GEOMETRIC REALIZATIONS OF HYPERELLIPTIC CURVES

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INTRODUCTION

Every elliptic curve $w^2 - z(z-1)(z-y) = 0, y \neq 0, 1$ is a torus and, in particular, can be represented as an identification space of a parallelogram. The gluing maps are translations. The present paper is concerned with the question of a corresponding realization of hyperelliptic curves

$$w^2 - \prod_{j=0}^{n}(z - y_j) = 0 \tag{0.1}$$

where $y \in \mathbb{C}^{n+1}, y_a \neq y_b, a \neq b$. The curve (0.1) has genus $[\frac{n}{2}]$, where $[\cdot]$ is the greatest integer function. We shall prove

Theorem 0.1. *Each curve (0.1) can be realized as the identification space of a centrally symmetric simple planar $2n$-gon P_y with opposite sides glued by translation. For an open set of y, of full measure in the parameter space, P_y can be taken to be convex.*

In genus one the exceptional set of Theorem 0.1 is empty. We shall prove

Theorem 0.2. *If $g > 0$, the curve $w^2 - (1 - z^{2g+1}) = 0$ cannot be realized as the identification space of a centrally symmetric convex $4g$-gon.*

The first statement in theorem 0.1 is a consequence of the analysis of a natural map from a certain space of polygons to the moduli space of punctured spheres. The second statement is shown in Section 4 to be a consequence of known facts about an action of $G = SL(2, \mathbb{R})$ on a circle bundle over the moduli space. Finally, Theorem 0.2 will be seen to be a consequence of a study of "periodic points" for this G action, points whose isotropy groups are lattices in G.

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1. SYMMETRIC POLYGONS

Fix $n > 1$, and define $\mathcal{P}(n)$ to be the set of pairs $p = (P, v)$ such that $P \subseteq \mathbb{C}$ is a simple, symmetric $2n$-gon and v is a vertex of P.

Given $p \in \mathcal{P}(n)$, set $v_0(p) = v$, and let $v_j(P)$, $0 \leq j < 2n$ be the remaining vertices of P, arranged in counterclockwise order. The map $H : \mathcal{P}(n) \to \mathbb{C}^n$, define by

$$H(p) = (v_0(p), \cdots, v_{n-1}(p)) \tag{1.1}$$

is a one-to-one map of $\mathcal{P}(n)$ onto an open subset of $\mathbb{C}^n$. In particular, $\mathcal{P}(n)$ carries the natural structure of a complex manifold of dimension n.

Continuing with $p \in \mathcal{P}(n)$, denote the edges of P by $e_j(p) = [v_{j-1}(p), v_j(p)]$, $1 \leq j \leq 2n$. Glue e_j to e_{j+n} by parallel translation. The identification space is a Riemann surface with ideal points corresponding to the vertices of P. As the gluing defined above sends $v_j(p)$ to $v_{j+n-1}(p)$, the equivalence class $[v_j]$ is identified as the set of v_k such that k is in the orbit of j under the map $i \to i + n - 1 \pmod{2n}$. When n is even, there is one vertex class, and when n is odd, there are two, $[v_0]$ and $[v_n]$.

$X(p)$ denote the Riemann surface defined above. The total angle of P at the vertex class is $(n-1)2\pi$ when n is even. When n is odd the total angle at each of the two vertex classes is $(\frac{n-1}{2})(2\pi)$ (because the isometry $z \to -z$ of P interchanges these classes). If w_p is the holomorphic 1-form on $X(p)$ determined by dz, then w_p has, by the total angle count just made, one or two zeros whose total order is $n - 2$ (n even) or $n - 3$ (n odd). As this total order must also be $2g - 2$, $g = \text{genus}(X(p))$, we have

$$g = g(X_p) = \left[\frac{n}{2}\right]. \tag{1.2}$$

The involution $\tau(z) = -z$ induces a holomorphic involution, also denoted τ, of $X(p)$. This involution has fixed points the (equivalence class of) points 0, $u_j(p) = \frac{1}{2}(v_j(p) + v_{j-1}(p)), 1 \leq j \leq n$. When n is even, the vertex class is fixed. As we have already mentioned, the vertex classes are interchanged by τ when n is odd.

Proposition 1.3. *If $p \in \mathcal{P}(n)$, then $X(p)$ is hyperelliptic.*

Proof. The involution τ has $n + 1$ fixed points if n is odd and $n + 2$ fixed points if n is even. In either case this number is $2\left[\frac{n}{2}\right] + 2 = 2g(X(p)) + 2$, and the proposition obtains.

