = -V'(x)$, *the prime denoting* d/dx. *That is: prove that the energy* $H = \frac{1}{2}\dot{x}^2 + V(x) =$ *(kinetic) + (potential). is constant.*

b) Relate the critical points of the potential V *to the critical points (zeros) of the corresponding vector field in the* $(x, \dot{x})$ *plane.*

c). Draw the phase portrait for the 'Mexican hat ' potential $V(x) = (1 - x^2)^2$ *making sure to indicate the fixed points and the homoclinic orbits.*

Exercise 3. *(a) Draw the phase portrait in the* (θ, v)*-plane, (* $v = \dot{\theta}$*) for the pendulum* $\ddot{\theta} = -\sin(\theta)$. *Do so with the help of the energy* $H(\theta, v) = \frac{1}{2}v^2 + V(\theta)$ *which will require you to find the potential energy* $V(\theta)$. *[* θ *is the angle that the pendulum bob makes with the vertical.]*

(b) Find the two critical points of the resulting vector field $(f_1(x, v), f_2(x, v))$ *of* $\frac{d}{dt}(\theta, v) = (f_1(x, v), f_2(x, v))$. *What is the linearized vector field at the fixed points? (The answer will be a 2 by 2 matrix associated to each fixed point.) What are the associated eigenvalues? Which fixed point is stable ? Which is unstable? Draw a 'physical picture' of the pendulum with an arrow 'Down' for gravity, and the angle* θ *indicated in your picture. Indicate the two fixed points in your picture (One corresponds to* $\theta = 0$. *) Indicate the homoclinic orbit on your phase portrait from part (a), and describe it in words, based on your physical picture.*

(c)[EXTRA CREDIT] FINDING THE HOMOCLINIC: If $P_u = (x_u, v_u)$ *is the unstable fixed point, solve the equation* $H(x, v) = H(P_u)$ *to get a differential expression of the form* $f(\theta)d\theta = dt$ *where* $f(\theta)$ *is free of square roots. Integrate the expression. Invert it to get an explicit formula* $\theta = \theta(t)$ *which desribes the homoclinic orbit (with* $v(t) = d\theta/dt$*).*

Exercise 4. *Add damping. Consider the damped pendulum* $\ddot{\theta} = -\sin(\theta) - \mu\dot{\theta}$, μ *small, positive.*

a) Draw the phase portrait.

b) Verify that the critical points of the previous exercise did not move. Their linearizations do change. How? (Write down 2 by 2 matrices with μ*'s somewhere in them.) How does the stability change? What happened to the homoclinic orbit?*

Exercise 5. *Now add forcing: consider the damped driven pendulum is the* $\ddot{\theta} = -\sin(\theta) - \mu\dot{\theta} + \epsilon f(t)$ *Here* $f(t)$ *is a periodic function of time called the forcing function, for example* $f(t) = \cos(\omega t)$. *This is now a non-autonomous system since time explicitly occurs on the r.h.s of the differential equation.*

a) turn the system into an autonomous vector field in 3 dimensions by the following trick. Introduce a variable τ *to play the role of* t. *(Set* $v = \dot{\theta}$ *as per usual.)*

Write out the expression G explicitly:

$$\dot{\tau} = 1$$
$$\dot{\theta} = v$$
$$\dot{v} = G(\theta, y, \tau) = ?$$

Exercise 6 (EXTRA CREDIT)**.** *Do the n-dimensional version of exer. 2 above: for $\vec{x} \in \mathbb{R}^n, V : \mathbb{R}^n \to \mathbb{R}$ smooth, consider the 2nd order ODE [Newton's eq.]: $\ddot{\vec{x}} = -\nabla V(\vec{x})$. Define the associated energy H. Show your H is constant along solutions.*

Exercise 7 (EXTRA CREDIT: Gradient flows)**.** *Again: $x \in \mathbb{R}^n, V : \mathbb{R}^n \to \mathbb{R}$ smooth. But now look at the 1st order gradient system: $\dot{\vec{x}} = -\nabla V(\vec{x})$ Prove that if $\vec{x}(t)$ is a solution then $V(\vec{x}(t))$ is strictly monotonically decreasing, unless $\vec{x}(t)$ is constant.*