TROPICAL CURVES, THEIR JACOBIANS AND THETA FUNCTIONS

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Abstract. We study Jacobian varieties for tropical curves. These are real tori equipped with integral affine structure and symmetric bilinear form. We define tropical counterpart of the theta function and establish tropical versions of the Abel-Jacobi, Riemann-Roch and Riemann theta divisor theorems.

1. Introduction

In this paper we study algebraic curves defined over the so-called tropical semifield

$$\mathbb{T} = \mathbb{R} \cup \{-\infty\}$$

equipped with the following *tropical arithmetic operations* for $x, y \in \mathbb{T}$.

$$\text{“}x + y\text{”} = \max\{x, y\}, \ \ \text{“}xy\text{”} = x + y.$$

Here we use the quotation marks to distinguish between the classical and tropical operations. It is easy to check that each of these operations is commutative and associative and that together they satisfy to the distribution law.

These arithmetic operations do not turn $\mathbb{T}$ into a field. The idempotency of addition, “$x + x$” $= x$ makes subtraction impossible. However it does admit a division: “$\frac{x}{y}$” $= x - y$ for $y \neq -\infty$. Furthermore, $\mathbb{T}$ has the additive zero “$0_{\mathbb{T}}$” $= -\infty$ and the multiplicative unit “$1_{\mathbb{T}}$” $= 0$. We say that $\mathbb{T}$ is a *semifield* as we have all the operations except for subtraction.

It is possible to define *algebraic varieties over* $\mathbb{T}$ in a similar way as one defines algebraic varieties over fields, e.g. $\mathbb{C}$ or $\mathbb{R}$, see [Mik]. In this paper, we are concerned only with tropical curves which we define in the next section.

Tropical arithmetic operations are related to classical ones via the so-called *dequantization*. Namely let us define a family of addition operations on $\mathbb{R} \cup \{-\infty\}$ by

$$x \oplus_t y = \log_t(t^x + t^y),$$

where t is the parameter ranging from e to $+\infty$. For any finite value of t this operation is (by its definition) induced from $\mathbb{R}_{\geq 0} = \{z \in \mathbb{R} \mid z > 0\}$. Clearly, $\log_t(t^x t^y) = x + y$ and thus $\mathbb{R} \cup \{-\infty\}$ equipped with $\oplus_t$ for addition and $+$ for

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multiplication is isomorphic (as a semifield) to $\mathbb{R}_{\geq 0}$ with the usual arithmetical operations. However, we have

$$\lim_{t \to +\infty} \log_t(t^x + t^y) = \max\{x, y\},$$

which exhibits $\mathbb{T}$ as the limit semifield.

This degeneration of addition in $\mathbb{R}_{\geq 0}$ underlies the following collapse of complex algebraic varieties (see [KS01] and [GW00] for the case of Calabi-Yau varieties) in $(\mathbb{C}^*)^n$). Let

$$K = \{a(t) = \sum_{j \in I} \alpha_j t^j \mid \alpha_j \in \mathbb{C}\},$$

here $I \subset \mathbb{R}$ is countable and well-ordered and $a(t)$ converges for $t \in [0, t_0]$, $t_0 > 0$. Such K is an example of the so-called non-Archimedean field: there exist a valuation, i.e. a function

$$\mathrm{val} : K = K \to \mathbb{T}$$

such that $\mathrm{val}^{-1}(-\infty) = 0_K$ and for any $z, w \in K$ we have $\mathrm{val}(z + w) \leq$ "$\mathrm{val}(z) + \mathrm{val}(w)$" $= \max\{\mathrm{val}(z), \mathrm{val}(w)\}$ and $\mathrm{val}(zw) =$ "$\mathrm{val}(z)\,\mathrm{val}(w)$" $= \mathrm{val}(z) + \mathrm{val}(w)$. We have the valuation map

$$\mathrm{Val} : (K^*)^n \to \mathbb{R}^n,$$

where $K^* = K \setminus \{0\}$, defined by $\mathrm{Val}(z_1, \ldots, z_n) = (\mathrm{val}(z_1), \ldots, \mathrm{val}(z_n))$.

Let $V \subset (K^*)^n$ be an algebraic variety over K. Following Kapranov [Kap00] one may associate to V its *non-Archimedean amoeba*, i.e. the image $\mathcal{A} = \mathrm{Val}(V) \subset \mathbb{R}^n$. For small values of $t > 0$ the variety V determines a complex algebraic variety $V_t \subset (\mathbb{C}^*)^n$ by plugging t to all converging series in t that form the coordinates of $V \subset (K^*)^n$. According to Gelfand, Kapranov and Zelevinski [GKZ94] one may also associate to V_t its amoeba by taking the image $\mathcal{A}_t = \mathrm{Log}_t(V_t) \subset \mathbb{R}^n$, where

$$\mathrm{Log}_t : (\mathbb{C}^*)^n \to \mathbb{R}^n$$

is defined by $\mathrm{Log}_t(z_1, \ldots, z_n) = (\log_t(z_1), \ldots, \log_t(z_n))$. Note that the base t of the logarithm coincides with the parameter of deformation of V_t. We have the following diagram

$$\begin{array}{ccc} V_t & \xrightarrow{\subset} & (\mathbb{C}^*)^n \\ \downarrow{\scriptstyle \mathrm{Log}_t} & & \downarrow{\scriptstyle \mathrm{Log}_t} \\ \mathcal{A}_t & \xrightarrow{\subset} & \mathbb{R}^n \end{array}$$

The limit of $\mathrm{Log}_t(V_t) \subset \mathbb{R}^n$ coincides with the non-Archimedean amoeba $\mathcal{A} \subset \mathbb{R}^n$ and is an example of tropical variety by degeneration of complex varieties.

It was shown by Kapranov [Kap00] that if $V \subset (K^*)^n$ is a hypersurface then $\mathcal{A} = \mathrm{Val}(V)$ depends only on the valuation of coefficients of the polynomial defining V. These valuations are elements of $\mathbb{T}$ and can be considered as coefficients of the *tropical polynomial* defining $\mathcal{A}$.

The construction above gives a restricted class of tropical varieties. In particular, they are embedded to $\mathbb{R}^n = (\mathbb{T}^*)^n$. As the subject of this paper is tropical curves we won't give a general definition of higher-dimensional tropical varieties here (see [Mik06]). On the other hand, in this paper we define tropical curves intrinsically and study their inner geometry that does not depend on a particular embedding to an ambient space.

To conclude the introduction we give the definition of a particularly useful higher-dimensional tropical variety.

Definition 1.1. The *tropical projective space* $\mathbb{TP}^n$ consists of the classes of $(n+1)$-tuples of tropical numbers such that not all of them are equal to $-\infty$ with respect to the following equivalence relation. We say that

$$(x_0 : \cdots : x_n) \sim (y_0 : \cdots : y_n),$$

if there exists $\lambda \in \mathbb{T}^*$ such that $x_j = y_j + \lambda$, $j = 0, \dots, n$.

Note that $\mathbb{TP}^n$ contains $\mathbb{R}^n = (\mathbb{T}^*)^n$: a point $(x_1, \dots, x_n)$ corresponds to $(0 : x_1 : \cdots : x_n)$. Thus we have an embedding

$$\iota_n : \mathbb{R}^n \subset \mathbb{TP}^n. \tag{1}$$

Also we have $n+1$ affine charts $\mathbb{T}^n \to \mathbb{TP}^n$, given by the (tropical) ratio with the jth coordinate.

2. Some tropical algebra

2.1. **Tropical modules.** Recall that a commutative semigroup with zero is a set V equipped with an arithmetic operation (called addition and denoted with "+") that is commutative, associative and such that there exists a neutral element (called zero in V and denoted with $-\infty$).

Definition 2.1. A tropical module or a $\mathbb{T}$-*module* V is a commutative semigroup with zero equipped with a map $\mathbb{T} \times V \to V$ (called multiplication by a scalar) such that

- "$c(v + w)$" = "$cv + cw$" for any $c \in \mathbb{T}$;
- "$c(d(v))$" = "$(cd)v$" for any $c, d \in \mathbb{T}$, $v \in V$;
- "$0v$" = "v" for the multiplicative unit $0 \in \mathbb{T}$ and any $v \in V$;
- if "cv" = "dv" for some $c, d \in \mathbb{T}$, $v \in V$ then either $c = d$ or $v = -\infty$.

A basic example of a tropical module is the free n-dimensional tropical module $\mathbb{T}^n$. The addition and multiplication by a scalar are the corresponding coordinate-wise tropical operations. However, unlike the case with honest vector spaces over fields, there do exist other tropical modules.

Example 2.2. Let $V_n \subset \mathbb{T}^n$ be the submodule generated by elements

$$e_1 = (-\infty, 0, \dots, 0),\ e_2 = (0, -\infty, 0, \dots, 0),\ \dots,\ e_n = (0, \dots, 0, -\infty).$$

Elements of V_n are linear combinations "$\sum_{j=1}^n c_j e_j$" for some $c_j \in \mathbb{T}$.

It is easy to see that (at least as a topological space with topology induced from $\mathbb{T}^n = [-\infty, +\infty)^n$), the module V_n is 2-dimensional yet it is distinct from $\mathbb{T}^2$ (or $\mathbb{T}^k$ with any other k). Indeed, suppose that $a_j \in \mathbb{T}$ and $a_1 \geq a_2 \geq ... \geq a_n$. Then

$$\text{“}\sum a_j e_j\text{”} = (a_2, a_1, \dots, a_1).$$

For algebraic considerations we need an intrinsic definition of dimension.

Definition 2.3. Let V be a tropical module. We say that its dimension is *smaller than* k if for any elements $v_1, \dots, v_k \in V$ and any linear combination

$$v = \sum_{j=1}^k c_j v_j,$$

$c_j \in \mathbb{T}$, the element v can be presented as a tropical linear combination of a proper subset of $\{v_1, \dots, v_k\}$.

We set *the dimension* to be equal to the maximal value of k such that the dimension is not smaller than k.

Proposition 2.4. *The dimension of* V_n *(see Example 2.2) equals 2 regardless of* n.

Proof. Suppose that $v = \sum_{j=1}^k c_j v_j$. We saw that each $c_j v_j$ is parameterized by two numbers $a_1^{(j)} \geq a_2^{(j)} \in \mathbb{T}$. In the collection $\{c_j v_j\}$ we keep the vector v_j with the maximal $a_1^{(j)}$ and another vector $v_{j'}$ with $c_{j'} v_{j'}$ having the maximal coordinate at the position of $a_2^{(j)}$. □

Let us introduce another useful characteristic of a $\mathbb{T}$-module V. The *rank* of V is the minimal number of generators of V. Note that even though the modules V_n have the same dimension, they have different ranks.

Proposition 2.5. *The dimension from Definition 2.3 coincides with the topological dimension of* V.

Proof. Fixing $v_1, \dots, v_k$ gives us a piecewise-linear continuous map $\mathbb{T}^k \to V$:

$$(c_1, \dots, c_n) \mapsto \text{“}\sum_{j=1}^k c_j v_j\text{”}.$$

Suppose it is a surjection to $U \subset V$. If the topological dimension of its image is d then $k \geq d$. On the other hand by the convexity argument U has to be contained in the image of the union of the d-dimensional coordinate subspaces of $\mathbb{T}^k$. □

2.2. **Projectivization of a tropical module.** We may generalize the construction of $\mathbb{TP}^n$ by projectivizing *any* tropical module V, not necessarily a free module $\mathbb{T}^{n+1}$.

Definition 2.6. Let V be a tropical module. Its *projectivization* $\mathbb{P}(V)$ consists of the classes of the non-zero elements of v with respect to the following equivalence relation. We say that $v \sim v'$, if there exists $\lambda \in \mathbb{T}^*$ such that $v =$ "$\lambda v'$".

Clearly we have $\mathbb{P}(\mathbb{T}^{n+1}) = \mathbb{TP}^n$.

Example 2.7. The projectivization $\mathbb{P}(V_n)$ of the module from Example 2.2 is called Γ_n. It will serve as our local model for a tropical curve. Note that by definition we have $\Gamma_n \subset \mathbb{TP}^{n-1}$.

