THE TROPICAL MANIN–MUMFORD CONJECTURE

DAVID HARRY RICHMAN

Abstract. In analogy with the Manin–Mumford conjecture for algebraic curves, one may ask how a metric graph under the Abel–Jacobi embedding intersects torsion points of its Jacobian. We show that the number of torsion points is finite for metric graphs of genus $g \geq 2$ which are biconnected and have edge lengths which are "sufficiently irrational" in a precise sense. Under these assumptions, the number of torsion points is bounded by $3g-3$. Next we study bounds on the number of torsion points in the image of higher-degree Abel–Jacobi embeddings, which send d-tuples of points to the Jacobian. This motivates the definition of the "independent girth" of a graph, a number which is a strict upper bound for d such that the higher-degree Manin–Mumford property holds.

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1. Introduction

Suppose X is a smooth algebraic curve, let $\mathrm{Jac}(X)$ denote the Jacobian of X, and let $\iota_q : X \to \mathrm{Jac}(X)$ denote the Abel–Jacobi map with basepoint $q \in X$. The Manin–Mumford conjecture, now a theorem due to Raynaud [18], states that if X has genus $g \geq 2$ then the image $\iota_q(X)$ intersects only finitely many torsion points of $\mathrm{Jac}(X)$.

1.1. **Statement of results.** The setup above makes sense when the algebraic curve is replaced with a metric graph. Say a metric graph Γ satisfies the *Manin–Mumford condition* if the image of the Abel–Jacobi map $\iota_q : \Gamma \to \mathrm{Jac}(\Gamma)$ intersects only finitely many torsion points of $\mathrm{Jac}(\Gamma)$, for every choice of basepoint $q \in \Gamma$. As with algebraic curves, the interesting case to consider is when Γ has genus $g \geq 2$.

Theorem 1.1 (Conditional uniform Manin–Mumford bound). *Let Γ be a connected metric graph of genus $g \geq 2$. If the set of torsion points $\iota_q(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}$ is finite, then we have the uniform bound*

$$\#(\iota_q(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}) \leq 3g - 3.$$

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Unlike the case of algebraic curves, for a metric graph the genus condition $g \geq 2$ is not sufficient to imply that $\#(\iota_q(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}})$ is finite. On a graph with unit edge lengths the degree-zero divisor classes supported on vertices form a finite abelian group, known as the *critical group* of the graph (see Section 2.1). In particular, vertex-supported divisor classes are always torsion.

Observation 1.2. *Suppose $\Gamma = (G, \ell)$ is a metric graph of genus $g \geq 2$ whose edge lengths are all rational, i.e. $\ell(e) \in \mathbb{Q}_{>0}$ for all $e \in E(G)$. Then Γ does not satisfy the Manin–Mumford condition.*

We say that a property holds for a *very general* point of some (real) parameter space if it holds outside of a countable collection of codimension-1 families. Recall that a graph G is *biconnected* (or *two-connected*) if G is connected after deleting any vertex.

Theorem 1.3 (Generic tropical Manin–Mumford)**.** *Let G be a finite connected graph of genus $g \geq 2$. If G is biconnected, then for a very general choice of edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$, the metric graph $\Gamma = (G, \ell)$ satisfies the Manin–Mumford condition.*

Say a metric graph Γ satisfies the *(generalized) Manin–Mumford condition in degree d* if the image of the degree d Abel–Jacobi map

$$\begin{aligned} \iota_D^{(d)} : \Gamma^d &\to \mathrm{Jac}(\Gamma) \\ (p_1, \dots, p_d) &\mapsto [p_1 + \cdots + p_d - D] \end{aligned}$$

intersects only finitely many torsion points of $\mathrm{Jac}(\Gamma)$, for every choice of effective, degree d divisor class $[D]$.

When $d = 1$ this is the Manin–Mumford condition on Γ. If the generalized Manin–Mumford condition holds in degree d, then it also holds in degree d' for any $1 \leq d' \leq d$. When $d \geq g$ and $g \geq 1$, the generalized Manin–Mumford condition cannot hold, since the higher Abel–Jacobi map will be surjective and $\mathrm{Jac}(\Gamma)_{\mathrm{tors}}$ is infinite.

Theorem 1.4 (Conditional uniform Manin–Mumford bound in higher degree)**.** *Let Γ be a connected metric graph of genus $g \geq 1$. If Γ satisfies the Manin–Mumford condition in degree d, then*

$$\#(\iota_D^{(d)}(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}) \leq \binom{3g-3}{d}.$$

Recall that the *girth* of a graph is the minimal length of a cycle; this number provides a constraint on d such that the degree d Manin–Mumford condition holds. Note that if G has genus ≥ 2 and girth 1, i.e. G has a loop edge, then G cannot be biconnected.

Observation 1.5. *Let G be a finite connected graph with girth γ. Then for any choice of edge lengths the metric graph $\Gamma = (G, \ell)$ does not satisfy the generalized Manin–Mumford condition in degree $d \geq \gamma$.*

We define the *independent girth* γ^{ind} of a graph as

$$\gamma^{\mathrm{ind}}(G) = \min_C \left(\#E(C) + 1 - h_0(G \backslash E(C)) \right)$$

where the minimum is taken over all cycles C in G, and h_0 denotes the number of connected components. Since the (usual) girth is by definition $\gamma(G) = \min_C(\#E(C))$ and $h_0 \geq 1$, we have the inequality $\gamma^{\mathrm{ind}} \leq \gamma$. The independent girth is invariant under subdivision of edges, so it is well-defined for a metric graph. The independent girth may be expressed in terms of the cographic matroid of G, see Section 2.4.

Say a graph G is *Manin–Mumford finite in degree d* if for very general edge lengths ℓ, the metric graph (G, ℓ) satisfies the degree d Manin–Mumford condition.

Theorem 1.6 (Generic tropical Manin–Mumford in higher degree)**.** *Let G be a finite connected graph of genus $g \geq 1$ with independent girth γ^{ind}. Then G is Manin–Mumford finite in degree d if and only if $1 \leq d < \gamma^{\mathrm{ind}}$.*

1.2. **Previous work.** Faltings's theorem (previously Mordell's conjecture) states that a smooth curve of genus $g \geq 2$ has finitely many rational points, i.e. points whose coordinates are all rational numbers. By analogy with Mordell's conjecture, Manin and Mumford conjectured that a smooth algebraic curve of genus 2 or more has finitely many torsion points. The Manin–Mumford conjecture was proved by Raynaud [17], which inspired several generalizations concerning torsion points in abelian varieties.

After Raynaud's work, it was still unknown whether the number of torsion points on a genus g curve could be bounded as a function of g; this became known as the "uniform Manin–Mumford conjecture." Baker and Poonen [3] extended Raynaud's result quantitatively by proving strong bounds on the number of torsion points that arise on a given curve as the basepoint for the Abel–Jacobi map varies. In particular, they showed that a curve X of genus $g \geq 2$ has finitely many choices of basepoint q so that $\#(\iota_q(X) \cap \mathrm{Jac}(X)_{\mathrm{tors}})$ has size greater than 2. Katz, Rabinoff, and Zureick-Brown [11] used tropical methods to prove a uniform bound on the number of torsion points on an algebraic curve of fixed genus, which satisfy an additional technical constraint on the reduction type. Kühne [13] (in characteristic zero) and Looper, Silverman, and Wilms [14] (in positive characteristic) recently proved uniform bounds on the number of torsion points on an algebraic curve.

Regarding the higher-degree Manin–Mumford conjecture, Abramovich and Harris [1] studied the question of when the locus $W_d(X)$ of effective degree d divisor classes on an algebraic curve X contains an abelian subvariety of $\mathrm{Jac}(X)$. This question was studied further by Debarre and Fahlaoui [8].

1.3. **Notation.** Here we collect some notation which will be used throughout the paper.

Γ	a compact, connected metric graph
(G, ℓ)	a combinatorial model for a metric graph, where $\ell : E(G) \to \mathbb{R}_{>0}$ is a length function on edges of G
G	a finite, connected combinatorial graph (loops and parallel edges allowed)
$E(G)$	edge set of G
$V(G)$	vertex set of G
$\mathcal{T}(G)$	set of spanning trees of G
$\mathcal{C}(G)$	set of cycles of G
D	a divisor on a metric graph
$\mathrm{PL}_{\mathbb{R}}(\Gamma)$	set of piecewise linear functions on Γ
$\mathrm{PL}_{\mathbb{Z}}(\Gamma)$	set of piecewise $\mathbb{Z}$-linear functions on Γ
$\Delta(f)$	the principal divisor associated to a piecewise ($\mathbb{Z}$-)linear function f
$\mathrm{Div}(\Gamma)$	divisors (with $\mathbb{Z}$-coefficients) on Γ
$\mathrm{Div}^d(\Gamma)$	divisors of degree d on Γ
$[D]$	the set of divisors linearly equivalent to D (i.e. the linear equivalence class of the divisor D)
$\lvert D\rvert$	the set of effective divisors linearly equivalent to D
$\mathrm{Pic}^d(\Gamma)$	divisor classes of degree d on Γ
$\mathrm{Eff}^d(\Gamma)$	effective divisor classes of degree d on Γ
$\mathrm{Jac}(\Gamma)$	the Jacobian of Γ, $\mathrm{Jac}(\Gamma) = \mathrm{Div}^0(\Gamma)/\,\Delta(\mathrm{PL}_{\mathbb{Z}}(\Gamma))$

2. Graphs and matroids

A *graph* $G = (V, E)$ consists of a finite set of vertices $V = V(G)$ and a finite set of edges $E = E(G)$, equipped with two maps *head* $: E \to V$ and *tail* $: E \to V$, which we abbreviate by $e^+ = \mathit{head}(e)$ and $e^- = \mathit{tail}(e)$. We say an edge e lies between vertices v, w if $(e^+, e^-) = (v, w)$ or $(e^+, e^-) = (w, v)$. We allow loops (i.e. e such that $e^+ = e^-$) and multiple edges. The *valence* of a vertex $v \in V$ is

$$\mathrm{val}(v) = \#\{e \in E : e^+ = v\} + \#\{e \in E : e^- = v\}.$$

Note that loop edge at v contributes 2 to the valence of v.

Given a subset of edges $A \subset E$, we let $G|A$ denote the subgraph of G whose vertex set is $V(G)$ and whose edge set is A. We let $G \setminus A$ denote the subgraph whose vertex set is $V(G)$ and whose edge set is $E \setminus A$.

For a connected graph G, the *genus* is defined as $g(G) = \#E(G) - \#V(G) + 1$. A *spanning tree* of G is a subgraph T with the same vertex set and whose edge set is a subset of the edges of G, such that T is connected and has no cycles. The number of edges in any spanning tree of G is $\#V(G) - 1 = \#E(G) - g(G)$.

2.1. **Critical group.** Fix a graph $G = (V, E)$ with n vertices, enumerated $\{v_1, \dots, v_n\}$. The *Laplacian matrix* of G is the $n \times n$ matrix L whose entries are

$$L_{ij} = \begin{cases} \#(\text{edges between } v_i \text{ and } v_j) & \text{if } i \neq j \\ -\sum_{k \neq i} \#(\text{edges from } v_i \text{ to } v_k) & \text{if } i = j. \end{cases}$$

(Other sources, e.g. [2], use the opposite sign convention for the graph Laplacian.) The matrix L is symmetric and its rows and columns sum to zero. The Laplacian defines a linear map $\mathbb{Z}^V \to \mathbb{Z}^V$, whose image has rank $n-1$ if G is connected. Let $\epsilon : \mathbb{Z}^V \to \mathbb{Z}$ denote the linear map which sums all coordinates. The image of the Laplacian matrix L lies in the kernel of ϵ.