In what follows we shall use $u_0(p) = 0, u_j(p), 1 \leq j \leq n$, to denote the points defined above as points in $X(p)$. The proposition implies there exists a unique holomorphic map

$$F : X(p) \to \mathbb{C} \cup \{\infty\} \tag{1.4}$$

such that (a) $F \circ \tau = F$, (b) $F(0) = 0$, $F(u_1) = 1$, and $F([v_0]) = \infty$ and (c) F is a biholomorphism modulo τ. Of course, $F([v_n]) = \infty$ when n is odd.

With notations as above, define $y \in \mathbb{C}^{n+1}$ by $y_0 = F(0)$ and $y_j = F(u_j), 1 \le j \le n$. Set up the quadratic differential

$$q_y(z) = \frac{dz^2}{\prod_{j=0}^{n}(z - y_j)}. \tag{1.5}$$

There exists $\pm\alpha \in \mathbb{C}^*/\pm 1$, $\mathbb{C}^* = \mathbb{C} \setminus 0$, such that

$$F^*(\alpha^2 q_y) = w_p^2. \tag{1.6}$$

Accordingly, we define

$$\Phi(p) = (y, (\pm\alpha)). \tag{1.7}$$

Define $\Omega_n \subseteq \mathbb{C}^{n+1}$ to be

$$\Omega_n = \{y \in \mathbb{C}^{n+1} | y_0 = 0, y_1 = 1, y_a \neq y_b, a \neq b\}.$$

The map

$$\Phi : \mathcal{P}(n) \to \Omega_n \times \mathbb{C}^*/\pm 1 \tag{1.8}$$

is now well-defined.

Remark 1.9. *Let $p = (P, v) \in \mathcal{P}(n)$ and $\beta \in \mathbb{C}^*$. If $\beta p = (\beta P, \beta v)$, and if $\Phi(p) = (y, \pm\alpha)$, then clearly*

$$\Phi(\beta p) = (y, \pm\alpha\beta). \tag{1.10}$$

Theorem 1.11. *The map Φ in (1.8) is holomorphic, locally one-to-one and surjective.*

Theorem 1.11 well be proved in Sections 2 and 3.

2. SURJECTIVETY OF Φ

Fix $n > 1$ and $y \in \Omega_n$. Denote by X the Riemann surface of the curve $w^2 - \prod_{j=0}^{n}(z - y_j) = 0$, and let τ be the hyperelliptic involution. The 1-form $\frac{dz}{w}$ determines, up to a factor ± 1, a holomorphic 1-form ω on X. The zero set E of ω lies above ∞ and has one element (of order $n-2$) when n is even and two elements (of order $\frac{n-3}{2}$) when n is odd.

Local solutions of $df = \omega$ determine an atlas $\mathcal{U}$ on $X \setminus E$ with transitions which are local translations. If $\theta \in \mathbb{R}$ let $\mathcal{F}(\theta)$ denote the oriented foliation of $X \setminus E$ which is the lift by $\mathcal{U}$ charts of the foliation of $\mathbb{C}$ by lines which make an angle θ with the horizontal. Leaves of $\mathcal{F}(\theta)$ are geodesic for the metric $|w|^2$. The fact $\tau^*\omega = -\omega$ implies $\mathcal{F}(\theta + \pi) = \tau\mathcal{F}(\theta)$.

We shall now make a construction. The first part of the construction will use only the properties (a) $\tau^*\omega = -\omega$, and (b) there exists $u_0 \in X \setminus E$ such that $\tau(u_0) = u_0$. (Of course, (a) and (b) imply $\tau^2 = Id$.)

Let γ_0 be an $|\omega|^2$-geodesic in $X \setminus E$ joining u_0 to a point $b \in E$. For example, a geodesic length minimizing path from u_0 to E will do. Replace ω by $\zeta\omega, |\zeta| = 1$, if necessary and r elabel so that γ_0 is a segment of an incoming separatrix of $\mathcal{F}(0)$ at b. Define $a = \tau(b)$, and parametrize the union $\gamma_0 \cup \tau(\gamma_0)$ as a geodesic path γ from a to b, i.e., an $\mathcal{F}(0)$ saddle connection.

For all but a countable set of θ the foliation $\mathcal{F}(\theta)$ admits no saddle connection. Fix such a θ with $0 < \theta < \pi$. The path γ above is transverse to $\mathcal{F}(\theta)$, and Poincaré map of $\mathcal{F}(\theta)$ on γ decomposes $X \setminus E$ into a set $R_1, \cdots, R_k$ of maximal flowboxes with bases $I_1, \cdots, I_k$ aligned along γ from left to right. (cf. [3]) The facts $a, b \in E$ and $\mathcal{F}(\theta)$ admits no saddle connection imply that k exceeds by one the number of incoming separatrices of $\mathcal{F}(\theta)$. Indeed, the boxes R_i and R_{i+1} are joined along a common segment of incoming separatrix at $x_i \in E$ (the zero set of ω).

