SURGERY IN DIMENSION $1+1$ AND HURWITZ NUMBERS

YURII BURMAN AND RAPHAËL FESLER

In memoriam S.M.Natanzon.

Abstract. Surgery in dimension $1+1$ describes how to obtain a surface with boundary (compact, not necessarily oriented) from a collection of disks by joining them with narrow ribbons attached to the boundary. Counting the ways to do it gives rise to a "twisted" version of the classical Hurwitz numbers and of the cut-and-join equation.

Introduction.

Introduction. A classical surgery in dimension 2 is a way to represent a compact surface as a connected sum of spheres, that is, to obtain it from a collection of spheres by gluing cylinders to them. In this paper we transfer this technique to surfaces with boundary, that are obtained from a collection of disks by gluing rectangles ("ribbons") to their boundary. Like with the connected sum, one is to choose the orientation of the boundary near both ends of the ribbon to be glued.

Diagonals of ribbons form a graph embedded into the surface (a.k.a. fat graph, ribbon graph, combinatorial map, etc.), with all its vertices on the boundary. The edges adjacent to a given vertex are thus linearly ordered left to right; this ordering defines the embedding up to homotopy (in the classical theory, if a graph is embedded into an oriented surface without boundary then its edges adjacent to a given vertex are only cyclically ordered).

The paper contains three sections, to be called "geometric", "algebraic" and "combinatorial", respectively. In the first, "geometric" section we study surfaces with boundary (rigged with marked points) glued of ribbons and related graphs (with numbered vertices and edges) properly embedded into the surface. Such graphs appear to be 1-skeleta of the surface, and the surface can be retracted to them (Theorem 1.8); also, the graphs behave nicely under the orientation cover of the surface (Theorems 1.10 and 1.12).

Graph embeddings into oriented surfaces were studied earlier in a number of works (see [7] for the disk and [8] for arbitrary surfaces); they are in one-to-one correspondence with sequences of transpositions in the symmetric group. The cyclic structure of the product of the transpositions describes faces of the graph (i.e. closures of the connected components of its complement); the number of graphs with given faces is called a Hurwitz number and has been studied intensively during the last decades (the research involving dozens of authors and hundreds of works; its thourough review is far outside the scope of this paper). In the "algebraic"

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Section 2 we develop an analog of this correspondence for ribbon decomposition of surfaces. Instead of permutations (products of transpositions) the correspondence uses left cosets of the symmetric group of double size by a type B reflection group it contains. This allows to define a "twisted" analog of Hurwitz numbers. Their generating function is shown to satisfy a PDE of parabolic type (Theorems 2.12 and 2.14) called twisted cut-and-join equation — just like standard Hurwitz numbers, whose generating function satisfies the "classical" cut-and-join.

In the last, "combinatorial" section of the paper we study the twisted cut-and-join equation. First, we prove that the operator in its right-hand side has a basis of eigenfunctions called twisted Schur polynomials (Corollary 3.9). Then we include the twisted cut-and-join into a one-parameter family of operators and formulate a conjecture (Conjecture 3.18) describing their eigenvectors (called parametric Schur functions). We are planning to write later a separate paper on combiatorics of the parametric cut-and-join and parametric Schur functions.

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We dedicate this article to the memory of our colleague Sergey Natanzon who fell victim of the COVID-19 pandemic. The subject of our research, to which Sergey was always attentive, matches some of his favourite scientific topics — Hurwitz numbers and manifolds with boundary.

1. Surgery

1.1. General definitions.

Definition 1.1. *Decorated-boundary surface* (DBS) is a compact surface (2-manifold) M with boundary, together with a finite set of n numbered points $a_1, \dots, a_n \in \partial M$ and a local orientation o_i of ∂M in the vicinity of every point a_i ($i = 1, \dots, n$), such that every connected component of M has nonempty boundary and every connected component of ∂M contains at least one marked point.

The DBS M and M' with the same number n of marked points are called equivalent if there exists a homeomorphism $h : M \to M'$ such that $h(a_i) = a_i'$ and $h_*(o_i) = o_i'$ for all $i = 1, \dots, n$; here a_i, a_i' are marked points and o_i, o_i' are orientations; h_* is the action of h on the local orientations of the boundary. The set of equivalence classes of DBS with n marked points will be denoted DBS_n.