The *critical group* $\mathrm{Jac}(G)$ of a connected graph G is the abelian group defined as

$$\mathrm{Jac}(G) = \ker(\epsilon) / \operatorname{im}(L).$$

The critical group is finite, and its cardinality is the number of spanning trees of G [4, Theorem 6.2]. For more on the critical group, see [2, 4] and the references therein.

Example 2.1. Let G be the theta graph, which has two vertices connected by three edges.

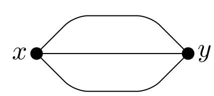

FIGURE 1. Theta graph.

The Laplacian matrix of this graph is $\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$, and $\mathrm{Jac}(G) \cong \mathbb{Z}/3\mathbb{Z}$.

Example 2.2. Let G be the Wheatstone graph shown below, which has four vertices and five edges.

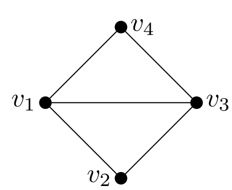

FIGURE 2. Wheatstone graph.

The Laplacian matrix of this graph is

$$\begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix},$$

and its critical group is $\mathrm{Jac}(G) \cong \mathbb{Z}/8\mathbb{Z}$.

We now give additional notation for constructing $\mathrm{Jac}(G)$, which connects the critical group to the Jacobian of a metric graph (see Section 3.2). Let the *divisor group* $\mathrm{Div}(G)$ be the free abelian group on the set of vertices of G;

$$\mathrm{Div}(G) = \{\sum_{v_i \in V} a_i v_i : a_i \in \mathbb{Z}\},$$

and let

$$\mathrm{PL}_{\mathbb{Z}}(G) = \{f : V \to \mathbb{Z}\}$$

denote the set of integer-valued functions on vertices of G, which also forms a free abelian group. (The notation PL is for *piecewise linear*; any $f : V \to \mathbb{Z}$ has a unique linear interpolation along the edges.) The set $\{\mathbf{1}_{v_i} : v_i \in V\}$ forms a basis for $\mathrm{PL}_{\mathbb{Z}}(G)$ where

$$\mathbf{1}_{v_i}(w) = \begin{cases} 1 & \text{if } w = v_i, \\ 0 & \text{if } w \neq v_i. \end{cases}$$

The divisor group by definition has the basis $\{v_i : v_i \in V\}$. Let $\Delta : \mathrm{PL}_{\mathbb{Z}}(G) \to \mathrm{Div}(G)$ be the linear map defined by the Laplacian matrix, with respect to the bases defined above.

The degree map $\deg : \mathrm{Div}(G) \to \mathbb{Z}$ is defined by $\deg(\sum_i a_i v_i) = \sum_i a_i$. Let $\mathrm{Div}^0(G)$ denote the kernel of the degree map. The critical group of G is

$$\mathrm{Jac}(G) = \mathrm{Div}^0(G)/\,\Delta(\mathrm{PL}_{\mathbb{Z}}(G)).$$

(Both $\mathrm{Div}(G)$ and $\mathrm{PL}_{\mathbb{Z}}(G)$ are isomorphic to $\mathbb{Z}^V$. $\mathrm{Div}(G)$ is naturally covariant with respect to $G_1 \to G_2$, while $\mathrm{PL}_{\mathbb{Z}}(G)$ is naturally contravariant.)

2.2. **Metric graphs.** A *metric graph* is a compact, connected metric space which comes from assigning the path metric to a finite, connected graph whose edges are identified with closed intervals $[0, \ell(e)]$ with positive, real lengths $\ell(e) > 0$. If the metric graph Γ comes from a combinatorial graph G by assigning edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$, we say (G, ℓ) is a *combinatorial model* for Γ and we write $\Gamma = (G, \ell)$. A single metric graph generally has many different combinatorial models.

The *genus* of a metric graph Γ is the dimension of the first homology,

$$g(\Gamma) = \dim H_1(\Gamma, \mathbb{R}).$$

Given a combinatorial model $\Gamma = (G, \ell)$, the genus agrees with the underlying graph,

$$g(\Gamma) = g(G) = \#E(G) - \#V(G) + 1.$$

The *valence* of a point x on a metric graph Γ, denoted $\mathrm{val}(x)$, is the number of connected components in a sufficiently small punctured neighborhood of x. If (G, ℓ) is a model for Γ, then for any $v \in V(G)$ the valence $\mathrm{val}(v)$ as a vertex of G agrees with $\mathrm{val}(v)$ as a point in the metric graph Γ.

2.3. **Stabilization.** The notion of stability is useful for our purposes because questions about Abel–Jacobi maps $\iota : \Gamma \to \mathrm{Jac}(\Gamma)$ maybe be reduced to $\iota : \Gamma' \to \mathrm{Jac}(\Gamma')$ where Γ' is a semistable metric graph. (See Section 3.2 for discussion of the Abel–Jacobi map.)

A connected graph G is *stable* if every vertex $v \in V(G)$ has valence at least 3, and *semistable* if every vertex has $\mathrm{val}(v) \geq 2$. A metric graph Γ is *semistable* if every point $x \in \Gamma$ has valence at least 2. Equivalently, Γ is semistable if it has a model (G, ℓ) where G is a semistable graph. We say (G, ℓ) is a (semi)stable model for Γ if G is (semi)stable.

Proposition 2.3. *A semistable metric graph Γ with genus $g \geq 2$ has a unique stable model (G, ℓ), (i.e. a model such that G is stable).*

Proof. The unique stable model has vertex set $V(G) = \{x \in \Gamma : \mathrm{val}(x) \geq 3\}$. The edges $E(G)$ correspond to connected components of $\Gamma \setminus V(G)$, which is isometric to a disjoint union of open intervals of finite length. $\square$

Proposition 2.4. *Suppose G is a stable graph of genus g. Then the number of edges in G is at most $3g-3$.*

Proof. Since every vertex has valence at least 3, we have

$$\#V(G) \leq \frac{1}{3} \sum_{v \in V(G)} \mathrm{val}(v) = \frac{2}{3} \cdot \#E(G).$$

By the genus formula $g = \#E(G) - \#V(G) + 1$, this implies

$$\#E(G) = g - 1 + \#V(G) \leq g - 1 + \frac{2}{3} \cdot \#E(G)$$

which is equivalent to the desired inequality $\#E(G) \leq 3g - 3$. $\square$

It follows from the previous proposition that a stable graph has genus $g \geq 2$.

Proposition 2.5 (Metric graph stabilization)**.** *Suppose Γ has genus $g \geq 1$. There is a canonical semistable subgraph $\Gamma' \subset \Gamma$ and a retract map $r : \Gamma \to \Gamma'$ such that r is a homotopy inverse to the inclusion $i : \Gamma' \to \Gamma$.*

We call the subgraph Γ' of Proposition 2.5 the *stabilization* of Γ, and denote it as $\mathrm{st}(\Gamma)$.

Example 2.6. Figure 3 shows the stabilization Γ' of a metric graph Γ of genus two. The retract map $\Gamma \to \Gamma'$ sends a point of Γ to the closest point of Γ' in the path metric.

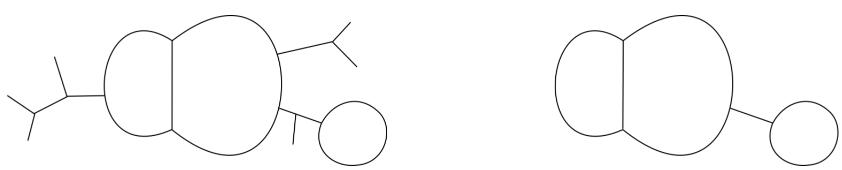

FIGURE 3. A metric graph (left) and its stabilization (right).

2.4. **Matroids.** In this section we review the definition of a matroid. In particular, we recall the graphic matroid and cographic matroid associated to a connected graph. Cographic matroids will be useful for understanding the structure of the Jacobian of a metric graph. For a reference on matroids, see [16] or [10].

A *matroid* $M = (E, \mathcal{B})$ is a finite set E equipped with a nonempty collection $\mathcal{B} \subset 2^E$ of subsets of E, called the *bases* of the matroid, satisfying the basis exchange axiom: for distinct subsets $B_1, B_2 \in \mathcal{B}$, there exists some $x \in B_1 \backslash B_2$ and $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$. In other words, from B_1 we can produce a new basis by exchanging an element of B_1 for an element of B_2. An *independent set* of a matroid $M = (E, \mathcal{B})$ is a subset of E which is a subset of some basis. A *cycle* of M is a subset of E which is minimal among those not contained in any basis, under the inclusion relation. The *rank* of a subset $A \subset E$ is the cardinality of a maximal independent set contained in A; we denote the rank by $\mathrm{rk}_M(A)$ or $\mathrm{rk}(A)$ if the matroid is understood from context.

Given a graph $G = (V, E)$, the *graphic matroid* $M(G)$ is the matroid on the ground set $E = E(G)$ with bases $\mathcal{B} = \{E(T) : T \text{ is a spanning tree of } G\}$. An independent set in $M(G)$ is a subset of edges which span an acyclic subgraph. (i.e. $h^1(G|A) = 0$.) A cycle in $M(G)$ is a cycle in the graph-theoretic sense, i.e. a subset of edges which span a subgraph homeomorphic to a circle. The graphic matroid $M(G)$ is also known as the *cycle matroid* of G.

Example 2.7. Suppose G is the *Wheatstone graph* shown in Figure 4. The bases of $M(G)$ are $\{abd, abe, acd, ace, ade, bcd, bce, bde\}$. The cycles are $\{abc, abde, cde\}$. (Here abc is shorthand for the set $\{a, b, c\}$.)

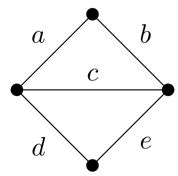

FIGURE 4. Wheatstone graph.

Given a graph $G = (V, E)$, the *cographic matroid* $M^\perp(G)$ is the matroid on the ground set $E = E(G)$ whose bases are complements of spanning trees of G. An independent set in $M^\perp(G)$ is a set of edges whose removal does not disconnect G (i.e. a set $A \subset E$ such that $G\backslash A$ is connected, equivalently $h^0(G\backslash A) = 1$). An edge set $A \subset E(G)$ is called a *cut* of G if $G\backslash A$ is disconnected. A cycle in $M^\perp(G)$ is a minimal set of edges A such that $h^0(G\backslash A) = 2$; this is called a *simple cut* or a *bond* of G. The cographic matroid is also known as the *cocycle matroid* or *bond matroid* of G. For more on cographic matroids, see [16, Chapter 2.3].

Note: when discussing the graphic or cographic matroid of a graph G, we always use "cycle of G" to refer to a cycle in the *graphic* matroid sense.

Example 2.8. Suppose G is the Wheatstone graph, shown in Figure 4. The bases of the cographic matroid $M^\perp(G)$ are $\{ac, ad, ae, bc, bd, be, cd, ce\}$. The cycles of $M^\perp(G)$ are $\{ab, acd, ace, bcd, bce, de\}$.

2.5. **Girth and independent girth.** Recall that the *girth* $\gamma = \gamma(G)$ of a graph is the minimal length of a cycle; a *cycle* is a subgraph homeomorphic to a circle. In other words,

$$\gamma(G) = \min_{C \in \mathcal{C}(G)} \{\#E(C)\} \tag{1}$$

where $\mathcal{C}(G)$ denotes the set of cycles of G.

