Nurowski twistors

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This is a report of my attempts to verify some facts that appear in the Nurowski-An paper on twistor space for rolling and what Pawel N. told me:

1. For each pseudo-riemanian oriented 4-manifold M of signature $(2, 2)$ Nurowski-An define $\mathbb{T}M$ as the set seld-dual null 2-planes in TM. It is a 5-manifold, in fact a bundle of real projective lines $p : \mathbb{T}M \to M$.

2. A rank-2 distribution $\mathcal{D}$ is defined on $\mathbb{T}M$ as follows: a point $\tilde{x} \in \mathbb{T}M$ stands for a null 2-plane $N_x \subset T_xM$ where $x = p(\tilde{x})$. Then $\mathcal{D}_{\tilde{x}} \subset T_{\tilde{x}}\mathbb{T}M$ is the horizontal lift of N_x wrt the Levi-Civita connection.

3. Some things to check:

   - $\mathcal{D}$ is integrable iff M is SD (or ASD, I forget)

   [Well, not exactly true, there might be some isolated points on the twistor fiber, max 4, where the distribution is integrable, ie fails to be bracket generating].

   - When M is the product of two surfaces with the diffrence metric, $\mathcal{D}$ is just the rolling distribution.

   - When $\mathcal{D}$ is not integrable it is automaticaly $(2, 3, 5)$.

   - $\mathcal{D}$ dependes only on the conformal class of the metric on M.

   Main question:

   - Find examples of irreducible M such that $\mathcal{D}$ is "flat" (G_2-symmetry).

   (Irreducible means not a product of surfaces with the difference metric).

4. Norowski-An have a recent additional paper where they propose looking at "Plebanski second heavenly metric", some class of metrics depending on an arbitrary function Θ of 4 variables, which appears somewhere in general realtivity, giving metrics which are SD and Ricci flat (I dont undersand the physical or geometrical motivation for introducing them); the flatness condition on $\mathcal{D}$, plus some simplifying assumptions on Θ, translate to a very complicated 8th order ODE for Θ, giving apparently many local solutions. The resulting metrics on M are irreducible (scalar flat reducible means the 2 surfaces are constant curvature, of same curvature).

5. Nurowski told me on the phone that Dennis The (now in Australia) gave a recent talk where he comes up with some class of homogeneous irreducible 2,2 metrics with flat twistor distribution. He uses a phenomenon in parabolic geometries called "symmetry gap", meaning in our case, that if a (2,3,5) distribution has "submaximal" symmetry group (or rather algebra, its all local) than the maximal dimension of the symmetry is 6 (if I am not mistaken). So its enough to present a 2,2 metric which is not SD, with 7 linearly independent conformal Killing fields.

6. **Some linear algebra with signature** $(2,2)$**.** Take $\mathbb{R}^4$ as the set of 2×2 matrices (linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$)

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

   The quadratic form $X \mapsto \det(X) = x_1x_4 - x_2x_3$ is of signature $(2,2)$.

   Note: Nurowski-An take the quadratic form $x_1x_2 - x_3x_4$, so their formulas look a little different.

7. The group $\tilde{G} = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts by

$$(g_+, g_-) \cdot X = g_+ X g_-^{-1},$$

   with ineffective kernel $\pm(I,I)$. Then $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})/\{\pm(I,I)\} \cong \mathrm{SO}_{2,2} \subset GL_4(\mathbb{R})$, the group of orientation preserving isometries of $\mathbb{R}^4$ (with respect to the quadratic form $\det(X)$). Let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$.

8. To each 2-plane in $\mathbb{R}^4$ correponds a line in $\Lambda^2(\mathbb{R}^2)$ (wedge the elemnts of a basis of this plane). A plane is said to be SD (self-dual) if the cooreponding line in $\Lambda^2(\mathbb{R}^2)$ is SD wrt the Hodge operator and the standard orientation. It turns out that each null plane is either SD or ASD.

9. A null vector in $\mathbb{R}^4$ is a matrix X such that $det(X) = 0$, hence of rank 1, i.e. with 1-dim kernel $L_- \subset \mathbb{R}^2$, or equivalently, a 1-dim image $L_+ \subset \mathbb{R}^2$. A null 2-plane is given by either the set of X with a common kernel L_- or a common image L_+. The former is an ASD plane and the later is SD (checked, tedious, ommited).

10. Let $N_0 = \{x_3 = x_4 = 0\}$ (the set of matrices with common image $L_0 = \mathbb{R}(1, 0)$ (the x-axis). More generaly, let $N_t = \{X \in \mathbb{R}^4 | \mathbb{R}(1, t)^t = Im(X)\}$, and $N_\infty = \{X \in \mathbb{R}^4 | \mathbb{R}(0, 1) = Im(X)\}$. Thus

$$N_t = \{X | x_3 - tx_3 = x_4 - tx_2 = 0\}, \quad t \in \mathbb{R}.$$

Let $\mathbb{T}$ be the space of SD null 2-planes in $\mathbb{R}^4$. From the description above of null planes, $\mathbb{T} = \{N_t | t \in \mathbb{R} \cup \{\infty\}\} = \mathbb{R}P^1$ (the set of 1-dim $L_+ \subset \mathbb{R}^2$).