3. Tropical curves

3.1. **Definitions.** Any affine-linear map $A : \mathbb{R}^k \to \mathbb{R}^{k'}$ is a composition of a linear map $L_A : \mathbb{R}^k \to \mathbb{R}^{k'}$ and a translation. We call A a $\mathbb{Z}$-*affine map* if its linear part L_A is defined over $\mathbb{Z}$ (i.e. given by a matrix with integer entries) no matter what is its translational part.

Let C be a connected topological space homeomorphic to a locally finite 1-dimensional simplicial complex.

Definition 3.1. A complete tropical structure on C is an open covering $\{U_\alpha\}$ of C together with embeddings

$$\phi_\alpha : U_\alpha \to \mathbb{TP}^{k_\alpha - 1}$$

called *the charts*, $k_\alpha \in \mathbb{N}$, subject to the following conditions that must hold for any α and β.

- We have $\phi_\alpha(U_\alpha) \subset \Gamma_{k_\alpha}$.
- If $U' \subset U_\alpha$ is an open subset then $\phi_\alpha(U')$ is open in $\Gamma_{k_\alpha} \subset \mathbb{TP}^{k_\alpha - 1}$.
- The "finite part" $\iota^{-1}_{k_\alpha - 1} \circ \phi_\alpha \circ \phi^{-1}_\beta \circ \iota_{k_\beta - 1}$ of the corresponding overlap maps is a restriction of a $\mathbb{Z}$-affine linear map $\mathbb{R}^{k_\beta - 1} \to \mathbb{R}^{k_\alpha - 1}$. Here we consider the map $\iota^{-1}_{k_\alpha - 1} \circ \phi_\alpha \circ \phi^{-1}_\beta \circ \iota_{k_\beta - 1}$ only where it is defined. The embedding $\iota_n : \mathbb{R}^n \to \mathbb{TP}^n$ is taken from (1).
- If $S \subset U_\alpha$ is a closed set in C then $f(S) \cap \iota_{k_\alpha - 1}(\mathbb{R}^{k_\alpha - 1})$ is a closed set in $\iota_{k_\alpha - 1}(\mathbb{R}^{k_\alpha - 1})$.

The space C equipped with a complete tropical structure is called a *tropical curve.* We are especially interested in compact curves. E.g. the first two curves on Figure 1 are not compact as each of them has an open end. Meanwhile, it is easy to compactify these curves by adding a point at infinity for each open end.

Clearly, the valence of a vertex of a 1-dimensional simplicial complex is a topological invariant. Furthermore, a point inside of an edge can be prescribed to be of valence 2 as it is possible to introduce a new 2-valent vertex at such point.

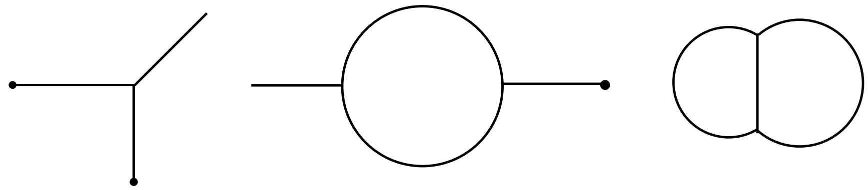

FIGURE 1. Examples of tropical curves of genus 0, 1 and 2.

Definition 3.2. The *finite part* C° of C is the complement of all 1-valent vertices in C. The 1-valent vertices are called the *endpoints* of C.

Proposition 3.3. *For any chart $\phi_\alpha : U_\alpha \to \mathbb{TP}^{k_\alpha - 1}$ we have*

$$\phi_\alpha^{-1}(\iota_{k_\alpha - 1}(\mathbb{R}^n)) = C^\circ \cap U_\alpha.$$

This proposition follows from the last condition in Definition 3.1, the one that is responsible for the completeness of the tropical structure. In other words, the 1-valent vertices must sit on the boundary part of $\mathbb{T}^N$.

Definition 3.4. Suppose that C_1 and C_2 are tropical curves. A continuous map

$$f : C_1 \to C_2$$

is called tropical if for every $x \in C_1$ there exists an open neighborhood $U \ni x$ and charts $U_\alpha \supset U$ and $U_\beta \ni f(x)$, $U_\alpha \subset C_1$, $U_\beta \subset C_2$, such that $\iota_{k_\beta} \circ \phi_\beta \circ f \circ \phi_\alpha^{-1} \circ \iota_{k_\alpha}$ is a restriction of a $\mathbb{Z}$-affine map $\mathbb{R}^{k_\alpha} \to \mathbb{R}^{k_\beta}$.

Tropical curves are called isomorphic if there exist mutually inverse tropical maps between them.

3.2. **Equivalence of tropical curves.** It turns out that some non-isomorphic tropical curves serve as two distinct models for the same variety. It is convenient to introduce an equivalence relation identifying such models.

Let C be a tropical curve, $x \in C^\circ$ be a point in the finite part of C. Restricting to another chart if needed we can find a chart $\phi_\alpha : U_\alpha \to \mathbb{TP}^{k_\alpha - 1}$ such that $\phi_\alpha(x) = (1_{\mathbb{T}} : \dots : 1_{\mathbb{T}}) = (0 : \dots : 0)$ is the "center" of the curve $\Gamma_{k_\alpha} \subset \mathbb{TP}^{k_\alpha - 1}$.

The projection along the last coordinate defines a continuous map $\lambda_{k_\alpha} : \mathbb{TP}^{k_\alpha} \setminus \{(-\infty : \dots : -\infty : 0)\} \to \mathbb{TP}^{k_\alpha - 1}$. The map

$$\lambda_{\Gamma_{k_\alpha}} : \Gamma_{k_\alpha + 1} \to \Gamma_{k_\alpha} \tag{2}$$

obtained by restricting λ_{k_α} to $\Gamma_{k_\alpha + 1}$ and extending it to $(-\infty : \dots : -\infty : 0)$ is continuous.

The map $\lambda_{\Gamma_{k_\alpha}}$ contracts the interval $[(0 : \dots : 0), (-\infty : \dots : -\infty : 0)]$ to $(0 : \dots : 0)$ and is a homeomorphism to its image when restricted to $\Gamma_{k_\alpha + 1} \setminus ((0 : \dots : 0), (-\infty : \dots : -\infty : 0)]$. Let

$$\tilde{U_\alpha} = \lambda_{\Gamma_{k_\alpha}}^{-1}(\phi_\alpha(U_\alpha)).$$

We form $\tilde{C} = C_x$ by replacing the neighborhood $\tilde{U_\alpha}$ with U_α. Clearly, the tropical structure naturally extends to the whole curve $\tilde{C}$. Topologically, $\tilde{C}$ is the result of gluing of a closed interval $[0, 1]$ to C by identifying $0 \in [0, 1]$ with $x \in C$. Denote with $\tilde{x} \in \tilde{C}$ the point corresponding to $1 \in [0, 1]$. (See [Mik06] for a more general procedure that can be used to define higher-dimensional tropical varieties.)

Definition 3.5. The map

$$\tilde{C} \to C \tag{3}$$

induced by (2) is called the *elementary equivalence* of tropical curves. Two curves C_1 and C_2 are called *tropically equivalent* if they can be connected with a sequence of elementary equivalences (in any direction).

This equivalence relation provides a convenient tool for working with curves with marked points. If $x \in C^\circ$ is in the affine part then we may replace C with $\tilde{C} = C_x$ and x with $\tilde{x}$. In this way we may replace a curve C with a collection of marked points on it with an equivalent curve $\bar{C}$, so that all the marked points will be the endpoints of $\bar{C}$. This trick turns out to be very useful for the setup of the Gromov-Witten theory, cf. [Mik].

Here we would just like to note that putting marked points to the end of the equivalent curve can be used to puncture a tropical curve. If we need to remove a point $x \in C^\circ$ then first we replace (C, x) with an equivalent pair $(\tilde{C}, \tilde{x})$ and then treat $\tilde{C} \setminus \tilde{x}$ as the result of puncturing C at x. Note that $\tilde{C} \setminus \tilde{x}$ is a tropical curve while $C \setminus x$ itself is not. Furthermore, the contraction map (3) can be considered as a morphism analogue of a Zariski-open set (in the Grothendieck topology style), see [Mik] for details.

3.3. **Tropical curves as metric graphs.** A tropical curve turns out to carry the same information as the so-called *metric graph*. Recall that the edges adjacent to one-valent vertices of a graph are called *leaves*. Other (non-leaf) edges are called *inner*. A metric graph (cf. e.g. [BHV01]) is a graph equipped with finite positive length for all its inner edges. The leaves of a metric graph are prescribed the (positive) infinite length.

Let Γ be a connected finite graph and $\mathcal{V}_1(\Gamma)$ be its 1-valent vertices (or *endpoints*). We may consider

$$\Gamma^\circ = \Gamma \setminus \mathcal{V}_1(\Gamma).$$

If Γ is a metric graph then Γ° is a complete metric space (with an inner metric). Vice versa, a complete metric space homeomorphic to Γ° for some finite graph Γ defines a metric graph.

Recall that if X is any topological space one can consider its maximal compactification $\bar{X}$ by adding a point "at infinity" for each end of X. Clearly, Γ and $\bar{\Gamma^\circ}$ are homeomorphic if Γ is a finite graph.

Proposition 3.6. *There is a natural 1-1 correspondence between compact tropical curves and metric graphs.*

Proof. Let $L \subset \mathbb{R}^k$ be a line of a rational slope and let $\xi \in \mathbb{Z}^k$ be a primitive (i.e. not divisible by an integer) vector parallel to L. Define the distance between points $x, y \in L$ to be $\frac{||x-y||}{||\xi||}$, where $||\ .\ ||$ stands for any norm in the vector space $\mathbb{R}^k$. This metric is preserved by the overlap maps $\phi_\alpha \circ \phi_\beta^{-1}|_{\phi_\beta(U_\alpha \cap U_\beta)\cap \mathbb{R}^{k_\beta}}$. This gives us a natural inner metric on a tropical curve.

Vice versa, if C is a metric graph then we set the metric-preserving charts to $\mathbb{R}$ for the interior of the edges; charts to $\mathbb{T}$ for small neighborhoods of 1-valent vertices and isometric embedding charts to Γ_k for small neighborhoods of multivalent vertices. □

3.4. **Regular and rational functions on curves.** By the very definition a tropical curve C is a topological space homeomorphic to a graph equipped with an additional geometric structure which we call *the tropical structure*. It allows one to define a sheaf of regular function on C. We say that a function $\mathbb{R}^n \to \mathbb{R} \subset \mathbb{T}$ is $\mathbb{Z}$ *affine-linear* if it is obtained from a function $\mathbb{R}^n \to \mathbb{R}$ that is linear over $\mathbb{Z}$ by adding a real constant.

Let $U \subset C$ be an open set.

Definition 3.7. A function $f : U \to \mathbb{T}$ is called *regular* if for any $x \in U$ there exists a chart

$$\phi_\alpha : U_\alpha \to \mathbb{T}^n \subset \mathbb{TP}^n,$$

$n = k_\alpha - 1$, $U_\alpha \ni x$, a tropical polynomial $g : \mathbb{T}^n \to \mathbb{T}$,

$$g(x_1, \ldots, x_n) = \text{“}\sum_{(j_1,\ldots,j_n)\in\mathcal{A}} a_{j_1\ldots j_n} x_1^{j_1} \ldots x_n^{j_n}\text{”} = \max_{(j_1,\ldots,j_n)\in\mathcal{A}} \{a_{j_1\ldots j_n} + j_1 x_1 + \cdots + j_n x^n\},$$

where $\mathcal{A}$ is a finite subset of $\mathbb{Z}^n_{\geq 0}$, such that the function $f \circ \phi_\alpha^{-1} - g$ (which is defined on $\phi_\alpha(U)$) restricted to $\mathbb{R}^n \subset \mathbb{T}^n$ is $\mathbb{Z}$ affine-linear. Here $\mathbb{T}^n \subset \mathbb{TP}^n$ is embedded given by $(x_1, \ldots, x_n) \mapsto (0 : x_1 : \cdots : x_n)$. If the function g may be chosen to be a constant then f is called *an affine-linear function on U*.