Remark 2.1. *When n, X and ω are as in the first paragraph of this section, one finds readily that there are $n-1$ incoming separatrices, whether n is even or odd, and therefore there are n flowboxes, i.e., $k = n$ above.*

For each j the flowbox R_j is a parallelogram in $\mathcal{U}$-coordinates. That is, there exists a $\mathcal{U}$-chart function which maps R_j to a parallelogram based upon the real axis and having one angle θ. Let $x_0 = a$, and construct a path δ from a to b by connecting x_{i-1} to x_i by a geodesics in $R_i, 1 \leq i \leq k$. The portion of $\bigcup_{j=1}^{k} R_j$ which lies above γ and below δ is denoted P_0. It is evident there exists a $\mathcal{U}$-chart (P_0, f) such that $f(P_0)$ is a polygon in the upper half plane with base on the real axis centered at 0.

As $\mathcal{F}(\theta)$ and $\mathcal{F}(\theta + \pi)$ coincide but for orientation of their leaves, the same parallelograms are flowboxes for $\mathcal{F}(\theta + \pi)$. Now the top or R_j relative to $\mathcal{F}(\theta)$ is the base of R_j relative to $\mathcal{F}(\theta + \pi)$. The construction which led to δ above yields a path, denoted ϵ, from b to a. ϵ has the same segments as δ, but the orders of appearance and orientations are not the same. The region between γ and ϵ (below γ), denoted Q_0, is the domain of $\mathcal{U}$-chart (Q_0, h) such that $h(Q_0)$ is a polygon in the lower half plane with base on the real axis coinciding with the base of $f(P_0)$ above.

Let $\mathcal{O}$ be the region consisting of P_0, Q_0 and γ. The chart functions f and h coalesce on $\mathcal{O}$ to give $\mathcal{U}$-chart $(\mathcal{O}, F)$ such that $F(\mathcal{O})$ is a polygon with a diameter on the real axis, centered at 0.

As $\tau\mathcal{F}(\theta) = \mathcal{F}(\theta + \pi)$, it must be that $\tau(P_0) = Q_0$, and this implies $F \circ \tau = -F$. Therefore, $F(\mathcal{O})$ is a simple symmetric $2k$-gon, $k =$ number of flowboxes above.

Thus far we have used only (a) and (b) above. It has been noted in Remark 2.1 that in the case of interest $k = n$. We shall find a further restriction imposed by the existence of $2g(X) + 2$ fixed points for τ in the hyperelliptic case. When n is even, this number is $n + 2$; when n is odd, it is $n + 1$. The center of $\mathcal{O}$ is fixed by τ, and no other τ-fixd point lies in $\mathcal{O}$. τ fixes the single vertex class when n is even or odd, the set $\delta \setminus E$ will contain n fixed point of τ. If the component δ_i of δ which connects x_{i-1} to x_i inside R_i contains a τ fixed point, then $\tau(R_i) = R_i, \tau(\delta_i) = \delta_i$, and R_i, i.e., δ_i, contains only one fixed point. As there are exactly n parallelograms, it must be that $\tau(R_i) = R_i$ for $1 \leq i \leq n$.

The path ϵ from b to a is in all cases comprised of segments $\tau(\delta_1), \tau(\delta_2), \cdots, \tau(\delta_n)$ in that order. However, when τ has $2g + 2$ fixed points, we have proved that $\tau(\delta_i) = \delta_i$ but for parametrization. It follows that X is the identification space of $F(\mathcal{O})$ obtained by gluing opposite edges, i.e., edges i and $i + n, 1 \leq i \leq n$.

Recall that we are seeking a pair $p = (P, v) \in \mathcal{P}(n)$ such that $\Phi(p) = (y, \pm\alpha)$, where y is as given above and $\alpha \in \mathbb{C}^*$. In our construction we have associated to (X, ω) a centrally symmetric simple polygon $P = F(\mathcal{O})$ which realizes X. The construction made use of an arbitrary Weierstrass point $u_0 \in X \setminus E$. It is therefore no loss of generality to suppose u_0 sits above 0 on the Riemann surface of $w^2 - \prod_{j=0}^{n}(z - y_j) = 0$.

Now choose $v \in P$ a vertex so that in the canonical ordering $u_1 = u_1(P,v)$ sits above $1 = y_1$. By construction of the map Φ there exists $\alpha \in \mathbb{C}^*$ such that $\Phi((P,v)) = (y, \pm\alpha)$. As $y \in \Omega_n$ is arbitrary, surjectivity of Φ is now a consequence of Remark 1.9.