Definition 2.9. The *independent girth* γ^{ind} of a graph is defined as

$$\gamma^{\text{ind}}(G) = \min_{C \in \mathcal{C}(G)} \{\, \text{rk}^\perp(E(C))\, \} \tag{2}$$

where $\text{rk}^\perp$ is the rank function of the cographic matroid $M^\perp(G)$. (See Section 2.4 for discussion of cographic matroids). If G has genus zero, we let $\gamma^{\text{ind}}(G) = \gamma(G) = +\infty$.

Equivalently,

$$\gamma^{\text{ind}}(G) = \min_{C \in \mathcal{C}(G)} \{\, \#E(C) + 1 - h_0(G\backslash E(C))\, \}$$

where $G\backslash E(C)$ denotes the subgraph obtained by deleting the interior of each edge in C, and h_0 denotes the number of connected components of a topological space.

Proposition 2.10. *(a) For any graph G, $\gamma^{\text{ind}}(G) \leq \gamma(G)$.*
(b) If (G, ℓ) and (G', ℓ') are combinatorial models for the same metric graph Γ, then $\gamma^{\text{ind}}(G) = \gamma^{\text{ind}}(G')$.

Proof. (a) The rank function of any matroid satisfies $\text{rk}(A) \leq \#A$. The claim follows from comparing definitions (1) and (2).

(b) The independent girth does not change under subdivision of edges, and any two combinatorial models of Γ have a common refinement by edge subdivisions. $\square$

Proposition 2.10(b) implies that γ^{ind} is a well-defined invariant for a metric graph; given a metric graph Γ we have

$$\gamma^{\text{ind}}(\Gamma) := \gamma^{\text{ind}}(G) \quad \text{for any choice of model } \Gamma = (G, \ell). \tag{3}$$

Note that γ^{ind} is also invariant under stabilization, i.e.

$$\gamma^{\text{ind}}(\Gamma) = \gamma^{\text{ind}}(\operatorname{st}\Gamma).$$

Example 2.11. Consider Figure 5. The graph on the left has seven simple cycles; their lengths are $\{4,4,4,6,6,6,6\}$, and their ranks in the cographic matroid are all 3. For this graph, $\gamma = 4$ and $\gamma^{\text{ind}} = 3$. After deleting a central edge, the resulting graph on the right has three simple cycles with lengths $\{4,6,6\}$ and cographic rank 2; hence $\gamma = 4$ and $\gamma^{\text{ind}} = 2$.

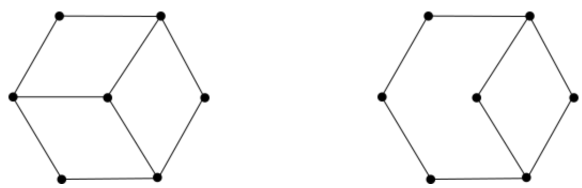

FIGURE 5. Graphs with independent girth 3 (left) and independent girth 2 (right).

Example 2.12. Consider Figure 6. This graph has $\gamma = 4$ and $\gamma^{\text{ind}} = 3$, with the minimum achieved on the 4-cycle in the middle. After deleting one of the horizontal edges in the middle cycle, the resulting graph has $\gamma = 4$ and $\gamma^{\text{ind}} = 4$.

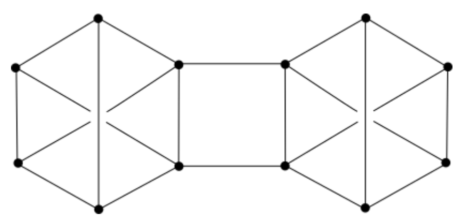

FIGURE 6. Graph with girth 4 and independent girth 3.

In general, under edge deletion we have $\gamma(G \backslash e) \geq \gamma(G)$ since $\mathcal{C}(G \backslash e) \subset \mathcal{C}(G)$. The examples above demonstrate that $\gamma^{\text{ind}}(G \backslash e)$ can increase or can decrease, relative to $\gamma^{\text{ind}}(G)$.

Theorem 2.13. *There exists a constant C such that for any stable graph G of genus $g \geq 2$, the girth $\gamma = \gamma(G)$ satisfies*

$$\gamma < C \log g.$$

Proof. Recall that the girth γ of a graph G is the minimal length of a (simple) cycle in G. Let v be a vertex in $V(G)$. Let $N_r(v)$ denote the neighborhood of radius r around v, in the graph G. For any radius $r < \frac{1}{2}\gamma$, the neighborhood $N_r(v)$ is a tree (i.e. $N_r(v)$ is connected and acyclic).

Recall that G is stable if every vertex has valence at least 3. Since G is stable, we may calculate a simple lower bound for the number of edges in $N_r(v)$. Namely,

$$\#E(N_r(v)) \geq 3 + 6 + \cdots + 3 \cdot 2^{r-1} = 3(2^r - 1).$$

This quantity is clearly a lower bound for the total number of edges $\#E(G)$. Moreover, by Proposition 2.4 we have $\#E(G) \leq 3g - 3$. Thus

$$3(2^r - 1) \leq \#E(G) \leq 3g - 3 \qquad \Rightarrow \qquad 2^r \leq g$$

for any integer $r < \frac{1}{2}\gamma$. Hence

$$2^{\gamma/2-1} < g \qquad \Leftrightarrow \qquad \gamma < 2\log_2 g + 2.$$

By the assumption $g \geq 2$, this bound implies $\gamma < 4\log_2 g$, as desired. □

Corollary 2.14. *There exists a constant C such that for any metric graph Γ of genus $g \geq 2$, the independent girth γ^{ind} satisfies $\gamma^{\text{ind}} < C\log g$.*

Proof. Combine Theorem 2.13 with Proposition 2.10(a) and (3). □

3. Divisors on metric graphs

In this section we recall the theory of divisors and linear equivalence on metric graphs.

On a metric graph Γ, the *divisor group* $\mathrm{Div}(\Gamma)$ is the free abelian group generated by the points of Γ. We also let $\mathrm{Div}_{\mathbb{R}}(\Gamma) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Div}(\Gamma)$. In other words,

$$\mathrm{Div}(\Gamma) = \{\sum_{x\in\Gamma} a_x x : a_x \in \mathbb{Z},\, a_x = 0 \text{ for almost all } x\},$$
$$\mathrm{Div}_{\mathbb{R}}(\Gamma) = \{\sum_{x\in\Gamma} a_x x : a_x \in \mathbb{R},\, a_x = 0 \text{ for almost all } x\}.$$

A divisor $D = \sum_{x\in\Gamma} a_x x$ is *effective* if $a_x \geq 0$ for every x. The *degree* map $\deg : \mathrm{Div}_{\mathbb{R}}(\Gamma) \to \mathbb{R}$ sends $D = \sum_{x\in\Gamma} a_x x$ to $\deg(D) = \sum_{x\in\Gamma} a_x$.

3.1. **Real Laplacian.** A *piecewsie linear function* on Γ is a continuous function $f : \Gamma \to \mathbb{R}$ which is linear on each edge of some combinatorial model, i.e. for some model $\Gamma = (G, \ell)$, on the interior of each edge e in $E(G)$ we have $\frac{d}{dt}f(t)$ is constant, where t is a length-preserving parameter on e. We say a model (G, ℓ) is *compatible with* a piecewise linear function f if f is linear on each edge of G. We let $\mathrm{PL}_{\mathbb{R}}(\Gamma)$ denote the set of all piecewise linear functions on Γ, which has the structure of a vector space over $\mathbb{R}$.

The *metric graph Laplacian* Δ is the $\mathbb{R}$-linear map from $\mathrm{PL}_{\mathbb{R}}(\Gamma)$ to $\mathrm{Div}_{\mathbb{R}}(\Gamma)$ defined as follows. For $f \in \mathrm{PL}_{\mathbb{R}}(\Gamma)$, let (G, ℓ) be a model for Γ which is compatible with f, and let

$$\Delta(f) = \sum_{v\in V(G)} a_v v \qquad \text{where} \qquad a_v = \sum_{\substack{e\in E(G)\\ e^+=v}} f'(t) + \sum_{\substack{e\in E(G)\\ e^-=v}} f'(t), \tag{4}$$

such that in each summand $f'(t) = \frac{d}{dt}f(t)$, the parameter t is directed away from the vertex v. Equivalently, the coefficient of v in $\Delta(f)$ is

$$\begin{aligned} a_v &= \sum_{\substack{e\in E(G)\\ e^+=v}} \frac{f(e^-) - f(e^+)}{\ell(e)} + \sum_{\substack{e\in E(G)\\ e^-=v}} \frac{f(e^+) - f(e^-)}{\ell(e)} \\ &= \sum_{\substack{e\in E(G)\\ e^+=v}} \frac{f(e^-)}{\ell(e)} + \sum_{\substack{e\in E(G)\\ e^-=v}} \frac{f(e^+)}{\ell(e)} - \left(\sum_{\substack{e\in E(G)\\ e^+=v\\ \text{or } e^-=v}} \frac{1}{\ell(e)} \right) f(v) \end{aligned}$$

For any piecewise linear function f there is a unique way to write $\Delta(f) = D - E$ where D and E are effective; we call $\Delta^+(f) = D$ the *divisor of zeros* of f and call $\Delta^-(f) = E$ the *divisor of poles* of f. Note that for any $f \in \mathrm{PL}_{\mathbb{R}}(\Gamma)$, the divisor $\Delta(f)$ has degree zero.

The Laplacian $\Delta : \mathrm{PL}_{\mathbb{R}}(\Gamma) \to \mathrm{Div}_{\mathbb{R}}(\Gamma)$ fits in an exact sequence

$$0 \to \mathbb{R} \to \mathrm{PL}_{\mathbb{R}}(\Gamma) \xrightarrow{\Delta} \mathrm{Div}_{\mathbb{R}}(\Gamma) \to \mathbb{R} \to 0,$$

where the image of $\mathbb{R} \to \mathrm{PL}_{\mathbb{R}}(\Gamma)$ is the set of constant functions on Γ, and $\mathrm{Div}_{\mathbb{R}}(\Gamma) \to \mathbb{R}$ is the degree map. In particular, for any points $y, z \in \Gamma$, the divisor $D = z - y$ has degree zero and there is a function f satisfying $\Delta(f) = z - y$, unique up to an additive constant.

Definition 3.1. The *unit potential function* j_z^y is the unique function in $\mathrm{PL}_{\mathbb{R}}(\Gamma)$ satisfying

$$\Delta(j_z^y) = z - y \quad \text{and} \quad j_z^y(z) = 0.$$

There are useful explicit formulas for the slopes of j_z^y due to Kirchhoff, which we discuss in Section 3.4.

Proposition 3.2 (Slope-current principle). *Suppose $f \in \mathrm{PL}_{\mathbb{R}}(\Gamma)$ has zeros $\Delta^+(f)$ and poles $\Delta^-(f)$ of degree $d \in \mathbb{R}$. Then the slope of f is bounded by d, i.e.*

$$|f'(x)| \leq d \qquad \text{for any } x \text{ where } f \text{ is linear.}$$

(This bound is attained only on bridge edges, and only when all zeros are on one side of the bridge and all poles are on the other side.)

Proof. See [18, Proposition 3.5]. □

Remark 3.3. The above proposition has a "physical" interpretation: f gives the voltage in the resistor network Γ when subjected to an external current of $\Delta^-(f)$ units flowing into the network and $\Delta^+(f)$ units flowing out. The slope $|f'(x)|$ is equal to the current flowing through the wire containing x, which must be no more than the total in-flowing (or out-flowing) current.

3.2. **Integer Laplacian and Jacobian.** Recall that $\mathrm{Div}(\Gamma)$ denotes the free abelian group generated by points of Γ. A *piecewise $\mathbb{Z}$-linear function* on Γ is a piecewise linear function whose slopes are integers, i.e. there exists some model $\Gamma = (G, \ell)$ such that $f'(t) \in \mathbb{Z}$ on the interior of each edge of G. We let $\mathrm{PL}_{\mathbb{Z}}(\Gamma)$ denote the set of all piecewise $\mathbb{Z}$-linear functions on Γ.