A function $h : U \to \mathbb{T}$ is called *rational* if there exist two regular functions $f_1, f_2 : U \to \mathbb{T}$ such that

$$h = \text{“}\frac{f_1(x)}{f_2(x)}\text{”} = f_1(x) - f_2(x)$$

for any $x \in U \cap C^\circ$.

Clearly, regular functions form a sheaf of tropical algebras on C. The same holds for rational functions. The sheaf of regular functions is called the *structure sheaf* and is denoted with $\mathcal{O}_C^{\mathbb{T}}$.

3.5. **Projective tropical curves, tropical curves in $\mathbb{R}^n$ and in other higher-dimensional tropical varieties.** Linear morphisms maybe used to embed or immerse tropical curves into *tropical toric varieties* that arise as compactifications of $\mathbb{R}^n = (\mathbb{T}^\times)^n$. Our main example of such compactification is $\mathbb{TP}^n$.

Let $f : C \to \mathbb{T}$ be an affine-linear function. For every edge $E \subset C$ we may define *the slope* of f once we fix an orientation of E. Indeed if ξ is a primitive tangent vector consistent with the choice of the orientation then the slope is just the partial derivative $\frac{\partial f}{\partial \xi}$. If we do not fix the orientation of E then the slope is only defined up to the sign.

Definition 3.8. A map $C \to \mathbb{T}^n$ is called a *linear morphism* if it is given in coordinates by affine-linear functions. A map $C \to \mathbb{TP}^n$ is called a linear morphism if locally (with respect to the charts $\mathbb{T}^n \subset \mathbb{TP}^n$) it is given by linear morphisms.

Remark. This definition agrees with a more general definition given in [Mik06]. It is easy to show that every map $C \to \mathbb{TP}^n$ locally given in coordinates by regular functions admits a resolution by a linear morphism $\tilde{C} \to \mathbb{T}^n$ for an equivalent tropical curve $\tilde{C}$ with a contraction $\tilde{C} \to C$.

Let C be a compact tropical curve and $h : C \to \mathbb{TP}^n$ be a linear morphism. The image $h(E)$ of an edge $E \subset C$ is either a point (if the slope of all coordinates is zero) or a straight interval with a rational slope (as the slopes of all coordinates are integers). We define *the weight* $w(E)$ as the GCD of the coordinate slopes of E. The collection of the slopes of all coordinates gives the *weight vector* $\xi_E \in \mathbb{Z}^n$ (defined up to a sign unless we specify the orientation of E). The ratio $\frac{\xi_E}{w(E)}$ is a primitive integer vector parallel to E.

The image $h(E)$, if it not a single point, is contained in the finite part $\mathbb{R}^n \subset \mathbb{TP}^n$ if and only if both vertices adjacent to E have valence greater than 1. All 1-valent vertices of C are mapped to the boundary part $\mathbb{TP}^n \setminus \mathbb{R}^n$. Let $P \subset C$ be a vertex and $E_1, \dots, E_k$ are the edges adjacent to P. If $k > 1$ then $h(P) \in \mathbb{R}^n$ and $\sum_{j=1}^k \xi_k = 0$.

Recall that a *leaf* of a graph is an edge adjacent to a 1-valent vertex. The *degree* of a projective curve is the number of leaves (counted with some weight, cf. [Mik07]) adjacent to any of the $(n+1)$ components of the boundary divisor $\mathbb{TP}^n \setminus \mathbb{R}^n$. Note that linear morphisms of different curves may have the same image, see Figure 2 for an example.

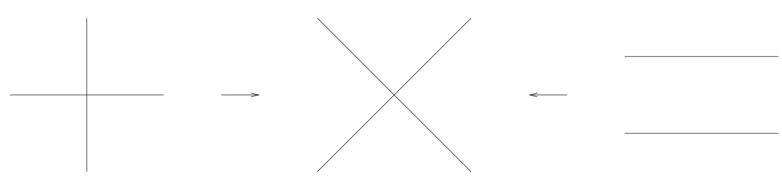

FIGURE 2. A cross figure (in the center) obtained as the image of two different linear morphisms.

If $X \subset \mathbb{TP}^n$ is a projective tropical variety (see [Mik06]) then we define a (parameterized) tropical curve in X as a linear morphism $C \to \mathbb{TP}^n$ such that its image is contained in X. Note that to embed X to a projective space we often need to pass to an equivalent model of X.

4. Divisors and line bundles

4.1. **Tangent vectors and 1-forms.** Let ξ be a primitive integer vector tangent to an interval in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^{k_\beta}$. Note that the overlap maps $\phi_\alpha \circ \phi_\beta^{-1}|_{\phi_\beta(U_\alpha \cap U_\beta) \cap \mathbb{R}^{k_\beta}}$ takes ξ to a primitive integer tangent vector to an interval in $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^{k_\alpha}$ (since the overlap map admits a $\mathbb{Z}$-affine inverse). Thus we have a well-defined notion of a *primitive tangent vector* to a point $p \in C$.

If p is a point in the interior of an edge then it has two primitive tangent vectors ξ_1 and ξ_2. We have $\xi_2 = -\xi_1$. An *integer vector tangent to C at p* is an integer multiple of ξ_1 (or ξ_2).

If p is a vertex of C then we can distinguish between outward and inward primitive tangent vectors at p. If p is a k-valent vertex then it has k outward tangent vectors $\xi_1, \dots, \xi_k$ with

$$\sum_{j=1}^{k} \xi_j = 0$$

(the curve C is an abstract curve now, but this equality makes sense in any affine chart of C). An integer vector tangent to C at p is an integer linear combination of $\xi_1, \dots, \xi_{k-1}$. It defines an integer vector in $\mathbb{R}^{k_\alpha}$ for any chart $U_\alpha \ni p$.

Note that there is no difference between a 2-valent vertex and an interior point of an edge. Both of them have affine charts to $\mathbb{R}$. At our convenience we may introduce extra 2-valent vertices by subdividing an edge or do the opposite operation.

Definition 4.1. Let $\mathrm{Aff}_\mathbb{R}$ be the sheaf of affine functions with real (i.e., not necessarily integral) slope. Define the *real* cotangent local system $\mathcal{T}^*$ on C by the following exact sequence of sheaves:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Aff}_\mathbb{R} \longrightarrow \mathcal{T}^* \longrightarrow 0,$$

A *1-form* on C is a global section of $\mathcal{T}^*$.

In particular, since a 1-form ω is locally constant it must be balanced at any $p \in C$:

$$\sum_{i=1}^{\mathrm{val}(p)} \omega(\xi_i) = 0, \tag{4}$$

where $\mathrm{val}(p)$ is the valence of p and the ξ_i's are the outward primitive integral tangent vectors at p.

Remark. The fact that the forms have to be constant on edges can be interpreted in the degeneration picture as follows. The regular forms on complex curves that survive in the limit are of the form $\frac{\alpha}{2\pi i}\frac{dz}{z}$ on long cylinders (edges to be). The coefficient α would be the value of the limiting form ω on the primitive vector tangent to the edge. The balancing condition (4) reflects the fact that the sum of residues of a rational form on $\mathbb{P}^1$ (a vertex to be) is zero.

4.2. **Divisors.** A *divisor* on C is a formal linear combination of points in C with integer coefficients: $D = \sum a_i p_i$, where $a_i \in \mathbb{Z}$. We define its *degree* in the usual way: $\deg(D) := \sum a_i$. The divisor D is called *effective* if all the coefficients are non-negative. Clearly, the degree of an effective divisor is non-negative. The set of all divisors form an abelian group which we will denote by $\mathrm{Div}(C)$.

Given a rational function f on an open subset $U \subset C$ one can consider the divisor

$$(f) := \sum_{p\in U}\left(\sum_{i=1}^{k(p)} \frac{\partial f}{\partial \xi_i}(p)\right) p. \tag{5}$$

If $U = C$, i.e. f is a global function, then (f) called *principal*. We say that two divisors are *linearly equivalent*: $D_1 \sim D_2$, if $D_1 - D_2$ is principal.

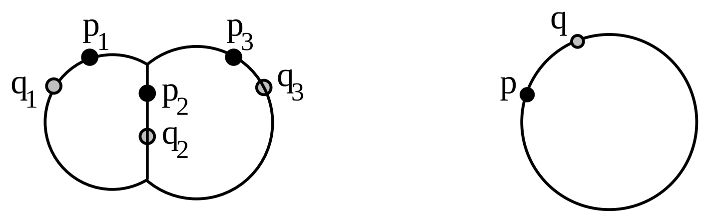

FIGURE 3. Linear equivalence: $p_1 + p_2 + p_3 \sim q_1 + q_2 + q_3$, but $p \not\sim q$.

Proposition 4.2. *Let U be an open set in a tropical curve C whose boundary consists of the points $z_1, \dots, z_k$. Let ν_i be the outward (that is pointing away from U) primitive tangent vectors at the z_i. Let f be a rational function on U with divisor $(f) = \sum a_j p_j$. Then*

$$\sum a_j = \sum_{i=1}^{k} \frac{\partial f}{\partial \nu_i}(z_i). \tag{6}$$

Proof. We observe that the formula is additive when connecting pieces of the curve at points other than the p_i. Note also, that the formula trivially holds for affine linear functions. On the other hand at the points p_i it is just the definition of the order $a_i = \sum_{j=1}^{\mathrm{val}(p_i)} \frac{\partial f}{\partial \xi_j}(p_i)$. □

Corollary 4.3. *A principal divisor on a compact curve has degree zero.*

4.3. **Line bundles.**

Definition 4.4. A *line bundle* on C is an $\mathcal{O}^*$-torsor, where the sheaf of affine functions $\mathcal{O}^*$ acts on the structure sheaf by tropical multiplication. Equivalently a line bundle L is a topological space together with the following data:

- A continuous projection $\pi : L \to C$ with fibers $\mathbb{T}$.
- Every point $p \in C$ has an open neighborhood $U_p \ni p$ and families of *trivialization* maps $\{\phi\}_V : \pi^{-1}(V) \cong V \times \mathbb{T}$ for any open $V \subset U_p$, which are the restrictions from $\{\phi\}_{U_p} : \pi^{-1}(U_p) \cong U_p \times \mathbb{T}$, and such that any two trivializations $\phi_1, \phi_2 \in \{\phi\}_V$ differ by (tropical multiplication by) an affine linear function in V.

Given a sufficiently fine open covering $\{U_\alpha\}$ of C a line bundle can be specified by transition functions $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}$ between local trivializations. The functions $f_{\alpha\beta}$ are affine linear and satisfy the usual cocycle condition. Another choice of trivializations will result in changing the cocycle by a coboundary. A continuous map $L_1 \to L_2$ which respects the projection and local trivialization up to affine functions is an isomorphism of line bundles. Thus, as in the classical geometry, the isomorphism classes of line bundles are parameterized by the first Čech cohomology $H^1(C, \mathcal{O}^*)$. This is a group and we will refer to it as the Picard group $\mathrm{Pic}(C)$.

Proposition 4.5. *Equivalent tropical curves have the same Picard groups.*

4.4. **Local sections, global sections and linear systems.**

Given an open subset $U \subset C$ a section $s : U \to \pi^{-1}(U_p)$ is *regular*, respectively *rational*, if for any open $V \subset U$ and a trivialization $\phi : \pi^{-1}(V) \cong V \times \mathbb{T}$, s becomes a regular, resp. rational, function on V. The notion does not depend on trivializations. Neither does the formal sum in (5). Thus a global rational section of L defines a divisor on C. There is the usual correspondence between line bundles and divisors up to equivalence:

Proposition 4.6. *Every divisor defines a line bundle together with its rational section. This section is defined uniquely up to adding a constant (i.e. up to tropical multiplication).*

Conversely, every line bundle L has a rational section. The divisors of any two rational sections of L are linearly equivalent.

Proof. Let $D = \sum a_i p_i$ be a divisor. We can cover C by open sets U_i so that each point p_i is contained in a unique U_i. Then for each U_i we choose a rational function f_i whose order of zero (pole) at p_i is a_i. The incompatibilities over the intersections define a Čech cocycle with values in affine linear functions. Hence we get a line bundle. The rational section is given by the collection $\{f_i\}$.