3. Φ IS A LOCAL BIHOLOMORPHISM

The purpose of this section is to prove that the map $\Phi : \mathcal{P}(n) \to \Omega_n \times \mathbb{C}^*/\pm 1$ is a local biholomorphism and thus to complete the proof of Theorem 1.11. We shall first prove Φ is holomorphic, but only to establish its continuity. Using this continuity and a determinant calculation from [4] we shall then prove Φ admits a right inverse on a neighborhood of each image point.

Fix $p = (P,v) \in \mathcal{P}(n)$, and let P be triangulated by a symmetric triangulation t whose vertex set is the set of vertices of P. If $q = (Q,u)$ is sufficiently close to p, then t determines a triangulation $t(q)$ of Q with similar properties. Let $F_{p,q} : X(p) \to X(q)$ be the canonical PL map determined by the PL-structures $t(p)$ and $t(q)$. F_{pq} preserves the ordering of Weierstrass points. A standard calculation (cf.[4]) show that the Beltrami differential $\mu_{p,q}$ associated to F_{pq} varies holomorphically in q for p fixed. Moreover, if τ_p, τ_q are the hyperelliptic involutions, symmetry of t implies $F_{pq} \circ \tau_p = \tau_q \circ F_{pq}$ and therefore $\mu_{p,q} \circ \tau_p = \mu_{p,q}$. It follows that F_{pq} induces a quasiconformal homeomorphism H_{pq} from $X(p)/\tau_p$ to $X(q)/\tau_q$ and that Beltrami differential varies holomorphically with q. If $\Phi(p) = (y(p), \pm\alpha(p))$ and $\Phi(q) = (y(q), \pm\alpha))$, the definitions imply $H_{pq}(y(p)) = y(q)$, and therefore $y(q)$ varies holomorphically with q. It follows easily from the definition of Φ that $\pm\alpha(q)$ varies holomorphically with q.

In order to construct local right inverses fix $p = (P,v)$ with $\Phi(p) = (y, \pm\alpha(p))$, as above. Let $F : X(p) \to \mathbb{C} \cup \{\infty\}$ be the map (1.4) which is used in the definition of Φ. For $1 \le j \le n$ let γ_j be the segment of ∂P from $u_j(p) = \frac{1}{2}(v_j(p), v_{j-1}(p))$ to $v_j(p)$. Also, let γ_0 be a smooth path in P from 0 to $v_0 = v$. Define $\delta_j = F(\gamma_j), 0 \le j \le n$. δ_j is a path from $y_j(p)$ to ∞. Choose a version of $\frac{dz}{w} = \eta$ such that $F^*(\alpha\eta) = \omega$, and declare η to have values on δ_j which are limits from the lefthand side of δ_j. We have

$$\begin{aligned} v_0(p) &= 2\alpha \int_{\delta_0} \eta \\ v_j(p) - v_{j-1}(p) &= \int_{\delta_j} \eta. \end{aligned} \tag{3.1}$$

As y varies in a small neighborhood of $y(p)$ it is possible to vary paths $\delta_j(y)$ (from y_j to ∞) and the definition of $\frac{dz}{w} = \eta_y$ in such a way that

$$\Psi(y,\beta) = \left\{ 2\beta \int_{\delta_j(y)} \eta_y \right\}_{1 \le j \le n} \tag{3.2}$$

is holomorphic in (y,β). We restrict the subscripts to $1 \le j \le n$ because the remaining integral is minus the sum of the other integrals. There are n integrals and n parameters (y,β). If (y,β) is sufficiently close to $(y(p), \alpha(p))$ then $\Psi(y,\beta)$ determines a symmetric polygon whose distinguished vertex is a function (negative sum) of the integrals (3.2). As Φ is continuous, the relation (3.1) implies Φ is locally biholomorphic as soon as Ψ has this property.

The Jacobian determinant of the map Ψ has been calculated in [4] in a more general setting. One finds

$$\left(\det\frac{\partial\Psi}{\partial(\beta,y)}\right)^2 = \left(\pi^{n+1}\Gamma^2\left(\frac{n-3}{2}\right)\beta^{n-1}\prod_{0\le k<l\le n}(y_k-y_l)^{-1}\right)^2. \tag{3.3}$$

It follows that Ψ is nowhere singular. Theorem 1.11 is thereby proved.