The Laplacian Δ, defined in Section 3.1, restricts to a map $\Delta : \mathrm{PL}_{\mathbb{Z}}(\Gamma) \to \mathrm{Div}(\Gamma)$. Two divisors D, E are *linearly equivalent* if there is some $f \in \mathrm{PL}_{\mathbb{Z}}(\Gamma)$ such that $D = E + \Delta(f)$. We let $[D]$ denote the linear equivalence class of a divisor D, i.e.

$$[D] = \{E \in \mathrm{Div}(\Gamma) : E = D + \Delta(f) \text{ for some } f \in \mathrm{PL}_{\mathbb{Z}}(\Gamma)\}$$

The *Picard group* $\mathrm{Pic}(\Gamma)$ is defined as the cokernel of $\Delta : \mathrm{PL}_{\mathbb{Z}}(\Gamma) \to \mathrm{Div}(\Gamma)$. The elements of $\mathrm{Pic}(\Gamma)$ are linear equivalence classes of divisors on Γ. The integer Laplacian map fits in an exact sequence

$$0 \to \mathbb{R} \to \mathrm{PL}_{\mathbb{Z}}(\Gamma) \xrightarrow{\Delta} \mathrm{Div}(\Gamma) \to \mathrm{Pic}(\Gamma) \to 0.$$

The degree of a divisor class is well-defined, so we have an induced degree map $\mathrm{Pic}(\Gamma) \to \mathbb{Z}$. The *Jacobian group* $\mathrm{Jac}(\Gamma)$ is the kernel of this degree map, so we have a short exact sequence

$$0 \to \mathrm{Jac}(\Gamma) \to \mathrm{Pic}(\Gamma) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

This short exact sequence splits, $\mathrm{Pic}(\Gamma) \cong \mathbb{Z} \times \mathrm{Jac}(\Gamma)$, but this isomorphism is not canonical. One way to obtain a splitting is to choose a point $q \in \Gamma$, and define $\mathbb{Z} \to \mathrm{Pic}(\Gamma)$ by sending n to the divisor class $[nq]$. We also denote $\mathrm{Jac}(\Gamma)$ by $\mathrm{Pic}^0(\Gamma)$, and use $\mathrm{Pic}^d(\Gamma)$ to denote the divisor classes of degree d.

The tropical Abel–Jacobi theorem, due to Mikhalkin and Zharkov [15, Theorem 6.2], identifies the structure of $\mathrm{Jac}(\Gamma)$ as a connected topological abelian group.

Theorem 3.4 (Tropical Abel–Jacobi). *Suppose Γ is a metric graph of genus g. Then*

$$\mathrm{Jac}(\Gamma) \cong \mathbb{R}^g / \mathbb{Z}^g.$$

Fix a basepoint q on the metric graph Γ. The *Abel–Jacobi map*

$$\iota_q : \Gamma \to \mathrm{Jac}(\Gamma) \tag{5}$$

sends a point $x \in \Gamma$ to the divisor class $[x - q]$. More generally, if D is a degree d divisor we have a higher-dimensional Abel–Jacobi map

$$\iota_D^{(d)} : \Gamma^d \to \mathrm{Jac}(\Gamma)$$

which sends a tuple $(x_1, \dots, x_d)$ to the divisor class $[x_1 + \cdots +_d - D]$,

Recall that $\mathrm{st}(\Gamma)$ denotes the stabilization of Γ (see Section 2.3).

Proposition 3.5. *(1) The retract $r : \Gamma \to \mathrm{st}(\Gamma)$ induces an isomorphism $\mathrm{Jac}(\Gamma) \to \mathrm{Jac}(\mathrm{st}(\Gamma))$ on Jacobians.*
(2) The inclusion $i : \mathrm{st}(\Gamma) \to \Gamma$ induces an isomorphism $\mathrm{Jac}(\mathrm{st}(\Gamma)) \to \mathrm{Jac}(\Gamma)$ on Jacobians.

For a proof and further motivation, see Caporaso [6].

3.3. **Cellular decomposition of the Jacobian.** In this section we recall how the geometry of the Abel–Jacobi map $\Gamma \to \mathrm{Jac}(\Gamma)$ is related to the cographic matroid $M^\perp(G)$, where (G, ℓ) is a model for Γ. We describe cellular decompositions of the subset of effective divisor classes inside $\mathrm{Pic}^k(\Gamma)$ for $k \geq 0$; each $\mathrm{Pic}^k(\Gamma)$ can be identified with $\mathrm{Jac}(\Gamma)$ by subtracting a fixed degree k divisor.

A consequence of Mikhalkin and Zharkov's proof [15] of the tropical Abel–Jacobi theorem (Theorem 3.4) is that the Abel–Jacobi map $\Gamma \to \mathrm{Jac}(\Gamma)$ is linear on each edge of Γ. The universal cover of $\mathrm{Jac}(\Gamma)$ is naturally identified with $H^1(\Gamma, \mathbb{R})$. The Abel–Jacobi map, restricted to a single edge $e \subset \Gamma$, lifts locally to $e \to H^1(\Gamma, \mathbb{R})$. The linear independence of the edge-vectors in the image $\Gamma \to \mathrm{Jac}(\Gamma)$ is exactly recorded by the cographic matroid $M^\perp(G)$, for any combinatorial model $\Gamma = (G, \ell)$.

Definition 3.6. Let $\Gamma = (G, \ell)$ be a metric graph. Given edges $e_1, \dots, e_k \in E(G)$, let $\mathrm{Div}(e_1, \dots, e_k) \subset \mathrm{Div}^k(\Gamma)$ denote the set of effective divisors formed by adding together one point from each edge e_i. Let $\mathrm{Eff}(e_1, \dots, e_k)$ denote the corresponding set of effective divisor classes,

$$\mathrm{Eff}(e_1, \dots, e_k) = \{[x_1 + \cdots + x_k] : x_i \in e_i\} \subset \mathrm{Pic}^k(\Gamma).$$

The following result relates these cells of effective divisor classes with the cograph matroid (see Section 2.4).

Theorem 3.7. *Let $\Gamma = (G, \ell)$ be a metric graph. The dimension of $\mathrm{Eff}(e_1, \dots, e_k)$ is equal to the rank of $\{e_1, \dots, e_k\}$ in the cographic matroid $M^\perp(G)$.*

Proof. For each edge $e_i \in E(G)$, let $v_i \in H^1(\Gamma, \mathbb{R})$ denote a vector parallel to the Abel–Jacobi image of e_i in $\mathrm{Jac}(\Gamma)$. Then according to Definition 5.1.3 of [7, p. 156], the set of vectors $\{v_i : e_i \in E(G)\}$ form a realization of the cographic matroid $M^\perp(G)$. This means that the cographic rank of $\{e_1, \dots, e_k\}$ agrees with the dimension of the linear span of $\{v_1, \dots, v_k\}$.

The subset $\mathrm{Eff}(e_1, \dots, e_k) \subset \mathrm{Pic}^k(\Gamma)$ is naturally identified with the Minkowski sum of the corresponding vectors $v_1, \dots, v_k \in H^1(\Gamma, \mathbb{R})$, so the claim follows. □

Corollary 3.8. *Let $\Gamma = (G, \ell)$ be a metric graph of genus g. For any integer d in the range $0 \leq d \leq g$, the space $\mathrm{Eff}^d(\Gamma)$ of degree d effective divisor classes has the structure of a cellular complex whose top-dimensional cells are indexed by independent sets of size d in the cographic matroid $M^\perp(G)$.*

3.4. **Kirchhoff formulas.** In this section we review Kirchhoff's formulas [12] for the unit potential functions j_z^y, which are fundamental solutions to the Laplacian map (see Definition 3.1). Expositions of this material are found in Bollobás [5, §II.1] and Grimmet [9, §1.2].

Theorem 3.9 (Kirchhoff). *Suppose $\Gamma = (G, \ell)$ is a metric graph with edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$. For vertices $y, z \in V(G)$, let $j_z^y : \Gamma \to \mathbb{R}$ denote the function in $\mathrm{PL}_{\mathbb{R}}(\Gamma)$ which satisfies $\Delta(j_z^y) = z - y$ and $j_z^y(z) = 0$. Then the following relations hold.*

(a) For any directed edge $\vec{e} = (e^+, e^-)$,

$$\frac{j_z^y(e^+) - j_z^y(e^-)}{\ell(e)} = \frac{\sum_{T \in \mathcal{T}(G)} sgn(T, y, z, \vec{e}) w(T)}{\sum_{T \in \mathcal{T}(G)} w(T)} \tag{6}$$

where $\mathcal{T}(G)$ denotes the spanning trees of G, the weight *$w(T)$ of a spanning tree is defined as*

$$w(T) = \prod_{e_i \notin E(T)} \ell(e_i),$$

and

$$sgn(T, y, z, \vec{e}) = \begin{cases} +1 & \text{if the path in } T \text{ from } y \text{ to } z \text{ passes through } \vec{e} \\ -1 & \text{if the path in } T \text{ from } y \text{ to } z \text{ passes through } -\vec{e} \\ 0 & \text{otherwise.} \end{cases}$$

(b) The total potential drop between y and z is

$$j_z^y(y) - j_z^y(z) = \frac{\sum_{T \in \mathcal{T}(G_0)} w(T)}{\sum_{T \in \mathcal{T}(G)} w(T)} \tag{7}$$

using the same notation as above, and where the graph G_0 (in the numerator) is the graph obtained from G by identifying vertices y and z.

Proof. For part (a), see Bollobás [5, Theorem 2, §II.1]. Part (b) follows from consideration of the graph G_+ obtained by adding an auxiliary edge to G between y and z, and then applying part (a) to G_+ with respect to the auxiliary edge. □

The expressions (6), (7) are both a ratio of homogeneous polynomials[1] in the variables $\{\ell(e_i) : e_i \in E(G)\}$. In (6), the numerator and denominator are homogeneous of degree g; in (7), the denominator has degree g while the numerator has degree $g + 1$. As a result, the expression (6) is invariant under simultaneous rescaling of edge lengths, while the expression (7) scales linearly with respect to simultaneously rescaling all edge lengths.

Example 3.10. Consider the theta graph shown in Figure 7, where $a = \ell(e_1)$, $b = \ell(e_2)$, $c = \ell(e_3)$ are edge lengths. The spanning trees are $\mathcal{T}(G) = \{e_3, e_2, e_1\}$ which have respective weights $\{ab, ac, bc\}$. The current along edge e_1 is

$$\frac{j_z^y(y) - j_z^y(z)}{a} = \frac{bc}{ab + ac + bc},$$

according to (6). We have

$$j_z^y(y) - j_z^y(z) = a\left(\frac{bc}{ab + ac + bc}\right) = \frac{abc}{ab + ac + bc}$$

in agreement with (7); G_0 consists of three loop edges. Note the symmetry in a, b, c.

[1] moreover, polynomials whose nonzero coefficients are all ± 1

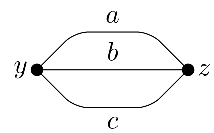

Figure 7. Theta graph with variable edge lengths.