Conversely, we can choose g disjoint open intervals N_i in the interiors of edges such that $C \setminus \cup N_i$ is a (connected) tree (cf. §4.5). Also choose g closed intervals $N_i^\epsilon \subset N_i$. Then a given line bundle L can be trivialized over $C \setminus \cup N_i^\epsilon$ and over each of the N_i. Choose a rational function f on $C \setminus \cup N_i^\epsilon$ which is affine linear on the

$N_i \setminus N_i^\epsilon$. To extend f to a section s of L over C one needs integral PL functions f_i on the N_i equal to given affine linear functions on the pairs of ends $N_i \setminus N_i^\epsilon$. This is always possible. The divisor of s is $(f) + (f_1) + \cdots + (f_g)$. □

If a divisor D has degree d we will say that the corresponding line bundle $L(D)$ has degree d as well. The groups of degree d divisors and line bundles will be denoted by $\mathrm{Div}^d(C)$ and $\mathrm{Pic}^d(C)$ respectively. Now we can reformulate Proposition 4.6.

Corollary 4.7. *For every d there is a canonical isomorphism* $\mathrm{Pic}^d(C) \cong \mathrm{Div}^d(C)/\sim$.

We will now look at the space of global regular sections of a given line bundle. Let D_0 be a divisor and let s_0 be the corresponding rational section (unique up to an additive constant) of the line bundle $L := L(D_0)$. Then any other rational section s of L is given by $s = s_0 + f$, where f is a rational function. Of course, the section s is regular if the divisor $D_0 + (f)$ is effective.

The space of global sections $\Gamma(C, L)$ has the structure of a $\mathbb{T}$-module. Given $s_1, s_2 \in \Gamma(C, L)$ one can take their tropical sum $\max\{s_1, s_2\}$ by choosing trivializations over open sets $U \subset C$ and considering s_1, s_2 as functions on U. The result is a section of the same bundle and it is independent of the trivializations. Adding a constant to a section is well defined as well.

Given a divisor D we denote by $|D|$ its complete linear system, that is the collection of all *effective* divisors linearly equivalent to D. Clearly, the set $|D|$ is the tropical projectivization of the $\mathbb{T}$-module $\Gamma(C, L)$ for the corresponding line bundle L. We will study linear systems in more details in the Riemann-Roch section.

4.5. **Universal covering and fundamental domains.** For the proof of Jacobi Inversion and Riemann Theorem it will be convenient to work with the universal covering $\hat{C} \to C$. Here is a way to fix a fundamental domain.

Let us choose g points $z_1, \ldots, z_g$ in C equipped with outward primitive tangent vectors $\eta_1, \ldots, \eta_g$. Let z_i^ϵ be the point on C which is distance ϵ from z_i into the direction η_i. For a small $\epsilon > 0$ we may consider the set $C \setminus \{z_1^\epsilon, \ldots, z_g^\epsilon\}$. If it is a (connected) tree then we refer to $\{z_i, \eta_i\}$ as a set of *break points* and call $D = z_1 + \cdots + z_g$ the *break divisor*.

The whole business with z_i^ϵ is only needed to serve the case when z_j are points of valence greater than 2. In this case some of z_j may coincide if the vectors η_j are different.

The choice of break points on C is equivalent to the choice of a connected fundamental domain $T \subset \hat{C}$ (see Fig. 4). Indeed, given T the break points are the missing boundary points of T with η_i pointing inside, see Fig. 4. On the other hand, given a collection of break points z_i, η_i, the fundamental domain T is defined by the property that the lifting map $C \to T \subset \hat{C}$ is continuous except at the z_i from the direction of η_i.

In general, given a (not necessarily connected) fundamental domain $T \subset \hat{C}$ we can define a *pseudo-break divisor* as follows. Using the identification of points in T

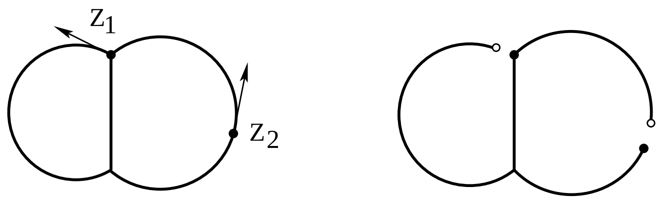

FIGURE 4. Break points and the fundamental domain T.

with those in C we set

$$D_T = \sum_{z\in C} (\mathrm{val}_C(z) - \mathrm{val}_T(z))z, \tag{7}$$

where $\mathrm{val}_C(z)$ and $\mathrm{val}_T(z)$ stand for the valence of z in C and T, respectively. Clearly a pseudo-break divisor D_T is a break divisor if T is connected (a tree). Also since T contains no cycles a simple Euler characteristic count gives the following criterion.

Lemma 4.8. *A pseudo-break divisor D_T is a break divisor if it is of degree g.*

We remark that given a break divisor $D = z_1 + \cdots + z_g$ there is more than one way to choose tangent vectors η_i to make z_i, η_i into a collection of break points. This choice will affect the choice of the fundamental domain T. In any case, T is equal to the limit of the fundamental domains T^ϵ (induced from the break points z_i^ϵ) when $\epsilon \to 0$, and the corresponding break divisors D_{T^ϵ} tend to D_T.

5. TORI AND POLARIZATIONS

A tropical *torus* X is the quotient $\mathbb{R}^n/\Lambda$, where $\mathbb{R}^n$ is considered with an integral affine structure, i.e. with a fixed integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, and $\Lambda \subset \mathbb{R}^n$ is (another) lattice. We will denote with $(\mathbb{R}^n)^*$ the dual space, and let $(\mathbb{Z}^n)^*$ and Λ^* be the dual lattices to $\mathbb{Z}^n$ and Λ, respectively.

Remark. Tropical tori, especially the ones endowed with positive polarization (see below) had been considered (under different names) in the framework of degenerations of Abelian varieties (cf. [Mum72], [FC90], [AN99], [Ale02]) long before tropical geometry came into existence.

5.1. **Line bundles and polarizations.** We can repeat most of our definitions and results about line bundles and sections for curves. A line bundle on a torus is defined by an element in $H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^* = \mathrm{Aff}$ is the sheaf of affine linear functions on X with integral slope.

Remark. Note however that as in classical geometry not every line bundle on a tropical torus has a rational section.

Now consider a short exact sequence of sheaves:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Aff} \longrightarrow \mathcal{T}^*_{\mathbb{Z}} \longrightarrow 0, \tag{8}$$

where $\mathbb{R}$ is locally constant sheaf and sections of $\mathcal{T}_{\mathbb{Z}}^* \cong (\mathbb{Z}^n)^*$ are locally constant integral 1-forms. The induced map of the long exact sequence

$$c : H^1(X, \mathrm{Aff}) \longrightarrow H^1(X, \mathcal{T}_{\mathbb{Z}}^*) \tag{9}$$

is called the *Chern class* map. A choice of a class $[c] \in H^1(X, \mathcal{T}_{\mathbb{Z}}^*) \cong \Lambda^* \otimes (\mathbb{Z}^n)^*$ which is in the image of the Chern class map is called a *polarization* of X. Using the natural isomorphism $\Lambda^* \otimes (\mathbb{Z}^n)^* \cong \mathrm{Hom}(\Lambda, (\mathbb{Z}^n)^*)$ one can think of $[c]$ either as a bilinear form on $\mathbb{R}^n$ or as a map $[c] : \Lambda \to (\mathbb{Z}^n)^*$.

We claim that the set of polarization classes is formed by the maps $\Lambda \to (\mathbb{Z}^n)^*$ which are *symmetric* as bilinear forms on $\mathbb{R}^n$. To see this we analyze the coboundary map in the induced long exact sequence:

$$\delta : H^1(X, \mathcal{T}_{\mathbb{Z}}^*) \longrightarrow H^2(X, \mathbb{R}). \tag{10}$$

By identifying $H^1(X, \mathcal{T}_{\mathbb{Z}}^*) \cong \Lambda^* \otimes (\mathbb{Z}^n)^*$ and $H^2(X, \mathbb{R}) \cong \wedge^2(\mathbb{R}^n)^*$ one can deduce that δ is the restriction of the skew-symmetrization map $(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \to \wedge^2(\mathbb{R}^n)^*$.

Remark. For a general affine manifold the coboundary map δ is given by a wedge product with some characteristic class ρ which takes values in the cohomology with coefficients in the tangent local system $\mathcal{T}$. This so called *radiance obstruction class* has the meaning of the translational part of the monodromy representation (cf. [KS06], Section 2.2). In our case the local system $\mathcal{T}$ is trivial because the linear part of the monodromy on the torus is trivial. Thus, $H^1(X, \mathcal{T})$ can be canonically identified with $(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, and ρ is the identity element there.

We can also describe the set of line bundles with a given Chern class. Notice that the map $H^0(X, \mathbb{R}) \to H^0(X, \mathrm{Aff})$ is an isomorphism. Hence $H^0(X, \mathcal{T}_{\mathbb{Z}}^*) \to H^1(X, \mathbb{R})$ is the natural inclusion $(\mathbb{Z}^n)^* \hookrightarrow (\mathbb{R}^n)^*$. Thus the Picard group of X with a fixed Chern class can be identified with the dual torus $(\mathbb{R}^n)^*/(\mathbb{Z}^n)^*$.

If the quadratic form defined by $[c]$ is positive definite we call $\mathbb{R}^n/\Lambda$ a *(polarized) tropical Abelian variety* (cf. [AN99], [Ale02]). The *index* of the polarization $[c]$ is the index of the image $[c](\Lambda)$ in $(\mathbb{Z}^n)^*$. We call a polarization *principal* if it has index one, that is, if the map $[c] : \Lambda \to (\mathbb{Z}^n)^*$ is an isomorphism.

Theorem 5.1. *Let L be a line bundle on $X = \mathbb{R}^n/\Lambda$ which defines a principal polarization. Then the space of sections $\Gamma(X, L)$ is one dimensional.*

Proof. Suppose L has the Chern class $[c] : \Lambda \to (\mathbb{Z}^n)^*$. A global section Θ of L can be viewed as a *convex* PL function on $\mathbb{R}^n$ subject to some quasi-periodicity condition:

$$\Theta(x + \mu) = \Theta(x) + \langle [c](\mu), x \rangle + \beta(\mu), \quad \text{for any } \mu \in \Lambda.$$

By applying quasi-periodicity twice we can see that β has to be quadratic. Namely,

$$\beta(\mu_1 + \mu_2) = \beta(\mu_1) + \beta(\mu_2) + [c](\mu_1, \mu_2).$$

That is, $\beta(\lambda) = \frac{1}{2}[c](\lambda,\lambda)$ + linear part. The linear part is responsible for shifts of the line bundle.

On the other hand, as a convex function $\Theta(x)$ is completely determined by its Legendre transform $\hat{\Theta} : (\mathbb{Z}^n)^* \to \mathbb{R}$. Now we show that a value of $\hat{\Theta}(0)$ determines $\hat{\Theta}(m)$ at any $m \in \mathrm{Im}[c] = (\mathbb{Z}^n)^*$. Say $m = [c](\mu)$ for some $\mu \in \Lambda$, then

$$\hat{\Theta}(-m) \;=\; \max_{x\in\mathbb{R}^n}\{\langle -[c](\mu), x\rangle \,-\, \Theta(x)\} \;=\; \max_{x\in\mathbb{R}^n}\{-\Theta(x) \,+\, \beta(\mu)\} \;=\; \hat{\Theta}(0) \,+\, \beta(\mu).$$

Performing the inverse of the Legendre transform gives

$$\Theta(x) \;\;:=\;\; \max_{m\in(\mathbb{Z}^n)^*}\{\langle m, x\rangle \,-\, \hat{\Theta}(m)\} \;\;=\;\; \max_{\lambda\in\Lambda}\{\langle -[c](\lambda), x\rangle \,-\, \hat{\Theta}(0) \,-\, \beta(-\lambda)\}.$$

This is the unique section of L determined upto a additive constant by $\hat{\Theta}(0)$. □

Remark. For a general tropical Abelian variety, a section of a positive line bundle L, thought of the quasi-periodic PL function Θ on $\mathbb{R}^n$, is completely determined by specifying values of its Legendre transform $\hat{\Theta}$ on every representative of the quotient $(\mathbb{Z}^n)^*/[c](\Lambda)$. Thus, in general, the dimension of $H^0(X, L)$ is given by the index of the polarization.