4. DYNAMICS OVER MODULI SPACE

Theorem 1.11 asserts that the map $\Phi : \mathcal{P}(n) \to \Omega_n \times \mathbb{C}^*/\pm 1$ is a surjective local biholomorphism. Let λ be the euclidean volume element on $\mathcal{P}(n)$. The determinant formula (3.3) suggests a prescription for a volume element ν on $\Omega_n \times C^*/\pm 1$

$$\nu = \left(\frac{i}{2}\right)^n |\beta|^{2n-2}\prod_{0\le k<l\le n}|y_k-y_l|(d\beta\wedge d\bar\beta)\bigwedge_{j=2}^n dy_j\wedge d\bar y_j. \tag{4.1}$$

There exists a constant $c(n) > 0$ such that

$$\lambda = \Phi^*(c(n)\nu). \tag{4.2}$$

It follows that λ projects to a volume element on the equivalence relation determined by Φ.

If $p = (P, v) \in \mathcal{P}(n)$, we denote the area of P by $N(p)$. If $\Phi(p) = (y, \pm\alpha)$, then but for a dimensional constant

$$N(p) = \frac{i}{2}\int_{\mathbb{C}}|\alpha|^2\frac{dz\wedge d\bar z}{|w|^2} \tag{4.3}$$

where $w^2 = \prod_{j=0}^n(z-y_j)$. In what follows we take (4.3) for the definition of $N(P)$. Also, express the right-hand side of (4.3) as $|\alpha|^2M(y)$ so that

$$N(p) = |\alpha|^2M(y). \tag{4.3'}$$

Set up the $(2n-1)$-form (with $\alpha = |\alpha|e^{i\theta}$) $\tilde\mu$, where

$$\tilde\mu = \frac{|\alpha|^{2n-1}}{M(y)}\prod_{0\le k<l\le n}|y_j-y_l|^{-2}\bigwedge_{j=2}^n\left(\frac{i}{2}dy_j\wedge d\bar y_j\right)\wedge d\theta$$

and observe that, up to a scale factor, $d(|\alpha|^2M(y))\wedge\tilde\mu = \nu$. The restriction of $\tilde\mu$ to the constant norm surface $|\alpha|^2M(y) = 1$ is the form

$$\mu = \frac{d\theta\wedge\bigwedge_{j=2}^n(\frac{i}{2}dy_j\wedge d\bar y_j)}{M(y)^n\prod_{0\le k<l\le n}|y_k-y_l|^2}. \tag{4.4}$$

Recall from [4]: if $\Lambda_n = \{(y,\pm\alpha)||\alpha|^2M(y) = 1\}$, then

$$\int_{\Lambda_n}\mu < \infty. \tag{4.5}$$

Identify μ with the measure it defines on Λ_n.

For a moment we shall use G to denote the group $SU(1,1) \cong SL(2,\mathbb{R})$ of 2×2 complex matrices $A = \begin{pmatrix} \xi & \bar{\eta} \\ \eta & \bar{\xi} \end{pmatrix}$ such that $\det A = 1$. G acts $\mathbb{R}$-linearly on $\mathbb{C}$ by $Az = \xi z + \bar{\eta}\bar{z}$, and this induces an action of G on $\mathcal{P}(n)$, e.g. coordinatewise in (1.1). Denote this latter action by $p \to T_A p$.

If $p \in \mathcal{P}(n)$ and $A \in G$, then T_A induces a quasiconformal homeomorphism from $X(p)$ to $X(T_A p)$. As $A^* dz = \xi dz + \bar{\eta} d\bar{z}$, $A = \begin{pmatrix} \xi & \bar{\eta} \\ \eta & \bar{\xi} \end{pmatrix}$, the homeomorphism h_A, which is real analytic away from the vertex class(es), satisfies $h_A^* \omega_{T_A p} = \xi \omega_p + \bar{\eta}\bar{\omega}_p$.

Let $p_1, p_2 \in \mathcal{P}(n)$ be such that $\Phi(p_1) = \Phi(p_2)$. The identity map on $\mathbb{C} \cup \{\infty\}$ lifts to a biholomorphism $\phi : X(p_1) \to X(p_2)$ such that $\phi^* \omega_{p_2} = \omega_{p_1}$ and ϕ preserves the ordering of the Weierstrass points. If $h_A^j : X(p_j) \to X(T_A p_j)$ are as above, the composition $\phi_A : X(T_A p_1) \to X(T_A p_2)$ where $\phi_A = h_A^2 \circ \phi \circ (h_A^1)^{-1}$ satisfies $\phi_A^* \omega_{T_A p_2} = \omega_{T_A p_1}$, and therefore $\Phi(T_A p_1) = \Phi(T_A p_2)$. The action of G on $\mathcal{P}(n)$ descends to an action on $\Omega_n \times \mathbb{C}^* / \pm 1$.

It is obvious from the definitions that the G-action preserves the volume element λ on $\mathcal{P}(n)$. As λ projects to the volume element ν ((4.1)), the action of G which is induced upon $\Omega_n \times \mathbb{C}^* / \pm 1$ by Φ must preserve ν. As $\det A = 1$, $A \in G$, the G action preserves the area function $N(p)$. It follows from (4.3′) and the definition (4.4) of the restriction volume from μ on Λ_n that μ is also G-invariant.