Example 3.11. Let G be the Wheatstone graph in Figure 8 (left), with edge lengths $a = \ell(e_1), \ldots, f = \ell(e_5)$. The spanning trees are

$$\mathcal{T} = \{345, 245, 234, 145, 135, 125, 124, 123\},$$

where 123 shorthand for spanning tree $\{e_1, e_2, e_3\}$, and the corresponding weights are $\{ab, ac, af, bc, bd, cd, cf, df\}$. The current along edge e_3 is

$$\frac{j_z^y(e_{3,+}) - j_z^y(e_{3,-})}{c} = \frac{ab + af}{ab + ac + af + bc + bd + cd + cf + df},$$

while the current along e_1 is

$$\frac{j_z^y(y) - j_z^y(z)}{a} = \frac{bc + bd + cd + cf + df}{ab + ac + af + bc + bd + cd + cf + df}.$$

The potential drop from y to z is

$$j_z^y(y) - j_z^y(z) = \frac{abc + abd + acd + acf + adf}{ab + ac + af + bc + bd + cd + cf + df},$$

in agreement with (7); the quotient graph G_0 is shown to the right in Figure 8.

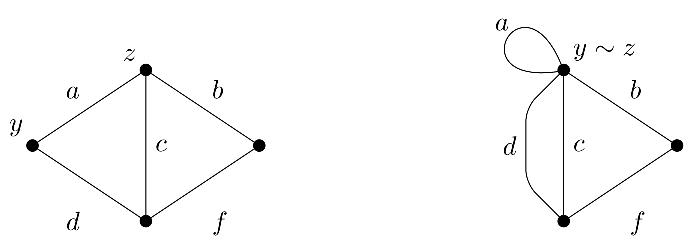

Figure 8. Wheatstone graph with variable edge lengths, and a quotient graph.

If we let $d = f = 0$, then we recover the formulas of Example 3.10.

4. Torsion points of the Jacobian

4.1. **Torsion equivalence.** Given an abelian group A, the *torsion subgroup* A_{tors} is the set of elements $a \in A$ such that $na = a + \cdots + a = 0$ for some positive integer n. For example, the torsion subgroup of $\mathbb{R}/\mathbb{Z}$ is $\mathbb{Q}/\mathbb{Z}$ and the torsion subgroup of $\mathbb{R}$ is $\{0\}$. Recall that $\mathrm{Jac}(\Gamma)$ is the abelian group of degree 0 divisor classes on Γ; we have

$$\mathrm{Jac}(\Gamma)_{\mathrm{tors}} = \{[D] : D \in \mathrm{Div}^0(\Gamma), \quad n[D] = 0 \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

We say points $x, y \in \Gamma$ are *torsion equivalent* if there exists a positive integer n such that $n[x - y] = 0$ in $\mathrm{Jac}(\Gamma)$. If two points x, y represent the same divisor class $[x] = [y]$, then x and y are torsion equivalent; hence this relation descends to a relation on $\mathrm{Eff}^1(\Gamma) = \{[x] : x \in \Gamma\}$. It will be convenient for us to consider this relation on $\mathrm{Eff}^1(\Gamma)$ rather than on Γ.

Lemma 4.1. *Torsion equivalence defines an equivalence relation on* $\mathrm{Eff}^1(\Gamma)$.

Proof. It is clear that torsion equivalence is reflexive and symmetric. Suppose n, m are positive integers such that $n[x-y] = 0$ and $m[y-z] = 0$ in $\mathrm{Jac}(\Gamma)$. Then $mn[x-z] = mn([x-y]+[y-z]) = 0$. This shows that torsion equivalence is transitive. $\square$

It is natural to extend this relation to divisor classes of higher degree: we say effective classes $D, E \in \mathrm{Eff}^d(\Gamma)$ are *torsion equivalent* if $n[D-E] = 0$ for some positive integer n. We call an equivalence class under this relation a *torsion packet*.

Definition 4.2. Given $[E] \in \mathrm{Eff}^d(\Gamma)$, the *torsion packet* $\{[E]\}_{\mathrm{tors}}$ is the set of divisor classes torsion equivalent to $[E]$, i.e.

$$\{[E]\}_{\mathrm{tors}} = \{[D] \in \mathrm{Eff}^d(\Gamma) \text{ such that } [D-E] \in \mathrm{Jac}(\Gamma)_{\mathrm{tors}}\}.$$

The terminology of torsion packets allows us to restate the Manin–Mumford condition in a basepoint-free manner.

Proposition 4.3.

(a) Given an effective divisor class $[D] \in \mathrm{Eff}^d(\Gamma)$*, there is a canonical bijection*

$$\{[D]\}_{\mathrm{tors}} \quad \leftrightarrow \quad \iota_D^{(d)}(\Gamma^d) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}$$

where $\iota_D^{(d)} : \Gamma^d \to \mathrm{Jac}(\Gamma)$ *is the Abel–Jacobi map* (5).

(b) A metric graph Γ *satisfies the degree* d *Manin–Mumford condition if and only if every torsion packet of degree* d *is finite.*

Proof. For part (a), we have

$$\begin{aligned}\{[D]\}_{\mathrm{tors}} &= \{[E] \in \mathrm{Eff}^d(\Gamma) : [E-D] \in \mathrm{Jac}(\Gamma)_{\mathrm{tors}}\} \\ &= \{[x_1 + \cdots + x_d] \in \mathrm{Eff}^d(\Gamma) : [x_1 + \cdots + x_d - D] \in \mathrm{Jac}(\Gamma)_{\mathrm{tors}}\} \\ &= \{[x_1 + \cdots + x_d] \in \mathrm{Eff}^d(\Gamma) : \iota_D^{(d)}(x_1, \ldots, x_d) \in \mathrm{Jac}(\Gamma)_{\mathrm{tors}}\} \\ &= \iota_D^{(d)}(\Gamma^d) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}.\end{aligned}$$

Part (b) follows directly from (a) and the definitions. $\square$

Recall that the potential function j_y^x is the unique piecewise $\mathbb{R}$-linear function satisfying

$$\Delta(j_y^x) = y - x \qquad \text{and} \qquad j_y^x(y) = 0.$$

Lemma 4.4. *Suppose* x, y *are two points on a metric graph* Γ*. Then* $[x-y]$ *is torsion in the Jacobian of* Γ *if and only if all slopes of the potential function* j_y^x *are rational.*

The above lemma is the special case $d = 1$ of the following statement.

Lemma 4.5. *Suppose* $D = x_1 + \cdots + x_d$ *and* $E = y_1 + \cdots + y_d$ *are effective divisors of degree* d *on a metric graph* Γ*. Let* $f \in \mathrm{PL}_{\mathbb{R}}(\Gamma)$ *be a function satisfying* $\Delta(f) = D - E$*. (Up to an additive constant,* $f = \sum_{i=1}^d j_{x_i}^{y_i}$*.)*

(a) The divisor class $[D-E] = 0$ *if and only if all slopes of* f *are integers.*

(b) The divisor class $[D-E]$ *is torsion if and only if all slopes of* f *are rational.*

Proof. For part (a), the "if" direction is a restatement of the definition of linear equivalence (Section 3.2). The "only if" direction follows from fact that for fixed divisors D and E, all solutions to $\Delta(f) = D - E$ (where $f \in \mathrm{PL}_{\mathbb{R}}(\Gamma)$) have the same slopes, since they all differ by an additive constant.

Part (b) follows from part (a) by linearity of the Laplacian Δ; more precisely, $[D-E]$ is torsion of order n iff $[n(D-E)] = [n\,\Delta(f)] = [\Delta(n \cdot f)] = 0$ iff all slopes of $n \cdot f$ lie in $\mathbb{Z}$ iff all slopes of f lie in $\frac{1}{n}\mathbb{Z}$. Conversely if all slopes of f are rational, then there exists an integer n such that all slopes of f lie in $\frac{1}{n}\mathbb{Z}$, since a piecewise linear function has finitely many slopes. $\square$

4.2. **Very general subsets.** A *very general* subset of $\mathbb{R}^n$ is one whose complement is contained in a countable union of distinguished Zariski-closed sets. A distinguished Zariski-closed set is the set of zeros of a polynomial function which is not identically zero. Given a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$, we denote

$$Z(f) = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : f(a) = 0\} \quad \text{and} \quad U(f) = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : f(a) \neq 0\}.$$

In this notation, a very general subset $S \subset \mathbb{R}^n$ is one which can be expressed as

$$S \supset \mathbb{R}^n \setminus \left(\bigcup_{i \in I} Z(f_i) \right) = \bigcap_{i \in I} U(f_i)$$

where I is a countable index set and each f_i is a nonzero polynomial. Note that the zero locus $Z(f)$ has Lebesgue measure zero if f is nonzero. Thus the complement of a (measurable) very general subset of $\mathbb{R}^n$ has Lebesgue measure zero. However, it is still possible that the complement of a very general subset is dense in $\mathbb{R}^n$.

If $D \subset \mathbb{R}^n$ is some parameter space with nonempty interior (with respect to the Euclidean topology), we say that a subset of D is *very general* if it has the form $D \cap S$ for a very general subset $S \subset \mathbb{R}^n$. In our applications, the relevant parameter space will be the positive orthant $D = (\mathbb{R}_{>0})^n$. We say that a property holds for a *very general* point of some real parameter space if it holds on a very dense subset.

Example 4.6.

(a) For a fixed nonconstant polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$, the set

$$U(f - \mathbb{Q}) = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : f(a_1, \ldots, a_n) \notin \mathbb{Q}\} \tag{8}$$

is very general, since $\{f - \lambda : \lambda \in \mathbb{Q}\}$ is a countable collection of nonzero polynomials.

(b) For polynomials $f, g \in \mathbb{Z}[x_1, \ldots, x_n]$ with $g \neq 0$ and f/g nonconstant, the set

$$U(\frac{f}{g} - \mathbb{Q}) = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : \frac{f(a_1, \ldots, a_n)}{g(a_1, \ldots, a_n)} \notin \mathbb{Q}\} \tag{9}$$

is very general, since $\{f - \lambda g : \lambda \in \mathbb{Q}\}$ is a countable collection of nonzero polynomials.

(c) The set

$$U^n_{\text{tr.}} = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : f(a_1, \ldots, a_n) \neq 0 \quad \text{for every } f \in \mathbb{Z}[x_1, \ldots, x_n] \setminus \{0\}\} \tag{10}$$

is very general, since $\mathbb{Z}[x_1, \ldots, x_n]$ is a countable set of polynomials. We call $U^n_{\text{tr.}}$ the set of *transcendental* points of $\mathbb{R}^n$. In particular, U^1_{tr} is the set of transcendental real numbers.

Note that in the above examples, the subsets (8) and (9) contain the transcendental points $U^n_{\text{tr.}}$. Conversely, $U^n_{\text{tr.}}$ is the intersection of (8) over all choices of f (resp. (9) over all choices of f and g).

In the later theorem statements (6.5 and 6.13) which concern very general edge lengths, the stated property holds when the edge lengths are transcendental (in the sense of (10), $n = \#E(G)$). More precisely, these conditions will hold on a finite intersection of sets of the form (9). The polynomials f, g will come from Kirchhoff's formulas (see Theorem 3.9 in Section 3.4).

5. Manin–Mumford conditions on metric graphs

Recall that a metric graph Γ satisfies the *Manin–Mumford condition* if the image of the Abel–Jacobi map $\iota_q : \Gamma \to \mathrm{Jac}(\Gamma)$ intersects only finitely many points of the torsion subgroup $\mathrm{Jac}(\Gamma)_{\text{tors}}$, for every choice of basepoint $q \in \Gamma$. Equivalently, Γ satisfies the Manin–Mumford condition if every degree one torsion packet $\{[x]\}_{\text{tors}}$ is finite.

A metric graph Γ satisfies the *degree d Manin–Mumford condition* if the image of the degree d Abel–Jacobi map

$$\begin{aligned} \iota_D^{(d)} : \Gamma^d &\to \mathrm{Jac}(\Gamma) \\ (p_1, \dots, p_d) &\mapsto [p_1 + \dots + p_d - D] \end{aligned}$$

intersects only finitely many points of $\mathrm{Jac}(\Gamma)_{\mathrm{tors}}$, for every choice of effective degree d divisor class $[D]$.