11. G acts transitively on $\mathbb{T}$. The stabilizer of N_0 is $H := U \times \mathrm{SL}_2(\mathbb{R}) / \{\pm(I, I)\}$, where $U \subset \mathrm{SL}_2(\mathbb{R})$ is the set of upper triangular 2×2 matrices of determinant 1 (stabilizer of L_0). Hence $\mathfrak{h} = \mathfrak{u} \oplus \mathfrak{sl}_2(\mathbb{R})$, where $\mathfrak{u}$ is the set of upper triangular matrices of trace 0.

    The action of G on $G/H = \mathbb{T} = \mathbb{R}P^1$ factors through the standard action of $P\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}P^1$ by Mobious transformations, $t \mapsto \frac{at+b}{ct+d}$.

12. We have the following picture: the set of null vectors in $\mathbb{R}^4$ define the null quadric, a quadratic surface $Q \subset \mathbb{R}P^3$, given in homogeneous coordinates by $x_1 x_4 - x_2 x_3 = 0$, or in affine coordinates, $x = x_1/x_4$, $y = x_2/x_4$, $z = x_3/x_4$, by the graph of the function $z = xy$ (it looks like a saddle point at the origin; the level curves are hyperbolas $y = const/x$). The null planes define on Q a double rulling, so that through each point of the quadric pass exactly two lines, one SD and one ASD. In the affine coordinates x, y, z, the lines on $z = xy$ through the point $(x_0, y_0, x_0 y_0)$ are $z = x_0 y$ and $z = x y_0$. The group G acts on Q, preserving both rullings.

13. Here is a representation theoretic description. Let $S = \mathbb{R}^2$ be the standard 2-dimesional real representation of $\mathrm{SL}_2(\mathbb{R})$. Note that it is self-dual, ie $S \cong S^*$, via the invariant area form $\omega = dx \wedge dy$ on S, $v = (a, b) \mapsto \iota_v \omega = ady - bdx$. The group $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ has 2 basic representations on $\mathbb{R}^2$, denoted by S_+, S_- (the left factor acts non-trivialy on S_+, the right factor on S_-). Let $\omega_\pm \in \Lambda^2(S^*_\pm)$ be the invariant area forms on $S_\pm$.

    Next $\mathbb{R}^4 = Mat_{2\times 2}(\mathbb{R}) = Hom(S_+, S_-) = S_+ \otimes S^*_- = S_+ \otimes S_-$ (that's quite long, sorry) , equiped with the 2-form $\omega_+ \otimes \omega_-$, which is G-invariant, symmetric, and of signature $(2, 2)$. The set of decomposable vectors $v_+ \otimes v_-$ is G-invariant and obiously null. A SD plane has the form $v_+ \otimes S_-$, and an ASD is $S_+ \otimes v_-$. Thus we see that the set of SD null 2-planes correpsonds to $P(S_+) \cong \mathbb{R}P^1$.

    [Can go on, find the Hodge star operator. . .]

14. Now let M be an oriented manifold with a metric of signature $(2, 2)$. For $x \in M$, an oriented orthonormal coframe in T_xM is a linear orientation preserving isometry $u : T_xM \to \mathbb{R}^4$. The set of such coframes is a principal G bundle $p : B \to M$.

15. Let $\mathbb{T}M \to M$ be the bundle of self dual null planes on M. That is

    $$\mathbb{T}M = \{u^{-1}N_0 | u \in B\}.$$

    Then $\mathbb{T}M$ is a 5-dimensional manifold, a bundle of real projective lines over M .

    Note that $\mathbb{T}M$ can be identified with B/H , $u^{-1}N_0 \mapsto uH$.

    Another description of $\mathbb{T}M$: it is the projectivization of the 2-plane bundle $\mathbb{S}_+ \to M$, associated to the standard representaion of left factor of $\tilde{G} = \mathrm{SL}_2 \times \mathrm{SL}_2$. (The bundle $\mathbb{S}_+$ may not exist globaly, but its projectivization does exist).

16. A rank 2 distribution $\mathcal{D}$ is defined on $\mathbb{T}M$ by parallel translation of SD null 2-planes along null directions. Given a point $\tilde{x} \in \mathbb{T}M$ it stands por a SD null 2-plane $N_x \subset T_xM$, where $x = p(\tilde{x})$. Then $\mathcal{D}(\tilde{x})$ is the horizontal lift (wrt the Levi Civita connection of M) of N_x to $T_{\tilde{x}}\mathbb{T}M$.

17. Let us see why $\mathcal{D}$ depends only on the conformal class of the metric on M and the orientation.

First note that the definition of null vectors and SD null 2-plane (ie $\mathbb{T}M$ itself) depend only on the conformal structure and the orientation.

Next, it is a standard fact that null geodesics on a pseudo riemannian manifold depend only on the conformal structure (a conformal change of the metric will result only in a reparametrization of null geodesics; ***add an argument or reference***). Also, as we saw before, any null vector is contained in a unique SD null 2-plane.