Theorem 4.6. *The triple (Λ_n, μ, G) is a real analytic, ergodic, finite measure preserving action.*

Proof. Finiteness has been noted above. (Λ_n, G) is a component of a stratum, in the sense of [6], and ergodicity of topological components is established in [6].

5. CLOSED ORBIT AND CONVEXITY

Theorem 4.6 implies that almost every $u \in \Lambda_n$ has a dense G-orbit. We turn now to a discussion of behavior at the opposite extreme, points u such that Gu is closed.

Denote by $\Gamma(u)$ the isotropy group of $u \in \Lambda_n$. We recall that $\Gamma(u)$ enjoys three properties for all u: 1. $\Gamma(u)$ is discrete. 2. $\Gamma(u)$ is not cocompact. 3. If $A \in \Gamma(u)$, then $\mathrm{tr}(A)$ is an algebraic integer. Generically, $\Gamma(u)$ is trivial. However, for a dense set of u $\Gamma(u)$ is a lattice. For example, by the Remark, p.579 in [5] $\Gamma(u)$ is commensurable with $SL(2,\mathbb{Z})$ when $u = \Phi(p)$ is such that $H(p) \in \mathbb{C}^* \mathbb{Q}^n$ ($H(\cdot)$ is defined in (1.1).) Consideration of the integrals (3.1)-(3.2) yields a corresponding criterion for $u = (y, \pm\alpha)$ which depends upon y and integrals of the 1-form $\frac{dz}{w}$, $w^2 = \prod_{j=0}^{n}(z - y_j)$.

In this section we shall make use of consequences of an isotropy group being a lattice to establish

Theorem 5.1. *Let $X(p)$ be the Riemann surface of the equation $w^2 = 1 - z^{2g+1}$, and let ω_g be a lift to $X(g)$ of the 1-form $\frac{dz}{w}$. The pair $(X(g), \omega_g)$ cannot be realized as $(X(p), \omega_p)$ for any $p = (P, v) \in \mathcal{P}(2g)$ such that P is a* **convex** *polygon.*

For each n let $\mathcal{P}_c(n)$ be the set of $p = (P, v) \in \mathcal{P}(n)$ such that P is convex. Clearly, $G\mathcal{P}_c(n) = \mathcal{P}_c(n)$ and $\mathcal{P}_c(n)$ contains a nonempty open set. If $\Lambda_{n,c} = \Phi(\mathcal{P}_c(n))$, then Theorem 4.6 implies $\Lambda_{n,c}$ is a dense set of full measure; indeed, its interior has this property.

Question 5.2. *Let $\Lambda_{n,b} = \Lambda_n \setminus \Lambda_{n,c}$. If $n > 3$, does there exist $u \in \Lambda_{n,b}$ such that Gu is not closed? If the answer is 'yes', does there exist $u \in \Lambda_{n,b}$ such that Gu is dense in $\Lambda_{n,b}$?*

We shall now give the proof of Theorem 5.1. To begin let (X, ω) be a pair consisting of a closed Riemann surface X and a nontrivial holomorphic 1-form ω. Let $\mathcal{U}$ be the atlas on $X \setminus E, E = \omega^{-1}0$, as in Section 2. Denote by $\mathrm{Aff}(\mathcal{U})$ the group of orientation preserving homeomorphism ϕ of X which are affine in $\mathcal{U}$-coordinates. $\mathcal{F}(\theta), \theta \in \mathbb{R}$, has the same meaning as in Section 2.

Let $\Gamma = \Gamma(\mathcal{U})$ be the image in G of $\mathrm{Aff}(\mathcal{U})$ under the map which assigns to $\phi \in \mathrm{Aff}(\mathcal{U})$ its derivative $D\phi$ in $\mathcal{U}$-coordinates. $\Gamma(\mathcal{U})$ enjoys the properties 1-3 which were listed above for $\Gamma(u)$([5]).

If $\theta \in \mathbb{R}$, define $A(\theta) \subseteq \mathrm{Aff}(\mathcal{U})$ to be the set of ϕ such that $D\phi(\cos\theta, \sin\theta) = (\cos\theta, \sin\theta)$. Notice that $A(\theta)$ is a subgroup and for each $\phi \in A(\theta)$ $D\phi \in \Gamma(\mathcal{U})$ is unipotent.

Lemma 5.3. *Assume $\Gamma(\mathcal{U})$ is a lattice. The following are equivalent:*

(A) $\mathcal{F}(\theta)$ admits a saddle connection.