5.1. **Failure of Manin–Mumford condition.** In this section, we consider cases when a metric graph fails to satisfy the Manin–Mumford condition, in degree one and in higher degree.

Proposition 5.1. *If $\Gamma = (G, \ell)$ is a metric graph whose edge lengths are all rational, then the Manin–Mumford condition fails to hold.*

Proof. Rescaling all edge lengths of Γ by the same factor does not change the validity of the Manin–Mumford condition, so we may assume that all edge lengths are integers. This means Γ has a combinatorial model $(G, \mathbf{1})$ with unit edge lengths. On a graph with unit edge lengths, the degree-0 divisor classes supported on vertices form a finite abelian group, known as the *critical group* of the graph (see Section 2.1). This implies that all vertices of G lie in the same torsion packet.

Now consider taking the k-th subdivision graph $G^{(k)}$ of G, meaning every edge if G is subdivided into k edges of equal length; the number of vertices is

$$\#V(G^{(k)}) = \#V(G) + (k-1)\#E(G).$$

The same reasoning implies that these new vertices are also in the same torsion packet of Γ. Taking $k \to \infty$ shows that Γ has an infinite torsion packet. $\square$

Proposition 5.1 also follows from part (a) of the following lemma. Recall that given edges $e_i \in E(G)$, $\mathrm{Eff}(e_1, \dots, e_k)$ denotes the set of effective divisor classes $[x_1 + \dots + x_k]$ which sum a point $x_i \in e_i$ from each edge (x_i is allowed to be an endpoint of e_i).

Lemma 5.2. *Let $\Gamma = (G, \ell)$ be a metric graph.*

(a) If an edge $e \in E(G)$ contains two points x, y such that $[x], [y]$ are distinct but in the same torsion packet, then the torsion packet $\{[x]\}_{\mathrm{tors}}$ is infinite.
(b) If $\mathrm{Eff}(e_1, \dots, e_d)$ contains distinct divisor classes $[D], [E]$ in the same degree d torsion packet, then the torsion packet $\{[D]\}_{\mathrm{tors}}$ is infinite.

Proof. (a) Suppose that an edge e contains distinct points x, y such that $[x - y]$ is torsion. Let z denote the midpoint of x and y; we claim $[x-z]$ is also torsion. The midpoint satisfies $[2z] = [x+y]$, hence $2[x - z] = [x - z] + [z - y] = [x - y]$. If n is a positive integer such that $n[x - y] = 0$, then $2n[x - z] = n[x - y] = 0$. This proves the claim that $[x - z]$ is torsion. By repeating this argument on the midpoint of x and z, we obtain infinitely many points on e in the same torsion packet $\{[x]\}_{\mathrm{tors}}$.

(b) Since the cell $\mathrm{Eff}(e_1, \dots, e_d)$ is convex, it contains a line segment connecting $[D]$ and $[E]$; this segment have positive length by the assumption $[D] \neq [E]$. Moreover, for

$$[F] = (\text{any rational affine combination of } [D] \text{ and } [E] \text{ along this line}),$$

the class $[D-F]$ is torsion. This guarantees infinitely many divisor classes $[F]$ in the torsion packet $\{[D]\}_{\mathrm{tors}}$, as claimed. $\square$

Proposition 5.3. *Suppose G has a cycle with d edges. Then for any edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$, the metric graph $\Gamma = (G, \ell)$ fails to satisfy the degree d Manin–Mumford condition.*

Proof. Let C be a cycle in G with edges $e_1, e_2, \ldots, e_d$ and vertices $v_1, v_2, \ldots, v_d$ in cyclic order, where edge e_i has endpoints v_i and v_{i+1} (where indices are taken modulo d). Consider the effective divisors $D = v_1 + \cdots + v_d$ and $E = x_1 + \cdots + x_d$ where x_i is the midpoint on edge e_i.

To show that $[D - E]$ is torsion, we construct a piecewise linear function f with $\Delta(f) = D - E$. Let $f : \Gamma \to \mathbb{R}$ be zero-valued outside of the cycle C, and $f(v_i) = 0$ for each vertex (potentially required by continuity of f). On each edge e_i, let f have slope $\frac{1}{2}$ in the directions away from v_i, so that at the midpoint $f(x_i) = \frac{1}{2}\ell(e_i)$. It is straightforward to verify that $\Delta(f) = D - E$ as desired.

By Lemma 4.5, the slopes $\pm\frac{1}{2}$ of f imply that $[D - E]$ is a nonzero, torsion divisor class. Moreover, both $[D]$ and $[E]$ lie in the same cell $\mathrm{Eff}(e_1, \ldots, e_d)$. Then Lemma 5.2(b) implies that the torsion packet $\{[D]\}_{\mathrm{tors}}$ is infinite, which violates the degree d Manin–Mumford condition. □

5.2. **Uniform Manin–Mumford bounds.** In this section, we show that if a metric graph satisfies the Manin–Mumford condition, then in fact the number of torsion points can be bounded uniformly in terms of the genus of Γ.

Theorem 5.4. *Suppose Γ is a metric graph of genus $g \geq 2$. If $\iota_q(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}$ is finite, then*

$$\#(\iota_q(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}) \leq 3g - 3.$$

Proof. The retract map $r : \Gamma \to \Gamma'$ from a metric graph to its stabilization induces an isomorphism on Jacobians $\mathrm{Jac}(\Gamma) \xrightarrow{\sim} \mathrm{Jac}(\Gamma')$ (Proposition 3.5) and hence on $\iota_q(\Gamma) \xrightarrow{\sim} \iota_{r(q)}(\Gamma')$, so we may assume that Γ is semistable and that (G, ℓ) is a stable combinatorial model for Γ. Proposition 2.4 states that $\#E(G) \leq 3g - 3$ since G is stable. Lemma 5.2(a) implies that a finite torsion packet has at most one point on a given edge of G. This proves that the size of a finite, degree 1 torsion packet is at most $3g - 3$. By Proposition 4.3, we are done. □

We next generalize the above argument to the higher-degree case.

Theorem 5.5. *Let $\Gamma = (G, \ell)$ be a connected metric graph of genus $g \geq 2$. If Γ satisfies the Manin–Mumford condition in degree d, then*

$$\#(\iota_D^{(d)}(\Gamma^d) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}) \leq \binom{3g-3}{d}.$$

Proof. The number $\#(\iota_D^{(d)}(\Gamma^d) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}})$ does not change under replacing Γ with its stabilization, so we may assume Γ is semistable and (G, ℓ) is a stable model. This means that the number of edges $\#E(G)$ is bounded above by $3g - 3$.

The image of $\iota_D^{(d)}(\Gamma^d)$ is homeomorphic to $\mathrm{Eff}^d(\Gamma)$. (They differ by a translation sending $\mathrm{Pic}^d(\Gamma)$ to $\mathrm{Pic}^0(\Gamma)$.) The maximal cells in the ABKS decomposition of $\mathrm{Eff}^d(\Gamma)$ are indexed by independent sets of size d in the cographic matroid $M^\perp(G)$, c.f. Corollary 3.8. The number of maximal cells is clearly bounded above by $\binom{\#E(G)}{d}$, the number of all size-d subsets of edges. Since we assumed G is stable, we have $\binom{\#E(G)}{d} \leq \binom{3g-3}{d}$.

From Lemma 5.2(b), we know that a finite degree d torsion packet contains at most one element from a given maximal cell of $\iota_D^{(d)}(\Gamma^d)$, which finishes the proof. □

6. Manin–Mumford for generic edge lengths

6.1. **Degree one.** In this section we prove our first main theorem, which gives conditions on when a metric graph satisfies the Manin–Mumford condition in degree 1. In this section, "torsion packet" will always mean a degree 1 torsion packet (c.f. Definition 4.2). Before addressing the general case, we demonstrate an example in small genus.

Example 6.1. Let G be the theta graph (see Figure 7) with vertices x, y and edges e_1, e_2, e_3, and consider the metric graph $\Gamma = (G, \ell)$ with edge lengths $a = \ell(e_1), b = \ell(e_2), c = \ell(e_3)$.

If a torsion packet contains two points on e_1, then Proposition 6.2 implies that $[x-y]$ is torsion on the deleted subgraph $\Gamma_1 = \Gamma \backslash e_1$. By Lemma 4.4, this would imply the potential function which sends current from x to y on the subgraph Γ_1 has rational slopes. We can compute these slopes directly: Γ_1 is a parallel combination of edges with lengths b and c, so the slope along e_2 is $\frac{c}{b+c}$. (This calculation also follows from Theorem 3.9.) To summarize:

$$\text{(some torsion packet contains} \geq 2 \text{ points of } e_1) \qquad \Rightarrow \qquad \frac{c}{b+c} \in \mathbb{Q}.$$

The contrapositive statement is that

$$\frac{c}{b+c} \notin \mathbb{Q}. \qquad \Rightarrow \qquad \text{(every torsion packet contains at most one point of } e_1).$$

To satisfy the Manin–Mumford condition, it suffices that every torsion packet $\{[x]\}_{\mathrm{tors}} \subset \mathrm{Eff}^1(\Gamma)$ contains at most one point of each edge e_1, e_2, e_3. Thus the Manin–Mumford condition holds for Γ if the edge lengths are in set

$$\{(a,b,c) \in \mathbb{R}^3_{>0} : \frac{b}{a+b} \notin \mathbb{Q} \text{ and } \frac{c}{a+c} \notin \mathbb{Q} \text{ and } \frac{c}{b+c} \notin \mathbb{Q}\}.$$

This is very general subset of $\mathbb{R}^3_{>0}$, c.f. Example 4.6(b).

Proposition 6.2. *Suppose Γ is a metric graph and points $x, y \in \Gamma$ lie on the same edge. Let Γ_0 denote the metric graph with the open segment between x and y removed. If $[x-y]$ is torsion on Γ and Γ_0 is connected, then $[x-y]$ is torsion on Γ_0.*

Proof. Suppose $[x-y]$ is torsion on Γ. Let j_x^y denote the potential function on Γ when one unit of current is sent from y to x. By Lemma 4.4, all slopes of j_x^y are rational. In particular, the slope of j_x^y on the segment between x and y is rational; let s denote this slope. Since Γ_0 is connected, we have $s < 1$.

Let Γ_0 denote the metric graph obtained from Γ by deleting the interior of edge e. It is clear that the restriction of j_x^y to Γ_0 has Laplacian $\Delta(j_x^y\big|_{\Gamma_0}) = (1-s)x - (1-s)y$.

Let $j_{x,0}^y$ denote the potential function on Γ_0 when one unit of current is sent from y to x. Since $j_{x,0}^y = (1-s)^{-1} j_x^y$, all slopes of $j_{x,0}^y$ are rational. By Lemma 4.4, this implies $[x-y]$ is torsion on Γ_0 as desired. □

Proposition 6.3. *Suppose x, y are two vertices on a graph G. Let j_x^y be the potential function on $\Gamma = (G, \ell)$, depending on variable edge lengths $\ell : E(G) \to \mathbb{R}$. Either:*

(1) all slopes of j_x^y are 1 or 0, independent of edge lengths; or

(2) there exists some edge e such that the slope of j_x^y along e is a non-constant rational function of the edge lengths.

Proof. Suppose there is a unique simple path in G from x to y. Let f be the piecewise linear function on Γ which has $f(x) = 0$, increases with slope 1 along the path from x to y, and has slope 0 elsewhere. Then f satisfies $\Delta(f) = x - y$ so we must have $potentialxy = f$ by uniqueness. Thus we are in case (1).