It follows that given any null geodesic $x(s)$ in M, there is a unique 1 parameter family of SD null 2-planes N_s along $x(s)$ (i.e. $N_s \subset T_{x(s)}M$) such that $\dot{x}(s) \in N_s$. Furthermore, since parallel transport is an isometry, the assignement $s \mapsto N_s$ is parallel along $x(s)$ and is unaltered by a conformal change of the metric.

Now take a point $\tilde{x} \in \mathbb{T}M$, and $N_x \subset T_xM$, $x = p(\tilde{x})$, the corresponding null 2-plane. A vector $v \in \mathcal{D}(\tilde{x})$ iff $v = \dot{\tilde{x}}(0)$, where $\tilde{x}(s)$ is a curve in $\mathbb{T}M$ such that $\tilde{x}(0) = \tilde{x}$, $x(s) = p(\tilde{x}(s))$ is a curve in M so that $p_*v = \dot{x}(0) \in N_x$, and the family of null 2-planes N_s corresponding to $\tilde{x}(s)$ is parallel transported along $x(s)$. Since $\dot{x} = p_*v$ is null, we can take wlog $x(s)$ to be a null geodesic, hence $\tilde{x}(s)$ is unaltered by a conformal change of the metric, and the same is true for $v = \dot{\tilde{x}}(0)$.

18. Let ω be the soldering form on B. It is an $\mathbb{R}^4$ valued 1-form with the (defining) property

$$\omega(u) = p^*u.$$

Let θ be the Levi-Civita connection form. It is a $\mathfrak{g} \subset \mathfrak{gl}_4(\mathbb{R})$ valued G-equivariant 1-form on B with the (defining) property

$$d\omega = -\theta \wedge \omega.$$

(The wedge product on the right means we are multiplying a 4×4 matrix of 1-forms by a column vector of 1-forms).

19. A better way to write the last equation is to use $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$, so that $\theta = (\theta^+, \theta^-)$, where $\theta^\pm$ are $\mathfrak{sl}_2(\mathbb{R})$-valued 1-forms on B (a traceless

2×2 matrix of 1-forms), and ω, $d\omega$ are $\mathfrak{gl}_2(\mathbb{R})$-valued 1- and 2-forms (resp.) on B. So the last equation becomes

$$d\omega = -\theta^+ \wedge \omega - \omega \wedge \theta^-,$$

where on the right we have the wedging of 2×2 matrices of 1-forms.

The forms ω, θ together determine a conframing of B.

20. Consider the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Then $\pi\theta$ is a $\mathfrak{g}/\mathfrak{h}$-valued H-equivariant 1-form on B, hence descends to a 1-form $\theta_{\mathbb{T}}$ on $\mathbb{T}M = B/H$ with values in the vertical bundle $V = B \times_H \mathfrak{g}/\mathfrak{h}$ (kernel of the derivative of the projection $\mathbb{T}M \to M$). The kernel of $\theta_{\mathbb{T}}$ is the horizontal distribution on $\mathbb{T}M$ induced by the Levi-Civita connection of M.

21. In terms of our model $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{h} = \mathfrak{u} \oplus \mathfrak{sl}_2(\mathbb{R})$, $\theta = (\theta^+, \theta^-)$, we can take

$$\theta_{\mathbb{T}} = \theta^+_{21}.$$

The H-action is given by

$$R_h^* \theta_{\mathbb{T}} = \mathrm{Ad}(h^{-1})\theta_{\mathbb{T}} = a^{-2}\theta_{\mathbb{T}},$$

where

$$h = (u, g^-), \quad u = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

22. A moving coframe on M is a (local) section of $M \to B$. Then $u = (\omega_1, \ldots, \omega_4)$, where the $\omega_i = u^* x_i$ form a basis of 1-forms such that

$$\langle v, v \rangle = \omega_1(v)\omega_4(v) - \omega_2(v)\omega_3(v), \quad v \in TM.$$

For each $x \in M$ and $t \in \mathbb{R} \cap \infty$, $u(x)^{-1} N_t \subset T_x M$ is a SD null 2-plane, ie an element of $p^{-1}(x) \subset \mathbb{T}M$, given by the common kernels of the two 1-forms

$$\omega_1 - t\omega_3, \quad \omega_2 - t\omega_4.$$

23. Pull back the ω_i's to $\mathbb{T}M$. Then a coframing of $\mathbb{T}M$ is given by

$$\omega_1, \ldots, \omega_4, \theta_{\mathbb{T}}$$

and the distribution $\mathcal{D}$ on $\mathbb{T}M$ is given by the common kernels of the three 1-forms

$$\alpha_1 = \omega_1 - t\omega_3, \quad \alpha_2 = \omega_2 - t\omega_4, \quad \alpha_3 = \theta_{\mathbb{T}}.$$

The integrability condition for $\mathcal{D}$ is given the condition

$$d\alpha_i \equiv 0 \pmod{\alpha_1, \alpha_2, \alpha_3}.$$

[Now comes this calculation; or rather, work on B, not $\mathbb{T}M$, basicaly thinking of B as a reduction of the frame bundle of $\mathbb{T}M$ to $H \subset GL_5(\mathbb{R})$.]