5.2. **Theta functions.** The unique (up to constant) section from Theorem 5.1 will be the tropical analog of the classical Riemann's *theta function.* Let Λ be a lattice in $\mathbb{R}^n$ and $Q : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$ be a positive definite symmetric bilinear form. As implicitly suggested in [AN99] we define

$$\Theta(x) := \max_{\lambda\in\Lambda}\{Q(\lambda, x) - \frac{1}{2}Q(\lambda,\lambda)\}, \quad x \in \mathbb{R}^n.$$

The maximum here always exists since Q is positive definite. From the definition we can readily see that $\Theta(x)$ is an even function: $\Theta(-x) = \Theta(x)$. It also satisfies the following functional equation:

Lemma 5.2. *$\Theta(x + \mu) = \Theta(x) + Q(\mu, x) + \frac{1}{2}Q(\mu, \mu)$, for any $\mu \in \Lambda$.*

Proof. This calculation is in a sense "Legendre dual" to the the proof of Theorem 5.1. By considering the effect of translation on each term we have:

$$\begin{aligned} &Q(\lambda, x + \mu) - \frac{1}{2}Q(\lambda,\lambda) \\ &\quad = Q(\lambda - \mu, x) + Q(\mu, x) + Q(\lambda, \mu) - \frac{1}{2}Q(\lambda - \mu, \lambda - \mu) - Q(\mu, \lambda) + \frac{1}{2}Q(\mu, \mu) \\ &\qquad\qquad = Q(\lambda - \mu, x) - \frac{1}{2}Q(\lambda - \mu, \lambda - \mu) + Q(\mu, x) + \frac{1}{2}Q(\mu, \mu). \end{aligned}$$

Relabeling the terms and combining them into the tropical sum completes the proof. □

The theta function above can be defined on $\mathbb{R}^n$ for an arbitrary positive symmetric bilinear form Q. But it is *regular (holomorphic)* in tropical sense only if the form Q is integral in the sense that the image of the induced map $\tilde{Q} : \Lambda \to (\mathbb{R}^n)^*$ ends up being inside the lattice $(\mathbb{Z}^n)^*$. From now on we restrict our attention to the integral forms Q only. This is when the theta function defines a polarization on the tropical torus $\mathbb{R}^n/\Lambda$. Moreover, we will be primarily interested in principal polarizations.

A tropical hypersurface in $\mathbb{R}^n$ defined by the corner locus of the theta function gives a honeycomb-like periodic cell decomposition of $\mathbb{R}^n$, the so-called Voronoi decomposition (cf. Fig. 5 and [AN99]). It is dual to the decomposition of $(\mathbb{R}^n)^*$ with vertices in $(\mathbb{Z}^n)^*$ induced by the Legendre transform $\hat{\Theta}(m)$, the so-called Delauney decomposition. The principal polarizations can be geometrically characterized by

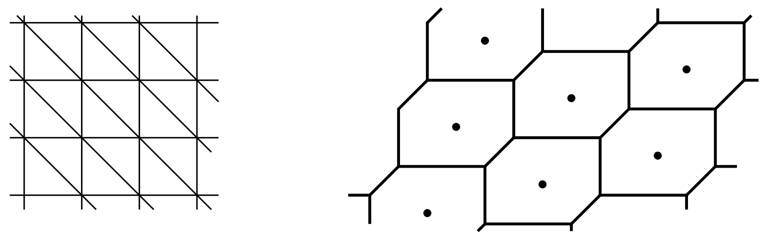

Figure 5. $\hat{\Theta}(m)$-induced decomposition of $(\mathbb{R}^n)^*$ and a corresponding Λ-periodic theta divisor.

that the volume of each maximal Voronoi cell is equal to the affine volume of the torus $\mathbb{R}^n/\Lambda$, which, in turn, is equal to $\det \tilde{Q}$.

The theta function can be considered as a section of the polarization line bundle L on X whose corresponding Čech 1-cocycle is defined by the "automorphy factors" $Q(\mu, x) + \frac{1}{2}Q(\mu, \mu)$ in the functional equation. Locally Θ is a tropical polynomial in $\mathbb{R}^g$ and thus defines a tropical hypersurface $[\tilde{\Theta}]$, see [Mik06]. Since the automorphy factor is affine linear the hypersurface $[\tilde{\Theta}]$ is periodic and thus descends to the quotient torus X. We call the resulting hypersurface in X the *theta divisor* $[\Theta] \subset X$.

6. Abel-Jacobi Theorem

In this section and throughout the rest of the paper we assume that C is a *compact* tropical curve of genus g.

6.1. **Tropical Jacobian.** Denote by $\Omega(C)$ the space of global 1-forms on C. Given a collection of break points (z_i, η_i) we can consider the values of forms on the η_i. This identifies $\Omega(C)$ with g dimensional $\mathbb{R}$-vector space. The integral lattice $\Omega_{\mathbb{Z}}(C) \subset \Omega(C)$ consists of the forms taking integer values on integer tangent vectors to the curve C.

Given a path γ in C, any 1-form ω on C pulls back to a piece-wise constant classical (old-fashioned) 1-form on an interval. Thus one can define the integral $\int_\gamma \omega \in \mathbb{R}$.

Let $\Omega(C)^*$ be the vector space of $\mathbb{R}$-valued linear functionals on $\Omega(C)$. Then the integral cycles $H_1(C, \mathbb{Z})$ form a lattice Λ in $\Omega(C)^*$ by integrating over them. As in the classical complex geometry we define the *Jacobian* of the curve C to be

$$J(C) := \Omega(C)^*/H_1(C, \mathbb{Z}) \cong \mathbb{R}^g/\Lambda.$$

Note that the space $\Omega(C)^*$ is naturally endowed with the (tautological) $\mathbb{Z}$-affine structure: the lattice $\Omega_{\mathbb{Z}}(C)^* \subset \Omega(C)^*$ is identified with the integer valued functionals on $\Omega_{\mathbb{Z}}(C)$. A set of break points gives a basis in $\Omega_{\mathbb{Z}}(C)$.

The metric on C defines a symmetric bilinear form Q on the space of paths in C by setting $Q(\ell, \ell) := \text{length}(\ell)$ for a simple (i.e., not self-intersecting) path ℓ and extending it to any pair of paths bilinearly, see Fig. 6. Since C is a 1-dimensional

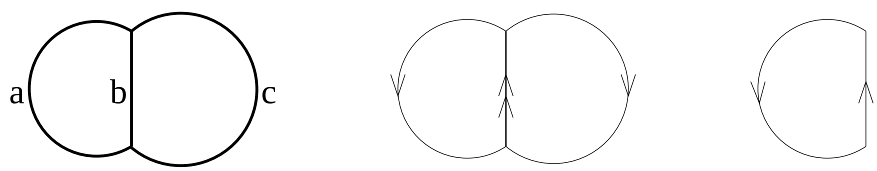

FIGURE 6. Two cycles γ_1 and γ_2 and their pairing $Q(\gamma_1, \gamma_2) = a + 2b$.

simplicial complex any 1-cycle homologous to zero is trivial at the simplicial chain level. Therefore, Q descends to a symmetric bilinear form on $H_1(C, \mathbb{Z})$.

Lemma 6.1. *The form Q is positive-definite.*

Proof. Since C is a simplicial complex any 1-chain is given as a sum $\sum_E a(E)E$, where $a(E) \in \mathbb{Z}$ and E runs over all edges of C. By bilinearity we have

$$Q(\sum_E a(E)E, \sum_E a(E)E) = \sum_E a(E)^2 l(E) > 0,$$

where $l(E)$ is the length of the edge E. □

Thus Q provides an isomorphism $\tilde{Q} : \Omega(C)^* \to \Omega(C)$. Under this isomorphism the lattice Λ is mapped isomorphically to the integral forms $\Omega_{\mathbb{Z}}(C) \subset \Omega(C)$. Thus the form Q makes $J(C)$ into a principally polarized tropical Abelian variety.

Remark. A version of the tropical Jacobian (as well as the tropical Picard group) for finite graphs was introduced in [BdlHN97] and [Nag97] even before the explicit appearance of Tropical Geometry. From a tropical viewpoint a finite graph is a graph defined over $\mathbb{Z}$ (i.e. with integer edge-lengths). The finite graph Jacobian can be interpreted as the integer points in the tropical Jacobian. The corresponding Abel-Jacobi Theorem was established in [BdlHN97].

6.2. **The Abel-Jacobi Theorem.** Once and for all let us fix a reference point $p_0 \in C$. Given a divisor $D = \sum a_i p_i$ we choose paths from p_0 to p_i. Integration along these paths defines a linear functional on $\Omega(C)$:

$$\hat{\mu}(D)(\omega) = \sum a_i \int_{p_0}^{p_i} \omega.$$

For another choice of paths the value of $\hat{\mu}(D)$ will differ by an element in Λ. Thus, we get a well-defined tropical analog of the *Abel-Jacobi map* $\mu : \mathrm{Div}^d(C) \to J(C)$. Note that the Abel-Jacobi map μ does not depend on the choice of a base point p_0 if

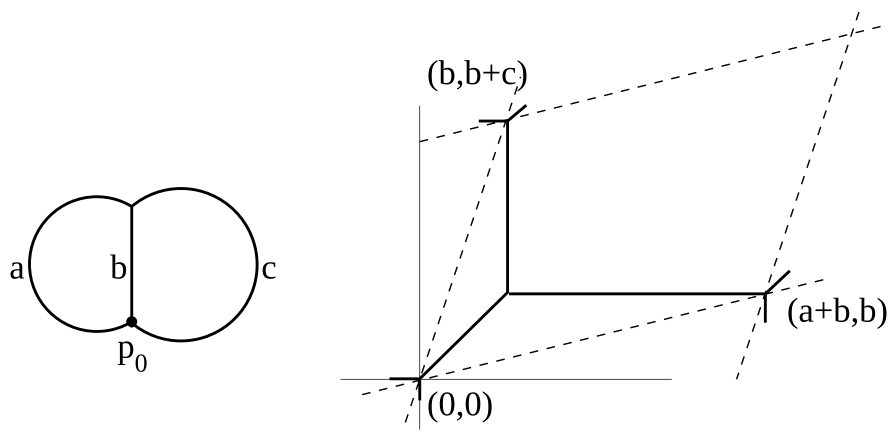

FIGURE 7. $\mu(C)$ in the tropical Jacobian $J(C)$.

the degree d is zero. The dependence on p_0 for $d \neq 0$ will reappear later in solution to the Jacobi inversion.

Theorem 6.2 (Tropical Abel-Jacobi)**.** *For each degree d the map μ factors through* $\mathrm{Pic}^d(C)$*:*

$$\begin{array}{ccc} \mathrm{Div}^d(C) & \longrightarrow & \mathrm{Pic}^d(C) \\ & \searrow^{\mu} & \downarrow \phi \\ & & J(C) \end{array}$$

so that ϕ is a bijection.

Proof. Here we only prove Abel's part: μ factors through and ϕ is injective. Theorem 6.5 gives an explicit solution to the Jacobi inversion.

Let $D = \sum p_i - \sum q_i$ be a divisor of a rational function f. The gradient field ∇f defines a linear functional on $\Omega(C)$ via integration along its integral trajectories t_j (counted with multiplicities). For some choice of paths from p_i to q_i this functional coincides with $\hat{\mu}(D)$. On the other hand, for any $\omega \in \Omega_{\mathbb{Z}}(C)$ we have

$$\sum_j \int_{t_i} \omega = \int_{\tilde{Q}^{-1}(\omega)} df = 0.$$

Thus, $\mu(D) = 0$ in $J(C)$.

Conversely, given any $D = \sum p_i - \sum q_i \in \mathrm{Div}^0(C)$ with $\mu(D) = 0$ we can choose k paths ℓ_i on C, such that each ℓ_i originates at p_i and ends at q_i and for any $\omega \in \Omega(C)$ we have $\sum_i \int_{\ell_i} \omega = 0$. Now we can define a rational function $f(x)$ on C by choosing a path $\ell(x)$ from p_0 to x:

$$f(x) := \sum Q(\ell_i, \ell(x)).$$

Since $\sum Q(\ell_i, \gamma) = \sum \int_{\ell_i} \tilde{Q}(\gamma) = 0$ for any closed loop γ in C the function $f(x)$ is independent of the choice of ℓ. By construction $(f) = \sum p_i - \sum q_i$. □

6.3. **Jacobi inversion.** Given a tropical map $\phi : C \to X$ we can pull back any meromorphic function on X to a meromorphic function on C, see e.g. [Mik06]. Let L be a line bundle, s be its section and D be the corresponding divisor. Pulling back local representatives of s defines a section $\phi^* s$ of some line bundle on C. The divisor $\phi^* D$ of $\phi^* s$ depends only on D and not on the choice of s (cf. [Mik06] and [AR07]).