On the other hand, suppose there are two distinct simple paths π_1, π_2 in G from x to y. Let e be an edge of G which lies on π_1 but not π_2. If we fix the lengths of edges in π_1 and send all other edge lengths to infinity, then the slope of j_x^y along e approaches 1. If we send the length $\ell(e)$ to infinity while keeping all other edge lengths fixed, then the slope of j_x^y along e approaches zero. Thus the slope of j_x^y along e is a non-constant function of the edge lengths. By Kirchhoff's formulas, Theorem 3.9, the slope (i.e. current) is a rational polynomial function of the edge lengths. This is case (2). □

Proposition 6.4. *Suppose x, y are two vertices on a graph G. Then for the metric graph $\Gamma = (G, \ell)$, either*

(1) $[x-y] = 0$ in $\mathrm{Jac}(\Gamma)$ for any edge lengths ℓ, or

(2) $[x-y]$ *is non-torsion in* $\mathrm{Jac}(\Gamma)$ *for very general edge lengths* ℓ.

Proof. If none of the slopes of j_x^y vary as a function of edge lengths, then by Proposition 6.3 all slopes of j_x^y are zero or one. This implies that $[x-y]=0$.

On the other hand, suppose for some edge e the slope of j_x^y along e is a non-constant rational function $\frac{p(\ell_1,\ldots,\ell_m)}{q(\ell_1,\ldots,\ell_m)}$. Then the subset

$$U = \left\{ (\ell_1,\ldots,\ell_m) \in \mathbb{R}^m_{>0} : \frac{p(\ell_1,\ldots,\ell_m)}{q(\ell_1,\ldots,\ell_m)} \not\in \mathbb{Q} \right\}$$

parametrizing edge-lengths where the slope at e take irrational values is very general, c.f. Example 4.6(b). By Lemma 4.4, $[x-y]$ is nontorsion on U, as desired. □

Theorem 6.5. *Suppose* G *is a biconnected metric graph of genus* $g \geq 2$. *For a very general choice of edge lengths* $\ell : E(G) \to \mathbb{R}_{>0}$, *the metric graph* $\Gamma = (G,\ell)$ *satisfies the Manin–Mumford condition.*

Proof. Let $m = \#E(G)$ and choose an ordering $E(G) = \{e_1, e_2, \ldots, e_m\}$, which induces a homeomorphism from the space of edge-lengths $\{\ell : E(G) \to \mathbb{R}_{>0}\}$ to the positive orthant $\mathbb{R}^m_{>0}$. We claim that for each edge e_i, there is a corresponding very general subset $U_i \subset \mathbb{R}^m_{>0}$ such that

$$\text{when edge lengths are chosen in } U_i\text{, every torsion packet of } \Gamma = (G,\ell) \text{ contains at most one point of } e_i. \tag{11}$$

Let e_i^+, e_i^- denote the endpoints of e_i, and let $G_i = G \backslash e_i$ denote the graph with edge e_i deleted. If the endpoints e_i^+, e_i^- are not connected by any path in G_i, this contradicts our assumption that G is biconnected. If the endpoints are connected by only one path π in G_i, then the union $\pi \cup \{e_i\}$ is a genus 1 biconnected component of G, which contradicts our assumption that G is biconnected and has genus $g \geq 2$. Thus e_i^+, e_i^- are connected by at least two distinct paths[2] in G_i.

Therefore, the divisor class $[e_i^+ - e_i^-] \neq 0$ in $\mathrm{Jac}(\Gamma_i)$ where $\Gamma_i = (G_i, \ell_i)$. By Proposition 6.4, $[e_i^+ - e_i^-]$ is nontorsion in $\mathrm{Jac}(\Gamma_i)$ on a very general subset $V_i \subset \mathbb{R}^{m-1}_{>0}$ of edge-length space. (Note that G_i has $m-1$ edges.) Finally, we let U_i be the preimage of V_i under the coordinate projection $\mathbb{R}^m_{>0} \to \mathbb{R}^{m-1}_{>0}$ forgetting coordinate i. The subset U_i is very general, and satisfies the claimed condition (11).

For any edge lengths in the intersection $U = \bigcap_{i=1}^m U_i$ a torsion packet of the corresponding $\Gamma = (G,\ell)$ can have at most one point on each edge e_i, giving the bound $\#\{[x]\}_{\mathrm{tors}} \leq m$. The subset U is very general, since it is a finite intersection of very general subsets. This completes the proof. □

6.2. **Higher degree.** In this section we address when a metric graph with very general edge lengths satisfies the Manin–Mumford condition in higher degree ($d \geq 2$).

The next proposition is a strengthening of Proposition 5.3. Recall that $M^\perp(G)$ denotes the cographic matroid of G (see Section 2.4).

Proposition 6.6. *Suppose* G *contains a cycle* C *whose edge set has rank* $d = \mathrm{rk}^\perp(E(C))$ *in the cographic matroid* $M^\perp(G)$. *Then for any edge lengths* $\ell : E(G) \to \mathbb{R}_{>0}$, *the metric graph* $\Gamma = (G,\ell)$ *fails the degree* d *Manin–Mumford condition.*

Proof. Suppose the given cycle of G consists of the edges $\{e_1,\ldots,e_k\}$ and vertices $\{v_1,\ldots,v_k\}$ in cyclic order; note that $k \geq d$. Let $D = v_1 + \cdots + v_k$ be the sum of the cycle's vertices. In the proof of Proposition 5.3, we showed that the degree-k torsion packet $\{[D]\}_{\mathrm{tors}}$ has infinite intersection with the cell $\mathrm{Eff}(e_1,\ldots,e_k)$, for any choice of edge lengths ℓ.

[2]The two paths may share edges in common.

Recall that $\mathrm{Eff}(e_1,\ldots,e_k)$ is the image of $\mathrm{Div}(e_1,\ldots,e_k)$ under the linear equivalence map $\mathrm{Div}^k(\Gamma)\to\mathrm{Pic}^k(\Gamma)$. The map $\mathrm{Div}(e_1,\ldots,e_k)\to\mathrm{Pic}^k(\Gamma)$ lifts to a linear map ϕ in the diagram

$$\begin{tikzcd} \prod_{i=1}^k [0,\ell(e_i)] \arrow[r, "\phi"] \arrow[d] & \mathbb{R}^g \arrow[d] \\ \mathrm{Div}(e_1,\ldots,e_k) \arrow[r] & \mathrm{Pic}^k(\Gamma), \end{tikzcd}$$

where $\prod_{i=1}^k[0,\ell(e_i)]\to\mathrm{Div}(e_1,\ldots,e_k)$ is the product of isometries $[0,\ell(e_i)]\to e_i$ and $\mathbb{R}^g\to\mathrm{Pic}^k(\Gamma)$ is an isometric universal cover. By Theorem 3.7, $\mathrm{Eff}(e_1,\ldots,e_k)$ has dimension $d=\mathrm{rk}^\perp(\{e_1,\ldots,e_k\})$ (where $d\le k$). This implies that ϕ has rank d, so the image of ϕ is covered by the restrictions of ϕ to the d-faces of $\prod_{i=1}^k[0,\ell(e_i)]$.

Thus $\mathrm{Eff}(e_1,\ldots,e_k)$ is covered by the corresponding images of the d-faces of $\mathrm{Div}(e_1,\ldots,e_k)$, which have the form

$$\mathrm{Eff}(e_i : i\in I)+[\sum_{i\notin I} v_i^\pm] \quad\subset\quad \mathrm{Eff}(e_1,\ldots,e_k), \tag{12}$$

where I is a size-d subset of $\{1,\ldots,k\}$ and $v_i^\pm\in\{v_i,v_{i+1}\}$ is an endpoint of e_i. (There are $\binom{k}{d}2^{k-d}$ such choices.)

Since $\mathrm{Eff}(e_1,\ldots,e_k)$ has infinite intersection with the torsion packet $\{[D]\}_{\mathrm{tors}}$, there is some choice of $I, v_i^\pm$ such that the subset (12) of $\mathrm{Eff}(e_1,\ldots,e_k)$ has infinite intersection with $\{[D]\}_{\mathrm{tors}}$. This implies that the degree-d torsion packet $\{[D-\sum_{i\notin I}v_i^\pm]\}_{\mathrm{tors}}$ has infinite intersection with $\mathrm{Eff}(e_i : i\in I)$, thus violating the degree d Manin–Mumford condition. $\square$

Next, we consider the converse situation of Proposition 6.6, i.e. when an edge set is acyclic after taking the closure in $M^\perp(G)$. Recall from Section 2.4 the notation $\mathrm{Div}(e_1,\ldots,e_k)$ and $\mathrm{Eff}(e_1,\ldots,e_k)$. Here we introduce a slight variation: let $\mathrm{Div}(e_1,\ldots,e_k)^\circ$ denote the set of effective divisors of the form $D=x_1+\cdots+x_k$ where x_i is in the *interior* e_i° of edge e_i; respectively let $\mathrm{Eff}(e_1,\ldots,e_k)^\circ$ denote the divisor classes of the form $[x_1+\cdots+x_k]$, where $x_i\in e_i^\circ$.

Proposition 6.7. *Suppose $e_1,\ldots,e_k$ are edges in G such that $\{e_1,\ldots,e_k\}$ is independent in $M^\perp(G)$ and the closure of $\{e_1,\ldots,e_k\}$ in $M^\perp(G)$ spans an acyclic subgraph of G. Then for very general edge lengths on $\Gamma=(G,\ell)$, distinct divisor classes in $\mathrm{Eff}(e_1,\ldots,e_k)^\circ\subset\mathrm{Pic}^k(\Gamma)$ are in distinct torsion packets.*

Before proving this statement, we introduce some lemmas and definitions.

Definition 6.8. Given a piecewise linear function f on $\Gamma=(G,\ell)$, say an edge of G is *current-active* with respect to f if the slope f' is nonzero in a neighborhood of its endpoints[3]; let $E^{\text{c.a.}}(G,f)$ denote the current-active edges,

$$E^{\text{c.a.}}(G,f)=\{e\in E(G) : f'\neq 0 \text{ in a neighborhood of } e^+,e^- \text{ in } e\}.$$

Say an edge is *voltage-active* with respect to f if the net change in f across e is nonzero; let $E^{\text{v.a.}}(G,f)$ denote the voltage-active edges,

$$E^{\text{v.a.}}(G,f)=\{e\in E(G) : f(e^+)-f(e^-)\neq 0 \quad \text{where } e=(e^+,e^-)\}.$$

Recall that a *cut* of G is a set of edges $\{e_1,\ldots,e_k\}$ such that the deletion $G\setminus\{e_1,\ldots,e_k\}$ is disconnected.

Lemma 6.9. *Consider a metric graph $\Gamma=(G,\ell)$ and $f\in\mathrm{PL}_{\mathbb{R}}(\Gamma)$. If $E^{\text{v.a.}}(G,f)$ is nonempty, it contains a cut of G.*

[3] if $e\cong[0,1]$, here a "neighborhood of the endpoints" means $[0,\epsilon)\cup(1-\epsilon,1]$ for some $\epsilon>0$

Proof. Suppose $e = (e^+, e^-)$ is voltage-active with respect to f, so that $f(e^+) > f(e^-)$ for some ordering of endpoints. Then we may partition $V(G)$ into two nonempty sets $V^+ \cup V^-$, where

$$V^+ = \{v \in V(G) : f(v) \geq f(e^+)\} \qquad \text{and} \qquad V^- = \{v \in V(G) : f(v) < f(e^+)\}.$$

It is clear that $E^{\text{v.a.}}(G, f)$ contains all edges between V^+ and V^-; such edges form a cut of G. □

Lemma 6.10. *On $\Gamma = (G, \ell)$, consider $f \in \mathrm{PL}_{\mathbb{R}}(\Gamma)$ such that $\Delta(f) = E - D$ for $D, E \in \mathrm{Div}(e_1, \ldots, e_k)^\circ$. If $E^{\text{c.a.}}(G, f)$ is nonempty, then it contains a cycle of G.*

Proof. Suppose $D = x_1 + \cdots + x_k$ and $E = y_1 + \cdots + y_k$ where $x_i, y_i \in e_i$. Since the divisor $\Delta(f)$ restricted to e_i has the form $y_i - x_i$, the slopes of f along e_i are as shown in Figure 9, where slopes are indicated in the rightward direction.

