HW 1. [Tent Map and Cantor set. D is weighted more heavily than A-C.]

Recall that the tent map : a continuous, piecewise linear map on the real line which has zeros at 0 and 1, and a maximum at $x = 1/2$, where its graph has a 'corner'. Write m for the slope of the tent map at $x = 0$. Then its slope at $x = 1$ is $-m$. Assume throughout $m > 0$. Write $T_m(x)$ for this tent map.

A) Write out a formula for the tent map $T_m(x)$. You will need to divide the formula into two cases, $x \leq 1/2$ and $x \geq 1/2$.

B) Prove: if $0 < m < 1$ then $x = 0$ is an attracting fixed point and the entire real line is its basin of attraction: that is: for all x we have $T^{\circ n}(x) \to 0$ as $n \to \infty$.

C). Prove: If $m > 1$ and x is outside the unit interval then the orbit of x tends to $-\infty$.

(*) D. Write B for the set of all x whose orbit is bounded. Prove that when $m = 3$ that B is *precisely* the standard Cantor set, obtained by successively removing middle thirds from intervals.

HW 2. [Conjugacy.] Two maps F, G of the real line to itself are *conjugate* if there is an invertible map ϕ such that $F \circ \phi = \phi \circ G$. Conjugate maps have the same dynamical behaviour. A quadratic map is one of the form $Ax^2 + Bx + C$, A, B, C constants.

A. Show that any quadratic map with $A > 0$ is conjugate to a quadratic map of the form $x^2 + c$, c a constant by looking for a conjugating map which is linear; $\phi(x) = ax + b$.

B. Show that any quadratic map is conjugate to a logistic map $kx(1-x)$ where again the conjugating map ϕ can be taken linear.

C. By A, we have that $kx(1-x)$ is conjugate to $x^2 + c$ provided $k < 0$. Find c as a function of k when the conjugation is implemented.

HW 3. [Quadratic family.] The quadratic family is the family of maps $Q_c(x) = x^2 + c$. Recall that $\mathbb{R} \cup \{\infty\} = S^1$ via stereographic projection (Math 128) so that we can think of ∞ as a real point. By continuity, we set $Q_c(\infty) = \infty$.

A. When $c = 0$, find the basin of attraction of the fixed point $x = 0$.

B. Show that ∞ is an attracting fixed point for all values of c,

C. Show that for c sufficiently large enough, ∞ is the only fixed point and that its basin of attraction of ∞ is the entire extended real line.

D. Find the set of parameter values c such that Q_c has exactly two fixed points besides $x = \infty$ and find these fixed points $p_-(c), p_+(c)$ (with $p_-(c) < p_+(c)$).

For the remaining problems, c is in the parameter range of problem D, and $I(c) = [p_-(c), p_+(c)]$.

E. Show that Q_c maps $I(c)$ onto an interval either equal to , or containing $I(c)$.

H. Show that for c sufficiently negative there exist points $x \in I(c)$ mapped outside of $I(c)$ by Q_c and show that the set of such points form an interval $J(c)$ strictly inside $I(c)$.

J. Show that there is a nonempty range of parameters such that $J(c) \neq \emptyset$ and that $|Q_c'(x)| > 1$ for $x \in I(c) \setminus J(c)$.

(*) K. Show that for c in the range of problem J, the set of points x whose orbits are bounded forms a subset of $I(c)$ which is homeomorphic to the Cantor set. Use: any set which is compact, totally disconnected and perfect.