On the other hand, another section of L defines a linearly equivalent divisor $D' \sim D$ on X, which in turn pulls back to a linearly equivalent divisor $\phi^* D' \sim \phi^* D$ on C. Thus the pull back line bundle $\phi^* L$ on C is well defined. Alternatively, to define $\phi^* L$ one can pull back a defining Čech cocycle of L.

Lemma 6.3. *The Abel-Jacobi map $\mu : C \to J(C)$ is tropical.*

Proof. We need to show that for any point $p \in C$ there is a local chart $U \subset R^{k-1}$ such that μ is the restriction of a $\mathbb{Z}$-affine map $\mathbb{R}^{k-1} \to \mathbb{R}^g$. But this is clear because affine linear coordinates near a k-valent vertex will also provide local coordinates in $\Omega(C)^*$ modulo some *linear* relations among cycles. □

Next we will prove a refined version of the residue formula (6) for curves in $\mathbb{R}^g$.

Lemma 6.4. *Let U be a connected open set of a curve tropically embedded in $\mathbb{R}^N$ with boundary $\partial U = \{z_1, \dots, z_k\}$ and $\nu_1, \dots, \nu_k$ – the corresponding outward primitive tangent vectors. Let f be a rational function on U with divisor $(f) = \sum a_j p_j$. Then*

$$\sum a_j p_j = \sum_{i=1}^{k} \left(\frac{\partial f}{\partial \nu_i}(z_i) \cdot z_i - f(z_i) \cdot \nu_i \right), \tag{11}$$

where the summation and equality takes place in the vector space $\mathbb{R}^N$, and p_j, z_i and ν_i are viewed as elements there.

Proof. The formula is additive with respect to gluing pieces and holds for affine functions. On the other hand, at $z = p_i$ the statement is the definition of a_i. □

For $\lambda \in \mathbb{C}^g$ let $\Theta_\lambda(x) := \Theta(x - \lambda)$ denote the translated theta function and let $[\Theta_\lambda]$ be its divisor on $J(C)$ (the corresponding line bundle L_λ is different, but defines the same polarization as L). Let $D_\lambda := \mu^*[\Theta_\lambda]$ denote the pull back of $[\Theta_\lambda]$ to the curve via the Abel-Jacobi map $\mu : C \to J(C)$. Note that both $[\Theta_\lambda] = [\Theta] + \lambda$ and D_λ are well defined for $\lambda \in J(C)$.

Theorem 6.5 (Jacobi Inversion)**.** *For any $\lambda \in J(C)$ the divisor D_λ is effective of degree g. There exists a universal $\kappa \in J(C)$ such that $\mu(D_\lambda) + \kappa = \lambda$ for all $\lambda \in J(C)$.*

Proof. D_λ is clearly effective since the theta function locally pulls back to regular sections. To calculate its degree we choose a set of break points z_i, η_i in the interiors of edges and disjoint from the support of D_λ. Let $T \subset \hat{C}$ be the associated fundamental domain, and let T^0 denote its interior and $\bar{T}$ – its closure.

Then ∂T^0 consists of g ordered pairs of break ends: $z_i^+ \in T$ and $z_i^- \in \bar{T} \setminus T$. The vectors η_i at the z_i provide an integral basis for $\Omega_\mathbb{Z}(C)^*$.

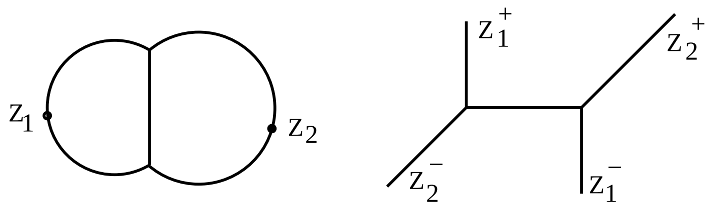

FIGURE 8. The interior of the fundamental domain T.

Let $\hat{\mu} : \hat{C} \to \mathbb{R}^g$ denote the lifting of the Abel-Jacobi map μ. The path in $\bar{T}$ from z_i^- to z_i^+ defines a cycle γ_i in C, that is $\hat{\mu}(z_i^+) = \hat{\mu}(z_i^-) + [\gamma_i]$ in $\mathbb{R}^g$. Differentiating the quasi-periodicity equation of Lemma 5.2 gives

$$d\Theta_\lambda(\hat{\mu}(z_i^+)) - d\Theta_\lambda(\hat{\mu}(z_i^-)) = \tilde{Q}(\gamma_i) \in \Omega_\mathbb{Z}(C).$$

Note that the left hand side is independent of a lift of λ to $\mathbb{R}^g$. The key observation is that $\tilde{Q}(\gamma_i)$ form a basis of $\Omega_\mathbb{Z}(C)$ dual to $\{\eta_i\}$, that is $\langle \tilde{Q}(\gamma_i), \eta_j \rangle = \delta_{ij}$. The residue formula (6) applied to the pullback $\hat{\mu}^*\Theta_\lambda$ on T^0 gives $\deg D_\lambda = \sum_{i=1}^g 1 = g$.

The second statement follows from the refined residue formula (11) applied to the theta function restricted to $\hat{\mu}(T^0) \subset \mathbb{R}^g$. Let $\nu_i = \hat{\mu}(\eta_i)$. Then we have

$$\hat{\mu}(D_\lambda) = \sum_{i=1}^g \left(\frac{\partial \Theta_\lambda}{\partial \nu_i}(\hat{\mu}(z_i^+)) \cdot \hat{\mu}(z_i^+) - \Theta_\lambda(\hat{\mu}(z_i^+))\nu_i \right) - \left(\frac{\partial \Theta_\lambda}{\partial \nu_i}(\hat{\mu}(z_i^-)) \cdot \hat{\mu}(z_i^-) - \Theta_\lambda(\hat{\mu}(z_i^-))\nu_i \right).$$

Differentiating the right hand side with respect to λ gives

$$\sum_{i=1}^{g}\left(d\Theta_\lambda(\hat{\mu}(z_i^+))\otimes\nu_i - d\Theta_\lambda(\hat{\mu}(z_i^-))\otimes\nu_i\right) = \sum_{i=1}^{g}\tilde{Q}(\gamma_i)\otimes\nu_i,$$

which is the identity element in $\mathrm{End}(\mathbb{R}^g)$. Thus, passing to the quotient $J(C)$ we get $\mu(D_\lambda) = \lambda + \text{const}$. □

Combining the Jacobi Inversion with the Abel-Jacobi Theorem we get the following statement.

Corollary 6.6. *Given a divisor D of degree d there is an effective divisor $D_\lambda = \mu^*[\Theta_{\mu(D)+\kappa}]$ of degree g linearly equivalent to $D + (g-d)p_0$. In particular, a degree g divisor has a* canonical *(independent of the base point p_0) effective representative in its class of linear equivalence.*

6.4. **Schottky problem and Torelli theorem.** The space of all principally polarized tropical Abelian varieties is the same as the space of symmetric positive definite matrices modulo some discrete automorphism group action. So its dimension is $\frac{g(g+1)}{2}$. On the other hand the space of tropical curves of genus $g \geq 2$ is $3g-3$. Hence the Jacobians form a subset $\mathcal{J}$ of positive codimension for $g \geq 3$ inside the space of all Abelian varieties. Description of this subset is the tropical counterpart of the classical Schottky problem.

The naïve statement of the classical Torelli theorem fails for tropical curves. For instance, in Fig. 9 the polarized Jacobian does not see the length of the connecting edge in the genus 2 curve. However we conjecture that following reformulation holds:

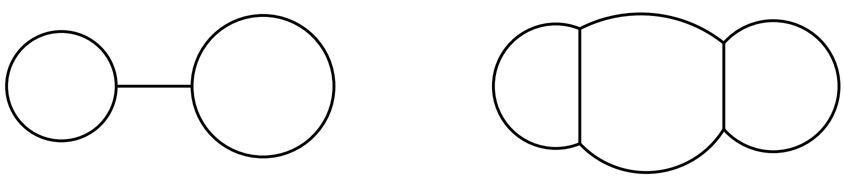

Figure 9. Counterexamples to Torelli in genus 2 and 3.

the map from the moduli space of tropical curves to $\mathcal{J}$ is tropical of degree 1.

7. The Riemann-Roch Theorem

In this section we continue to assume that C is compact of genus g and look at the space of effective divisors realizing the same class in the Picard group $\mathrm{Pic}(C)$. Recall that for any divisor D of degree d we associate the linear system $|D|$ consisting of all effective divisors linearly equivalent to D. By Theorem 6.2 $|D|$ is the inverse image of a point under the Abel-Jacobi map $\mathrm{Sym}^d C \to J(C)$. Thus $|D|$ has the structure of a compact (in topology induced from metric on C) CW-complex.

It is easy to see that (unlike the classical situations) the complex $|D|$ is not of a pure dimension (cf. [GK06], Example 1.11). Nevertheless, we may speak of its dimension $\dim|D|$. Namely, the following defintion was suggested by Baker and Norine.

Definition 7.1 ([BN07])**.** Let D be an effective divisor on C. The *dimension* $\dim|D|$ of the linear system $|D|$ is the maximal integer $r \geq 0$ with the following property: for any effective divisor R of degree r the space $|D-R|$ is non-empty. We set $\dim|D| = -1$ when $|D| = \emptyset$.

The definition immediately implies that for effective D and D' one has

$$\dim|D| + \dim|D'| \leq \dim|D+D'| \leq \dim|D'| + \deg D. \tag{12}$$

Plugging $D' = \emptyset$ into the second inequality gives us $\dim|D| \leq d$. By the Jacobi inversion every divisor of degree $d \geq g$ is linearly equivalent to an effective one. Again, directly by Definition 7.1 this implies that

$$\dim|D| \geq d - g. \tag{13}$$

If (13) turns into equality then D is called *regular*. Otherwise we define the discrepancy number

$$\rho(D) := \dim|D| - d + g.$$

An effective D with $\rho(D) > 0$ is called *special*.

Lemma 7.2. *Every divisor D of degree $d \geq 2g-1$ is regular, i.e.* $\dim|D| = d - g$.

Proof. Suppose $\dim|D| = s > d-g$. Consider a degree $d-s$ divisor D' with $|D'| \neq \emptyset$ (it exists since the Abel-Jacobi map is not surjective on $\mathrm{Sym}^k C$ for $k < g$). It can be written as $D' = D - (D - D') = D - S$, where S is of degree $s \geq g$ and hence $|S| \neq \emptyset$. Consequently, $|D'| \neq \emptyset$ which is a contradiction. $\square$

The divisor

$$K := \sum_{p \in C} (\mathrm{val}(p) - 2)p, \tag{14}$$

is called the *canonical divisor* of C. Here $\mathrm{val}(p)$ is the valence of p. By Euler's formula $\deg K = 2g-2$. It is effective unless C has one-valent vertices.

Remark. In the degeneration picture a k-valent vertex p of C corresponds to the Riemann sphere with k punctures. This explains why the canonical divisor is defined as above: the number of zeros (the multiplicity of p in K) of a rational form on $\mathbb{P}^1$ is the number of poles (valence of p) minus 2. In [Mik] all Chern classes are defined (in a similar combinatorial way) for higher-dimensional tropical varieties.

Since C is a curve we may avoid dealing with higher-dimensional cohomology by postulating the Serre duality. By introducing the cohomological notation $h^0(D) = \dim|D| + 1$ we set $h^1(D) = h^0(K - D)$.