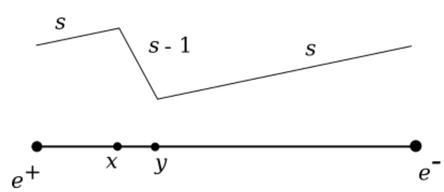

FIGURE 9. Slopes on edge e where $\Delta(f) = y - x$.

Edge e_i is current-active iff the corresponding slope s $(= s_i)$ is nonzero. In particular, if $e_i \in E^{\text{c.a}}(G, f)$ it is current-active *at both endpoints*.

On the other hand, consider an edge $e \in E(G) \setminus \{e_1, \ldots, e_k\}$. Then $\Delta(f)$ is not supported on e, so f does not change slope on e. Again in this case, if $e \in E^{\text{c.a.}}(G, f)$ then it is current-active at both endpoints.

By assumption that divisors $D, E \in \mathrm{Div}(e_1, \ldots, e_k)^\circ$, $\Delta(f)$ is supported away from the vertex set $V(G)$. This means that around a vertex v, the outward slopes of f sum to zero. The number of nonzero terms in the sum must be 0 or ≥ 2, and each nonzero term corresponds to a current-active edge incident to v. Thus

$$\begin{gathered} E^{\text{c.a.}}(G, f) \text{ spans a subgraph of } G \text{ where} \\ \text{every vertex has } \operatorname{val}(v) = 0 \text{ or } \operatorname{val}(v) \geq 2. \end{gathered}$$

The claim follows. □

Lemma 6.11. *Consider $D, E \in \mathrm{Div}(e_1, \ldots, e_k)^\circ$ and $f \in \mathrm{PL}_{\mathbb{R}}(\Gamma)$ such that $\Delta(f) = E - D$. If $D \neq E$, then $E^{\text{v.a.}}(G, f)$ or $E^{\text{c.a.}}(G, f)$ is nonempty (or both are).*

Proof. If $D = x_1 + \cdots + x_k$ is not equal to $E = y_1 + \cdots + y_k$, then there is some index i such that $x_i \neq y_i$. Consider the illustration of f in Figure 9, applied to the edge with $x_i \neq y_i$. We have

$$f(e_i^-) - f(e_i^+) = s \cdot \ell(e_i) - \ell([x_i, y_i]), \tag{13}$$

where $\ell([x_i, y_i])$ is the distance between x_i and y_i on e_i. If $s = 0$, then e_i is not current-active but is voltage-active. If $s = \ell([x_i, y_i])/\ell(e_i)$, then e_i is not voltage-active but is current-active. □

Lemma 6.12. *Consider a fixed vertex-supported $\mathbb{R}$-divisor $D = \lambda_1 v_1 + \cdots + \lambda_r v_r$ of degree zero on G, so $v_i \in V(G)$, $\lambda_i \in \mathbb{R}$ and $\sum_{i=1}^r \lambda_i = 0$. On $\Gamma = (G, \ell)$, suppose $f \in \mathrm{PL}_{\mathbb{R}}(\Gamma)$ satisfies $\Delta(f) = D$ and f has nonzero slope on $e \in E(G)$. If e is not a bridge, then the slope on e is a nonconstant rational function of edge lengths of Γ.*

Proof. Suppose we let $\ell(e) \to \infty$ for the chosen edge and fix the lengths of all other edges $e' \neq e$; we claim that the slope of f across e approaches zero.

The slope-current principle, Proposition 3.2, states that the slope of f is bounded above in magnitude by Λ, where $\Lambda = \frac{1}{2}\sum_i |\lambda_i|$ does not depend on the edge lengths.[4] Since $e = (e^+, e^-)$ is not a bridge edge, there is a simple path π from e^+ to e^- which does not contain e. By integration along π, $|f(e^-) - f(e^+)|$ is bounded above by $\Lambda \cdot \ell(\pi)$, which implies the bound

$$|f'(e)| = \left| \frac{f(e^-) - f(e^+)}{\ell(e)} \right| \leq \frac{\Lambda \cdot \ell(\pi)}{\ell(e)}.$$

If we let $\ell(e) \to \infty$ and keep $\ell(e')$ constant for each $e' \in E(G) \setminus \{e\}$, this upper bound approaches zero as claimed.

Thus the slope of f along e is a non-constant function of the edge lengths. It is a rational function by Kirchhoff's formulas, Theorem 3.9. □

Proof of Proposition 6.7. Suppose $D = x_1 + \cdots + x_k$ and $E = y_1 + \cdots + y_k$ are divisors in $\mathrm{Div}(e_1, \ldots, e_k)^\circ$. Let f be a piecewise linear function such that $\Delta(f) = E - D$. By Lemma 4.5, $[D]$ and $[E]$ lie in the same torsion packet if and only if all slopes of f are rational.

Let Γ_0 (resp. G_0) denote the metric graph (resp. combinatorial graph) obtained from deleting the interiors of edges $e_1, \ldots, e_k$ from Γ (resp. G). Let $f_0 = f\big|_{\Gamma_0}$ denote the restriction of f to Γ_0. We have

$$\Delta(f_0) = \lambda_1 w_1 + \cdots + \lambda_r w_r, \tag{14}$$

where $\{w_1, \ldots, w_r\} \subset V(G)$ is the set of endpoints of edges $e_1, \ldots, e_k$ and $\lambda_i \in \mathbb{R}$.

First, suppose the tuple $(\lambda_1, \ldots, \lambda_r) = (0, \ldots, 0)$. Then f_0 is constant, so every edge of G_0 is neither current-active nor voltage-active with respect to f. Since the edges $\{e_1, \ldots, e_k\}$ are assumed independent in $M^\perp(G)$, they do not contain a cut of G so the inclusion $E^{\text{v.a.}}(G, f) \subset \{e_1, \ldots, e_k\}$ implies that $E^{\text{v.a.}}(G, f) = \varnothing$ by Lemma 6.9. Since the edges $\{e_1, \ldots, e_k\}$ do not contain a cycle of G, the inclusion $E^{\text{c.a.}}(G, f) \subset \{e_1, \ldots, e_k\}$ implies that $E^{\text{c.a.}}(G, f) = \varnothing$ by Lemma 6.10. Then Lemma 6.11 implies that $D = E$.

Next, suppose the tuple $(\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r$ from (14) is nonzero. This means that some edge of G_0 must be current-active, so $E^{\text{c.a.}}(G, f)$ is nonempty. By Lemma 6.10, $E^{\text{c.a.}}(G, f)$ contains a cycle of G. The closure of $\{e_1, \ldots, e_k\}$ with respect to the cographic matroid $M^\perp(G)$ is equal to

$$\{e_1, \ldots, e_k\} \cup \{b_1, \ldots, b_j\} \quad \text{where } \{b_1, \ldots, b_j\} \text{ are the bridge edges of } G_0.$$

By assumption that $\{e_1, \ldots, e_k\} \cup \{b_1, \ldots, b_j\}$ is acyclic, $E^{\text{c.a.}}(G, f)$ must contain an edge $e_* \notin \{e_1, \ldots, e_k\}$ which is not a bridge in G_0. (The edge e_* depends on the tuple $(\lambda_1, \ldots, \lambda_r)$.)

Now consider applying Lemma 6.12 to the graph G_0, the divisor (14), and the edge $e_* \in E(G_0)$. The lemma concludes that as a function of the edge-lengths of Γ_0, the slope of f_0 (equivalently f) on e_* is a nonconstant ratio of polynomials. In particular,

$$V(\lambda_1, \ldots, \lambda_r) = \{\text{edge lengths of } \Gamma_0 \text{ such that } f_0' \text{ is irrational on } e_*\} \tag{15}$$

is a very general subset of $\mathbb{R}^{m-k}_{>0} \cong \{\ell_0 : E(G_0) \to \mathbb{R}_{>0}\}$, and on this subset we have $[D]$ and $[E]$ are in distinct torsion packets.

Finally, let $U(\lambda_1, \ldots, \lambda_r)$ be the preimage of $V(\lambda_1, \ldots, \lambda_r)$ under the projection $\mathbb{R}^m_{>0} \to \mathbb{R}^{m-k}_{>0}$, which is very general, and let

$$U = \bigcap_{\substack{(\lambda_1, \ldots, \lambda_r) \\ \in \mathbb{Q}^r \setminus (0, \ldots, 0)}} U(\lambda_1, \ldots, \lambda_r) \quad \subset \quad \mathbb{R}^m_{>0}.$$

The subset U is very general, as a countable intersection of very general subsets.

If edge lengths of $\Gamma = (G, \ell)$ are chosen such that there are distinct divisors $D, E \in \mathrm{Eff}(e_1, \ldots, e_k)^\circ$ where $[D]$ and $[E]$ are in the same torsion packet, then the tuple $(\lambda_1, \ldots, \lambda_r)$ as in (14) must be

[4] Since $\sum \lambda_i = 0$, we have $\Lambda = \sum\{\lambda_i : \lambda_i > 0\} = -\sum\{\lambda_i : \lambda_i < 0\}$.

rational and nonzero. Then the chosen edge lengths on $G_0 \subset G$ are excluded from the subset (15), hence the edge lengths are excluded also from U, as desired. $\square$

Theorem 6.13. *Let G be a connected graph of genus $g \geq 1$ and independent girth γ^{ind}. The metric graph $\Gamma = (G, \ell)$ satisfies the degree d Manin–Mumford condition for very general edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$ if and only if $1 \leq d < \gamma^{\text{ind}}$.*

Proof. If $d \geq \gamma^{\text{ind}}$, then $d \geq \text{rk}^{\perp}(E(C))$ for some cycle C of G. Proposition 6.6 states that Γ fails the Manin–Mumford condition in degree $d' = \text{rk}^{\perp}(E(C))$, so the condition also fails in degree $d \geq d'$.

Conversely if $d < \gamma^{\text{ind}}$, then for each d-subset of edges $\{e_1, \ldots, e_d\}$, its closure in $M^{\perp}(G)$ does not contain a cycle of G. In particular, the edges for each maximal cell $\text{Eff}(e_1, \ldots, e_d)$ of $\text{Eff}^d(\Gamma)$ satisfy the hypotheses of Proposition 6.7, so there is a very general subset of edge lengths of Γ for which every degree d torsion packet has at most one element in the chosen cell $\text{Eff}(e_1, \ldots, e_d)$. Since there are finitely many maximal cells (cf. Corollary 3.8), this implies that for very general edge lengths there are finitely many elements in each degree d torsion packet. $\square$

Corollary 6.14. *Let Γ be a metric graph of genus $g \geq 1$, and suppose Γ satisfies the Manin–Mumford condition in degree d. Then*

$$d < C \log g$$

for some constant C.

Proof. This follows from Proposition 6.6, which implies that $d < \gamma^{\text{ind}}$, and the bound $\gamma^{\text{ind}} < C \log g$ from Corollary 2.14. $\square$

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