(B) $DA(\theta)$ is nontrivial.

(C) $\mathcal{F}(\theta)$ partitions $X \setminus E$ into cylinders of closed leaves.

A proof of the lemma may be found in [5].

Lemma 5.4. *Let (X, ω) be such that $\Gamma(\mathcal{U})$ is a lattice with a single cusp. If $\theta_1, \theta_2 \in \mathbb{R}$ are such that $\mathcal{F}(\theta_j)$ admits a saddle connection for $j = 1, 2$, there exists $\phi \in \mathrm{Aff}(\mathcal{U})$ such that $\phi\mathcal{F}(\theta_1) = \mathcal{F}(\theta_2)$ or $\mathcal{F}(-\theta_2)$.*

Proof. The assumptions combine to imply $DA(\theta_1)$ and $DA(\theta_2)$ are conjugate in $\Gamma(\mathcal{U})$. Choose $\psi \in \mathrm{Aff}(\mathcal{U})$ such that $(D\psi)^{-1}DA(\theta_2)(D\psi) = DA(\theta_1)$. If $\psi\mathcal{F}(\theta_1) \stackrel{\text{def}}{=} \mathcal{F}(\theta)$ then θ is such that $DA(\theta) = DA(\theta_2)$, and this implies $\theta = \theta_2$ or $-\theta_2$ modulo 2π. That is, $\psi\mathcal{F}(\theta_1) = \mathcal{F}(\theta_2)$ or $\mathcal{F}(-\theta_2)$. The lemma is proved.

To apply the lemma let (X, ω) be such that $\Gamma(\mathcal{U})$ is a lattice with one cusp, and let $\theta_0 \in \mathbb{R}$ be such that $\mathcal{F}(\theta_0)$ decomposes $X \setminus E$ into cylinders of closed leaves. Denote the maximal such cylinders by $C_1, \cdots, C_r$, and let their heights be denoted $h_1, \cdots, h_r$. Now suppose $\theta \in \mathbb{R}$ is such that $\mathcal{F}(\theta_0)$ admits a saddle connection. Lemma 5.4 implies that there exists $\phi \in \mathrm{Aff}(\mathcal{U})$ and a choice of $\pm$ such that $\phi\mathcal{F}(\theta_0) = \mathcal{F}(\pm\theta)$. As $D\phi$ is a linear transformation, there exists $t > 0$ such that the cylinder ϕC_j has height $th_j, 1 \leq j \leq r$, relative to $\mathcal{F}(\theta)$.

Proof of Theorem 5.1. We take $(X, \omega) = (X_g, \omega_g)$, $g > 1$. It is proved in [5] that $\Gamma(\mathcal{U})$ is a $(2, 2g+1, \infty)$ triangle group and, in particular, $\Gamma(\mathcal{U})$ has only one cusp. In [5] it is shown that for one choice of θ $\mathcal{F}(\theta)$ has exactly g maximal cylinders of closed leaves which, up to a common constant factor have lengths $h_j = \sin\left(\frac{2j-1}{2g+1}\pi\right), 1 \leq j \leq g$. If the cylinders are denoted $C_1, \cdots, C_g$, there are two additional facts to record for later reference. A. $\partial C_1 \subseteq C_2, \partial C_g \subseteq C_{g-1}$ and $\partial C_j \subseteq C_{j-1} \cup C_{j+1}, 1 < j < g$. B. If $1 < j < g$, then up to a common constant factor each side of C_j is comprised of a pair of saddle connections of lengths $\sin\frac{2\pi j}{2g+1}$ and $\sin\frac{2\pi(j-1)}{2g+1}$. We also record the elementary inequality

$$2\sin\frac{2\pi(j-1)}{2g+1} > \sin\frac{2\pi j}{2g+1} \qquad (2 \le j \le g) \tag{5.5}$$

($\sin\frac{2\pi j}{2g+1} < \sin\frac{2\pi(j-1)}{2g+1} + \sin\frac{2\pi}{2g+1} \le \sin\frac{2\pi(j-1)}{2g+1}, 2 \le j \le g$).

Now suppose $p = (P, v) \in \mathcal{P}(2g)$ is such that P is convex and $(X(p), \omega_p)$ is isomorphic to $(X(g), \omega_g)$. We shall prove this leads to a contradiction. The vertices of P are ordered as $v = v_0, v_1, \cdots$ in the usual way.