Theorem 7.3 (Tropical Riemann-Roch, cf. [BN07], [GK06] and [MZ06])**.**

$$\dim|D| - \dim|K-D| = d - g + 1.$$

The first version of our preprint [MZ06] had a rather lengthy and involved independent proof of this statement by means of tropical geometry. However now we've learned of a much simpler proof of this statement (in the case of finite graphs) by Baker and Norine [BN07] which makes an elegant use of the chip-firing games technique (see e.g. [BLS91], [Big99]). Since [BN07] appeared before [MZ06] and contains a simpler proof we see no point of pursuing our initial cumbersome argument here. Below we just reproduce a geometrically adapted version of [BN07].

There is already a couple of other papers that observed that the restriction of [BN07] to the case of finite graphs is not essential. In particular, [GK06] (where this observation appeared first) formally deduces Theorem 7.3 from [BN07], and [HKN07] adapts the argument of [BN07] to the case of metric graphs. Note that one useful advantage of the approach of [BN07] is the explicit description of all non-effective divisors of degree $g-1$ (Corollary 7.10).

The main idea is to look for a canonical representative of a divisor in its linear equivalence class.

Let us fix a point $p \in C$. Any effective degree d divisor on $C \setminus \{p\}$ may be considered just as an unordered d-tuple of points. Thus to each such divisor we may associate a non-decreasing sequence of distances from these points to p. Given two divisors we may compare the corresponding sequences in the lexicographic order (inserting some number of zeroes in front of one of the sequences to make them equal length).

Definition 7.4. We say that a divisor D on C is *p-reduced* if its restriction to $C\setminus\{p\}$ is effective and minimizes the distance to p among such in the equivalence class of D.

Proposition 7.5. *For any fixed $p \in C$ there exists a unique p-reduced representative in every class in* $\mathrm{Pic}(C)$*.*

Proof. Given D of degree d consider the minimal $m \in \mathbb{Z}$ such that $|D + mp| \neq \emptyset$. Such m is finite since it is obviously bounded from below by $-d$ and from above by $g - d$ by Jacobi Inversion. Note that any effective $D' \sim D + kp$ for $k > m$ and with support disjoint form p has strictly larger distance to p than any element of $|D + mp|$ because of its higher degree. Thus it suffices to minimize the distance to p among elements of $|D + mp|$. Compactness of $|D + mp|$ guarantees existence.

To show uniqueness let us suppose that D and D' have the same distance sequence to p and there exists a *non-constant* rational function f such that $D' = D + (f)$. Since C is compact the function $f : C \to \mathbb{R}$ has a maximum M. Let $F_+ = f^{-1}(M)$ be the locus of points where f reaches its maximum. Clearly F_+ is a subgraph of C (possibly consisting of a single point). Interchanging the rôles of D and D' if needed we may assume that F_+ is disjoint from p.

All the boundary points of F_+ are the poles of f of some order, thus these points are contained in D with some positive multiplicities. Denote the divisor formed by them with D_M. The points of $f^{-1}(M-\epsilon)$ can be enhanced with natural multiplicities equal to the slope of f on the corresponding edges of C. The resulting divisor $D_{M-\epsilon}$ is linearly equivalent to D_M. In the same time, since $p \notin F_+$ the distance from $D_{M-\epsilon}$ to p is smaller then the distance from D_M to p. Therefore D cannot be p-reduced. ☐

Remark. The minimal m argument for existence also shows that if $|D| \neq \emptyset$ then the p-reduced form is effective.

Definition 7.6. A divisor K_+ on C is called a *moderator* if there exists an acyclic orientation on C (i.e. a presentation of C as a 1-dimensional simplicial complex with a choice of an acyclic orientation on the edges) such that

$$K_+ = \sum_{p \in C} (\mathrm{val}_+(p) - 1)p, \tag{15}$$

where val_+ stands for the number of outgoing edges.

Analogously we can define $K_- := \sum_{p \in C} (\mathrm{val}_-(p) - 1)p$ by counting *incoming* edges, which is also a moderator for the reversed orientation. The notation is justified by the following proposition.

Proposition 7.7. *The degree of K_+ and K_- is $g-1$, and $K_+ + K_- = K$.*

Proof. By definition we have

$$K_+ + K_- = \sum_{p \in C} (\mathrm{val}_+(p) - 1 + \mathrm{val}_-(p) - 1)p = \sum_{p \in C} (\mathrm{val}(p) - 2)p = K.$$

To calculate the degree we observe every edge and every vertex (with negative sign) enters in the coefficients of either K_+ or K_- exactly once. ☐

Lemma 7.8. *$|K_+| = \emptyset$ for any moderator K_+.*

Proof. Suppose that $K_+ + (f)$ is effective for some rational function f. Take the same maximal locus subgraph $F_+ \subset C$ as in the proof of Proposition 7.5. The presentation of K_+ as a moderator induces an acyclic orientation of the graph F_+. Thus there must exist a sink vertex $q \in F_+$. If f is locally constant in the neighborhood of q then it is also a sink in C and q enters $K_+ + (f)$ with a negative coefficient. Otherwise q enters in (f) with the coefficient $-m$ where m is not smaller than the number n of edges of $C \setminus F_+$ adjacent to q. In the same time the coefficient of q in K_+ is not greater than n. ☐

Lemma 7.9. *Given a divisor D on C exactly one of the following two holds. Either $|D| \neq \emptyset$, or there is a moderator K_+ such that $|K_+ - D| \neq \emptyset$.*

Proof. By Proposition 7.5 we may assume that D is p-reduced. Consider a presentation of C as a graph (with no loop edges) whose set of vertices $\{v_j\}$ contains p, the support points of D and points C of valence greater than 2. To choose an acyclic orientation it suffices to order the vertices v_j. We do it inductively starting with $v_0 = p$.

Suppose that the first k vertices are already chosen. Let us look at the edges $E_j^{(k)}$ connecting these vertices with the remaining vertices. Note that p-reducibility implies that one of the remaining vertices, say v, enters D with the coefficient smaller than the number of edges $E_j^{(k)}$ adjacent to it. Otherwise we could move all the outer endpoints of all $E_j^{(k)}$ by some distance $\epsilon > 0$ towards p (this does not change the linear equivalence class) and obtain a contradiction to p-reducibility. We set $v_{k+1} = v$.

Orienting each edge towards a smaller vertex gives a moderator K_+ with $K_+ - D$ effective, except possibly at p. If $|D| = \emptyset$ then D is not effective and it must have a negative coefficient at p and thus we get $K_+ - D$ effective at p. If both $|D|$ and $|K_+ - D|$ were non-empty we would get a contradiction with Lemma 7.8. □

This proposition has a notable corollary that also provides the statement converse to Lemma 7.8.

Corollary 7.10. *Let D be a divisor of degree $g-1$. If $|D| = \emptyset$ then D is linearly equivalent to a moderator. Moreover, if in addition D is p-reduced, then D is a moderator.*

Proof. By Lemma 7.9 there exists a moderator K_+ with $|K_+ - D| \neq \emptyset$. But the degree of $K_+ - D$ is zero. Thus $D \sim K_+$.

If D is p-reduced, then $K_+ - D$ is effective, hence trivial. □

Corollary 7.11. *If D is a divisor of degree $g-1$ then $|D| = \emptyset$ if and only if $|K - D| = \emptyset$.*

Proof. If $D = K_+$ is a moderator then $K - D = K_-$ is also a moderator. □

Corollary 7.12. *If $d < g-1$ and $|D+q| \neq \emptyset$ for every point $q \in C$ then $|D| \neq \emptyset$.*

Proof. By Lemma 7.9 if $|D| = \emptyset$ then there exists a moderator K_+ with $|K_+ - D| \neq \emptyset$. Since $\deg(K_+ - D) > 0$ there exists a point q such that $|K_+ - D - q|$ is still non-empty. Thus $|D+q| = \emptyset$. □

Proof of Riemann-Roch theorem. Because of Lemma 7.2 the Riemann-Roch holds for $d < 0$ and $d > 2g-2$. By symmetry between D and $K - D$ it suffices to prove only the inequality

$$\dim |K - D| \geq \dim |D| - d + g - 1 \tag{16}$$

for $0 \leq d \leq 2g-2$. The inequality is trivial for $\rho(D) = 0$. Also applying (13) to $K - D$ shows that (16) holds if $|D| = \emptyset$. So from now on we assume that D is special.

We need to show that given any effective R of degree $\rho(D)-1$ we have $|K-D-R| \neq \emptyset$. Replacing D with $D+R$ reduces the statement to showing that $|K-D| \neq \emptyset$ for any special D of degree $d \geq g-1$.

The case $d = g-1$ follows from Corollary 7.11. If D is a special divisor with $d \geq g$ we have $|D-R| \neq \emptyset$ for any effective R of degree $d-g+1$, and hence, $|K-D+R| \neq \emptyset$ by Corollary 7.11. Repetitive application of Corollary 7.12 shows that $|K-D| \neq \emptyset$. □

8. Riemann's Theorem

By taking the sum in the group $J(C)$ we may extend the Abel-Jacobi map to $\mu : C \times \cdots \times C \to J(C)$. This extension is still a tropical map. Furthermore, since taking the sum is commutative we also have a map $\mu : \mathrm{Sym}^k(C) \to J(C)$ (note though that $\mathrm{Sym}^k(C)$ is *not* a tropical variety in the usual sense for any $k > 1$).

The tropical counterpart of Riemann's theorem identifies the subset $W_{g-1} = \mu(\mathrm{Sym}^{g-1} C) \subset J(C)$ with a translate of the theta divisor. We establish it as a corollary of Jacobi's inversion. We start with a couple of statements which themselves may be of independent interest.

Definition 8.1. The support $\mathrm{supp}\,|D|$ of the linear system $|D|$ is the set of points $q \in C$ such that $|D-q| \neq \emptyset$.

Let Γ be a subgraph of C. We denote by $D_{\partial\Gamma}$ its boundary divisor, that is the divisor consisting of the boundary point of Γ.

Lemma 8.2. *Let Γ be a proper connected subgraph of C and D_b be any break divisor of Γ (cf. §4.5). Then $\Gamma \subset \mathrm{supp}\,|D_b + D_{\partial\Gamma}|$.*

Proof. Let T_Γ be a fundamental domain of Γ associated to some choice of break points constituting D_b (as usual, we may identify T_Γ with Γ). By taking the closure of one of the connected components of $T_\Gamma \setminus D_{\partial\Gamma}$ instead of Γ if needed we may assume that the multiplicity of any boundary point v in $D = D_b + D_{\partial\Gamma}$ is equal to the valence of v in the graph Γ. Indeed, $\mathrm{val}_\Gamma(v) > \deg D_b|_v$, since v can support at most $\mathrm{val}_\Gamma(v) - 1$ break points. On the other hand, if $\mathrm{val}_\Gamma(v) \geq \deg D_b|_v + 2$ then v is not an endpoint of T_Γ and it breaks T_Γ into several components.

Let $\epsilon > 0$ be the distance between $\partial\Gamma$ and the vertices of $\Gamma \setminus \partial\Gamma$. Here we assume presentation of the graph Γ such that its set of vertices contains the support points of D and points of Γ of valence greater than 2. Consider the divisor D' obtained from D by moving the points $D|_{\partial\Gamma}$ inside Γ by distance ϵ, and define the proper subgraph $\Gamma' \subset \Gamma$ by by removing swept out edges: $\Gamma' := \{x \in \Gamma \ : \ \mathrm{dist}(x, \partial\Gamma) \geq \epsilon\}$.

Clearly, the boundary divisor of Γ' is contained in the moved part of D'. On the other hand, the interior break points from D_b in Γ form a set of break points for Γ'. Hence $D' - D_{\partial\Gamma'}$ contains a break divisor on Γ'. The lemma follows now by induction replacing Γ and D with Γ' and D'. □

Let D be a degree g divisor on C. We set $\lambda = \mu(D) \in J(C)$. As in Corollary 6.6 we denote with $D_\lambda = \mu^*[\Theta_{\lambda+\kappa}]$ the canonical effective form of D.