Pick marked points $a_i, a_j \in \partial M$, and let $\varepsilon_i, \varepsilon_j \in \{+, -\}$. Consider points $a_i', a_j' \in \partial M$ lying near a_i, a_j and such that the boundary segment $a_i a_i'$ is directed along the orientation o_i if $\varepsilon_i = +$ and opposite to it if $\varepsilon_i = -$; the same for j. Take then a rectangle, called henceforth a *ribbon*, with the vertices A_i, A_i', A_j, A_j' (listed counterclockwise) and glue bijectively its sides $A_i A_i'$ and $A_j A_j'$ to the segments $a_i a_i'$ and $a_j a_j'$, respectively (vertices to namesake endpoints). The result of gluing is homeomorphic to a surface M' with the boundary $\partial M' \ni a_1, \dots, a_n$. The boundary of M' near a_i and a_j contains a segment of the boundary of M (the "old" part) and a segment of a side of the ribbon (the "new" part); define local orientations o_i', o_j' of $\partial M'$ near these points so that the orientations of the "old" parts be the same as o_i and o_j prescribe. For $k \neq i, j$ the boundary of M' near a_k is just a segment of ∂M, so take $o_k' = o_k$ by definition. The surface M', points $a_1, \dots, a_n \in \partial M'$ and the orientations $o_1', \dots, o_n'$ form a DBS — thus, we defined a mapping $G[i,j]^{\varepsilon_i,\varepsilon_j}$:

$\mathrm{DBS}_n \to \mathrm{DBS}_n$ called *ribbon gluing*. The ribbon gluing $G[i,j]^{\varepsilon_i,\varepsilon_j}$ will be called twisted if $\varepsilon_i \neq \varepsilon_j$, and non-twisted otherwise.

If $M' = G[i,j]^{\varepsilon_i,\varepsilon_j}(M)$ then the boundary $\partial M'$ is obtained from ∂M by the standard surgery in dimension 1: segments $a_i a_i'$ and $a_j a_j'$ are replaced by $a_i a_j'$ and $a_j a_i'$. This suggests the name "$1+1$-dimensional surgery" for the whole set of operations $G[i,j]^{\varepsilon_i,\varepsilon_j}$. Unlike the classical 1-dimensional surgery, gluing the ribbon is not an involution: if $M'' = G[i,j]^{\varepsilon_i,\varepsilon_j}(M')$, then $\partial M'' = \partial M$ (more precisely, there exists a natural decoration-preserving homeomorphism between the boundaries, unique up to homotopy), but M'' is obtained from M by gluing two segments of its boundary to two bases of a cylinder. (In particular, if both M and M'' are oriented then their genera differ by 1.)

Let $M \in \mathrm{DBS}_n$, $\varepsilon \in \{+,-\}$, and let γ be a smooth simple (i.e. non-selfintersecting) curve on M joining a_i and a_j and transversal to ∂M in its endpoints. Local orientations o_i and o_j of ∂M thus define orientations of the normal bundle to γ (the bundle is trivial because γ is simple and not closed); we call γ *non-twisting* if the orientations are the same, and *twisting* otherwise.

Take now a point $a_j' \in \partial M$ near a_i such that the segment $a_i a_j' \subset \partial M$ is directed along the orientation o_i if $\varepsilon = +$ and opposite to it if $\varepsilon = -$. Then draw a smooth simple curve γ_j joining a_j with a_j' and homotopic to (and going near) the union of the curve γ and the segment $a_i a_j'$. Also draw a simple smooth curve γ_i joining a_i with a point $a_i' \in \partial M$ near a_j and "parallel" to γ_j — i.e. such that γ_j, the segment $a_j a_j'$, γ_i and the segment $a_i a_i'$ form the boundary of a rectangle $\Pi \subset M$. It is easy to see that the segment $a_j a_j'$ is directed along the orientation o_j if $\varepsilon = +$ and the curve γ is non-twisting or $\varepsilon = -$ and the curve is twisting; in the other cases the segment is directed opposite to o_j.