Let l_j denote the oriented segment $\overrightarrow{v_{4g-j}v_j}, 1 \le j \le 2g$. We observe first that these segments cannot be pairwise parallel. For if they are, the foliation $\mathcal{F}(\theta)$, where θ is the common direction, has cylinders of lengths $\|l_1\|, \|l_1\| + \|l_2\|, \cdots, \|l_{g-1}\| + \|l_g\|$. Moreover, central symmetry and convexity imply $\|l_1\| \le \|l_2\| \le \cdots \le \|l_g\|$. When $g = 2$, one concludes that $h_2 > 2h_1$, contradicting $h_j = \sin\left(\frac{2j-1}{5}\pi\right), j = 1, 2$. When $g > 2$, one concludes $h_1 < h_2 < \cdot < h_g$ contradicting $h_j = \sin\left(\frac{2j-1}{2g+1}\pi\right)$.

Let k be the first positive integer such that l_k is not parallel to l_1. The edges $e_1, \cdots, e_{k-1}$ are cross-sections of cylinders $D_1, \cdots, D_{k-1}$ of closed leaves, and these cylinders have on each side a pair of saddle connections of lengths l_j and $l_{j-1}, 1 \le j \le k-1 (l_0 = 0)$. Property B and the fact every leaf of $\mathcal{F}(\theta)$ has length at least $\|l_1\|$ imply $\|l_j\| = \sin\frac{2\pi j}{2g+1}, 1 \le j < k$.

The left side of l_{k-1} is one of the saddle connections on the right hand side of D_k. Parallel segments from the vertices v_{2g+k} and v_{2g-k} do not coincide because, by assumption, l_k and l_{k-1} are not paralell. It follows that the length of the second saddle connection on the right side of D_k has length at least $2l_{k-1}$. This implies $\sin\frac{2\pi(k-1)}{2g+1} \le \sin\frac{2\pi k}{2g+1}$ contradicting (5.5). We have reached a contradiction, and Theorem 5.1 is proved.

Remark 5.6. *It is not difficult to see that each $p \in \mathcal{P}(3)$ admits a convex equivalent. We believe that an argument similar to the one above will show that the curves $w^2 = 1 - z^{2g+2}, g > 1$ equipped with $\frac{dz}{w}$, do not admit convex representatives in $\mathcal{P}(2g+1)$.*

6. CHARACTERIZATION OF CLOSED ORBITS

For any $u \in \Lambda_n$ the canonical map $G/\Gamma(u) \to Gu$ is continuous. Consideration of codimension three transversals to the G action shows that this map is a homeonorphism when Gu is closed in Λ_n.

Lemma 6.1. *If Gu is closed, and if $\Gamma(u)$ is viewed as a Fuchsian group in the disc, the limit set of $\Gamma(u)$ is all of S^1.*

Proof. Let K be the rotation subgroup of G, and let $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $t \in \mathbb{R}$. According to [1] it is true for almost all $k \in K$ that the ω-limit set of ku (in Λ_n) relative to $\{g_t | t \in \mathbb{R}\}$ is nonempty. For the disc picture this translates to the statement that for almost all k the geodesic from zero in direction k does not diverge to ∞ in $\Gamma \setminus \Delta, \Delta =$disc. It follows in particular that $\Gamma(u)$ has no domain of discontinuity on S^1. That is, $\Gamma(u)$ has S^1 for its limit set.

Proposition 6.2. *If Gu is closed, and if $\Gamma(u)$ is finitely generated, then $\Gamma(u)$ is a lattice.*

Proof. A finitely generated Fuchsian group with limit set S^1 must be a lattice.

Question 6.3. *If $u \in \Lambda_n$, is $\Gamma(u)$ finitely generated?*

J.Smillie has observed that Proposition 6.2 is true without the assumption that $\Gamma(u)$ is finitely generated. Question 6.3 appears to be open.

We shall give an outline of a proof of Smillie's theorem (for the setting of (Λ_n, G)):

1. If ν_n is the probability measure on Λ_n which is the image of normalized Haar measure on K under $k \to ku$, the orbit $G\nu_u$ is relatively compact in the weak-$*$ topology of *probability* measures on Λ_n. This is implicit in [1] and follows from its techniques.

2. If ν is a cluster point of $\{g_t \nu_u | t \to +\infty\}$, then ν is invariant under the group N of upper triangular unipotent matrices. This is also from [1].

3. Let $D = \{v \in \Lambda_n | \lim_{t\to\infty} g_t v = \infty\}$. The facts $G\nu_u$ is relatively compact in the space of *probability* measure and ν above is a cluster point imply $\nu(D) = 0$. In particular, a.e. ergodic component ν_e of ν satisfies $\nu_e(D) = 0$.

4. If ν_e is as in 3., then Ratner's Main Theorem [2] implies $G\nu_e = \nu_e$.

Step 4 establishes the fact that when Gu is closed, $G/\Gamma(u)$ supports a finite G-invariant measure. Therefore, $\Gamma(u)$ is a lattice.

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