Recall that the theta divisor $[\Theta_{\lambda+\kappa}]$ defines the Voronoi cell decomposition of $\mathbb{R}^g$. Let Q_0 be the interior of a maximal cell. We build a fundamental domain $Q \supset Q_0$ in $\mathbb{R}^g$ by including relative interiors of some boundary cells to satisfy the following "boundary connectedness" criterion: if a cell τ is included then so is any cell of larger dimension incident to τ. One way to choose such Q is to take a generic linear

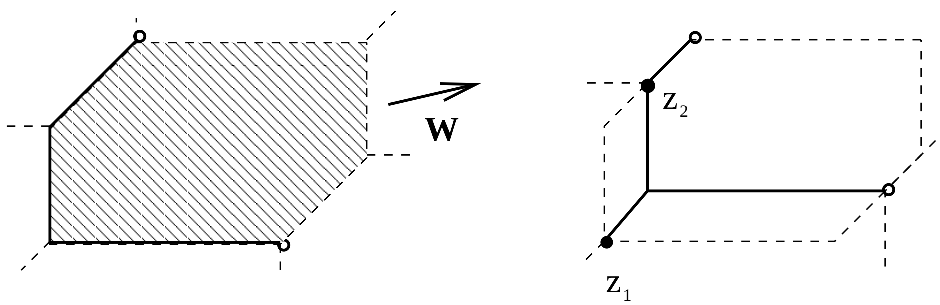

FIGURE 10. The fundamental domains Q, T and the break divisor $D_\lambda = z_1 + z_2$.

functional W on $\mathbb{R}^g$ and include those boundary cells whose baricenters are minimal with respect to W in their corresponding equivalence classes (see Fig. 10).

Furthermore, given a face σ in Q we can choose W such that, in addition, the converse to the above criterion holds for σ to the best possible extent. Namely, for any face $\tau \subset \sigma$ its class representative τ' in Q is a face of σ. On Fig. 10 this holds for the leftmost face. In such case we say that Q is *σ-complete*.

As before we denote by $\hat{\mu} : \hat{C} \to \mathbb{R}^g$ the lifting of the Abel-Jacobi map. Let $G = [\pi_1(C), \pi_1(C)] \subset \pi_1(C)$ be the commutator subgroup of the fundamental group of the curve, and let $\tilde{C} = \hat{C}/G$ be the intermediate abelian covering space. By definition $\hat{\mu}$ factors thought the abelian covering $\tilde{\mu} : \tilde{C} \to \mathbb{R}^g$.

Let $\tilde{T} = \tilde{\mu}^{-1}(Q) \subset \tilde{C}$. By the Abel-Jacobi theorem $\tilde{T}$ contains no cycles, hence it can be continuously lifted to a fundamental domain $T \subset \hat{\mu}^{-1}(Q) \subset \hat{C}$ of the curve. Let D_T be the associated pseudo-break divisor given by (7).

Lemma 8.3. *T is connected and D_T is a break divisor. Furthermore, $D_T = D_\lambda$.*

Proof. Suppose $z \in C$ enters in D_T with multiplicity k. Consider the continuous lift of a neighborhood of z to $\hat{C}$ such that $\hat{z} \in T$. Let $\eta_{i_1}, \dots, \eta_{i_k}$ be the break vectors, that is the primitive tangent vectors at $\hat{z}$ which are *not* in T. Also choose lifts of κ and λ to $\mathbb{R}^g$ such that $\Theta_{\lambda+\kappa} = 0$ on $\bar{Q}_0$. By our boundary connectedness criterion for Q the vectors $\eta_{i_1}, \dots, \eta_{i_k}$ are precisely those primitive tangent vectors at $\hat{z}$ which are mapped outside of $\bar{Q}_0$. Then as we saw in the proof of the Jacobi inversion theorem $\Theta_{\lambda+\kappa}$ must have slope 1 into the directions $\hat{\mu}(\eta_{i_1}), \dots, \hat{\mu}(\eta_{i_k})$ and 0 along other primitive vectors. Thus, the local pull back section $\mu^*\Theta_{\lambda+\kappa}$ has degree k at z, that is z enters in D_λ with multiplicity k, and consequently, $D_T = D_\lambda$. Finally, Lemma 4.8 implies connectedness of T. □

Lemma 8.3 identifies $\hat{T} := \hat{\mu}^{-1}(Q) \subset \hat{C}$ as a disjoint union of fundamental trees on which G acts faithfully and transitively by deck transformations. In fact, the components of $\hat{T}$ are fairly sparse in $\hat{C}$ in the following sense.

Lemma 8.4. *A connected fundamental domain T' in $\hat{C}$ intersects at most one component of $\hat{T}$.*

Proof. Suppose T' intersects two components $T_1, T_2 \subset \hat{T}$. Then there is a path $\hat{\gamma}$ in $T_1 \cup T' \cup T_2$ from $q_1 \in T_1$ to $q_2 \in T_2$, with q_1 and q_2 projecting to the same point in $\widetilde{C}$. Let γ be the image loop of $\hat{\gamma}$ in C. By considering a subpath of $\hat{\gamma}$ if needed we may assume that $\hat{\gamma}$ is simple and no two interior points of $\hat{\gamma}$ in $T_1 \cup T_2$ project to the same point in γ.

Consider a point $\hat{q} \in \hat{\gamma}$ which lies in $T' \cap (T_1 \cup T_2)$. Let $q \in C$ be its image in γ. Then γ passes only once through q. This is impossible since γ is trivial on the chain level in C. □

Theorem 8.5. $q \in \operatorname{supp}|D| \iff \mu(q) \in [\Theta_{\lambda+\kappa}]$.

Proof. Let us choose the fundamental domains Q and T as above.

($\Longrightarrow$) Let a point q be in $C_0 := \mu^{-1}(J(C) \setminus [\Theta_{\lambda+\kappa}]) \subset C$ (note that C_0 is not empty by our assumption) and let $\hat{q}$ be its lift in $T \subset \hat{C}$. We can orient the infinite tree $\hat{C}$ such that $\hat{q}$ is the sink (the root) and every other vertex has exactly one outgoing edge. Thus every leaf of $T \subset \hat{C}$ is oriented inward. By identifying T with C we get an orientation on C, which by (15) defines a degree $g-1$ divisor K_+. Each open leaf of T contributes the corresponding break point into K_+. Thus, by Lemma 8.3 we have $K_+ = D_\lambda - q$. Next we show that the described orientation on C is acyclic. Then $D_\lambda - q$ is a moderator and, consequently, $q \notin \operatorname{supp}|D_\lambda| = \operatorname{supp}|D|$ by Lemma 7.8.

Suppose there is an oriented cycle γ. Then its lift $\hat{\gamma}$ in $T \subset \hat{C}$ is a (connected) path. Let $v_1 \in T$ and $v_2 \in \bar{T} \setminus T$ be the two ends of $\hat{\gamma}$. Then the path from v_2 to $\hat{q}$ in the tree $\hat{C}$ goes through v_1. Note that $\hat{\mu}(v_1)$ lies in the relative interior of some boundary cell, say σ_1, of Q. Hence $\hat{\mu}(v_2) = \hat{\mu}(v_1) - [\gamma]$ lies in $\sigma_2 = \sigma_1 - [\gamma]$, a boundary cell of $\bar{Q}_0$ not included in Q. By continuity, there is a cell $\tau \succ \sigma_2$ which contains some part of $\hat{\mu}(\hat{\gamma})$.

Then for a suitable choice of the linear functional W' the new fundamental domain $Q' \subset \bar{Q}_0$ contains σ_2 and τ but does not contain σ_1. We choose the associated fundamental tree $T' \subset \hat{\mu}^{-1}(Q')$ to contain $\hat{q}$. Let $v_2' \in T'$ be the lift of $\hat{\mu}(v_2)$, and let $\hat{\gamma}' \subset \hat{T}$ be the lift of γ whose missing end is v_2'. Then T' contains $\hat{q}$, v_2' and some part of $\hat{\gamma}'$, but it does not contain v_1', the other end of $\hat{\gamma}'$. Hence $\hat{\gamma}'$ is oriented opposite to $\hat{\gamma}$, and thus has to lie in a component of $\hat{T}$ different from T. But this contradicts Lemma 8.4.

($\Longleftarrow$) Let $\sigma \subset Q$ be the boundary face (closed in Q) whose interior contains the lift of q. We may assume that our choice of Q is σ-complete. Consider the subgraph $C_\sigma \subset C$ which is the projection of $\tilde{\mu}^{-1}(\sigma) \subset \tilde{C}$ under the covering map $\tilde{C} \to C$. Since Q is σ-complete the subgraph C_σ is closed.

Let v be a boundary point of C_σ and let $\xi_1, \dots, \xi_k$ be the exterior primitive tangent vectors at v. Consider the continuous lift of a neighborhood of v to $\hat{C}$ such that $\hat{v} \in T$, then $\hat{\mu}(\hat{v}) \in Q$. Neither of vectors $\hat{\mu}(\xi_i)$ can be inside the cone over σ at $\hat{\mu}(\hat{v})$. Hence not all of them can be inside the cone over Q, otherwise it would violate the tropical balancing condition at $\hat{\mu}(\hat{v})$.

Hence, every boundary point of C_σ supports at least one break point of C whose tangent vector is exterior to C_σ. Consequently, $D_\lambda - D_{\partial C_\sigma}$ contains a break divisor for C_σ (we may replace C_σ by its connected component if needed). Thus we can apply Lemma 8.2 to conclude that $C_\sigma \subset \operatorname{supp}|D_\lambda|$. □

Remark. In the case $q \in [\Theta_{\lambda+\kappa}]$ the corresponding orientation on C always contains an oriented cycle, that is $D_\lambda - q$ is *not* a moderator. Indeed, the last paragraph of ($\Longleftarrow$) shows that every boundary point of C_σ has an outward oriented exterior edge, but there is no sink in $C \setminus C_\sigma$.

Note that if $\mu(C)$ intersect $[\Theta_{\lambda+\kappa}]$ in isolated points then D is rigid, that is the linear system $|D|$ consists of a single element, namely D_λ. On the other hand, if $\mu(C) \subset [\Theta_{\lambda+\kappa}]$ then we get an analog of the classical result that the linear system $|D|$ is special. However, contrary to the classical situation, there are intermediate cases.

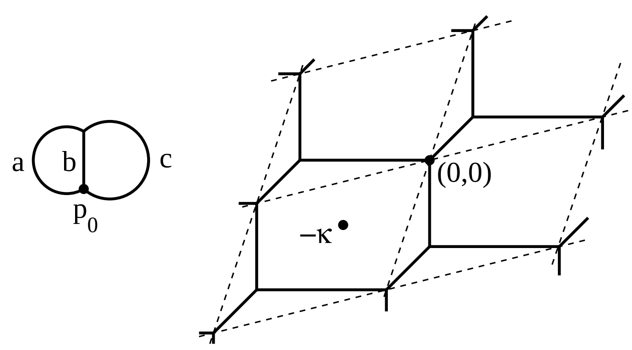

FIGURE 11. The theta divisor is the shift by $\kappa = (\frac{a+b}{2}, \frac{c+b}{2})$ of the $W_1 = \mu(C)$.

Corollary 8.6 (Riemann's Theorem). *$W_{g-1} + \kappa = [\Theta]$, where $\kappa \in J(C)$ is the Jacobi Inversion constant.*

Proof. By Corollary 6.6 a degree $g-1$ divisor D is linearly equivalent to effective if and only if $|D_\lambda - p_0| \neq \emptyset$. By Theorem 8.5 this is equivalent to $\mu(p_0) = 0 \in [\Theta_{\lambda+\kappa}]$. Hence we have

$$\lambda \in W_{g-1} \Longleftrightarrow 0 \in [\Theta_{\lambda+\kappa}] \Longleftrightarrow 0 \in [\Theta_{-\lambda-\kappa}] \Longleftrightarrow \lambda \in [\Theta] - \kappa,$$

where the second equivalence is because of $\Theta(-x) = \Theta(x)$. □

As a by-product we may now identify $\kappa \in \mathrm{Pic}^{g-1}$. By Corollary 7.11 we have $W_{g-1} = \mu(K) - W_{g-1}$, and consequently

$$W_{g-1} + \kappa = [\Theta] = -[\Theta] = -W_{g-1} - \kappa = W_{g-1} - \mu(K) - \kappa.$$

But the theta line bundle defines a principal polarization, hence $[\Theta]$ cannot be stable under any non-trivial translation. Thus, $2\kappa = -\mu(K)$, i.e. κ can be interpreted as a tropical spin structure.

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