Define now an operation of "ribbon removal" $R[\gamma]^\varepsilon : \mathrm{DBS}_n \to \mathrm{DBS}_n$ as follows. The set $M' = M \setminus \mathrm{int}(\Pi)$ is homeomorphic to a surface with the boundary containing $a_1, \dots, a_n$. A local orientation o_i' of $\partial M'$ near a_i is defined by the same rule as for the ribbon gluing: o_i and o_i' coincide on the intersection $\partial M' \cap \partial M$ near a_i. The local orientation o_j' is defined similarly, and $o_k' \stackrel{\text{def}}{=} o_k$ for all $k \neq i,j$. The operation $R[\gamma]^\varepsilon$ is a sort of inverse to ribbon gluing, due to the following obvious statement:

Proposition 1.2. (1) *Let $i,j \in \{1,\dots,n\}$, $\varepsilon_i, \varepsilon_j \in \{+,-\}$ and γ be a diagonal of the ribbon joining a_i and a_j. Then $R[\gamma]^{\varepsilon_i} G[i,j]^{\varepsilon_i,\varepsilon_j} = \mathrm{id}_{\mathrm{DBS}_n}$.*

(2) *Let γ be a simple smooth curve on M joining a_i and a_j and transversal to the boundary, and $\varepsilon_i \in \{+,-\}$. Let $\varepsilon_j \in \{+,-\}$ be defined as $\varepsilon_j = \varepsilon_i$ if the curve γ is non-twisting and $\varepsilon_j = -\varepsilon_i$ otherwise. Then $G[i,j]^{\varepsilon_i,\varepsilon_j} R[\gamma]^{\varepsilon_i} = \mathrm{id}_{\mathrm{DBS}_n}$.*

Remark 1.3. Gluing a ribbon decreases the Euler characteristics of the surface by 1 and removal, increases it by 1.

1.2. **Ribbon decompositions.** It follows from Definition 1.1 that every connected component of a DBS contains a marked point. $M \in \mathrm{DBS}_n$ is called *stable* if every its connected component either contains at least two marked points or is a disk (with one marked point only).

Denote by $E_n \in \mathrm{DBS}_n$ a union of n disks with one marked point on the boundary of each.

Proposition 1.4. *$M \in \mathrm{DBS}_n$ is stable if and only if it can be obtained by gluing several ribbons to E_n. For a stable DBS one has $\chi(M) \le n$, and the number of ribbons is equal to $n - \chi(M)$.*

Proof. If a surface with a ribbon glued has a component with only one marked point then the gluing left this component intact. So, gluing a ribbon to a stable DBS keeps its stability, which proves the 'only if' part of the proposition (E_n is stable by definition).

To prove the 'if' part we will need a lemma:

Lemma 1.5. *Let $n \ge 2$. Then for any $M \in \mathrm{DBS}_n$ which is connected and stable but is not a disk there exists a simple smooth nonseparating curve γ joining two marked points.*

"Nonseparating" here means that the complement of γ is connected, too.

Proof of the lemma. M is connected and stable and not a disk, so it contains at least two marked points. If the boundary ∂M is not connected then take two marked points on different components of ∂M and join them with a simple smooth curve γ, which is always nonseparating.

Let now ∂M be connected. Then M is a connected sum of a disk with a nonzero number of either handles or Moebius bands. Let $S^1 \subset M$ be a circle separating the disk from a handle or from a Moebius band, and let $p, q \in S^1$ be its opposite points. There exists a curve δ inside the handle or the Moebius band joining p and q and not separating. Now pick a curve γ_1 joining p with one marked point and γ_2 joining q with another one. Then the union $\gamma \stackrel{\mathrm{def}}{=} \gamma_1 \cup \delta \cup \gamma_2$ is nonseparating as required. □

Corollary 1.6. *If $M \in \mathrm{DBS}_n$ is stable and $M \ne E_n$ then there exists a simple smooth curve γ on M joining two marked points and such that $M' \stackrel{def}{=} R[\gamma]^\varepsilon(M)$ is stable (regardless of ε).*

Proof of the corollary. A stable DBS different from E_n contains a component with two or more marked points. If this component is a disk then take for γ any curve joining these points. If it is not a disk then take for γ the nonseparating curve of Lemma 1.5. □

The proposition is now proved using induction on the Euler characteristic of M. Every component of M is a manifold with nonempty bounbdary, so the 2-nd Betti number of M is zero and $\chi(M) = h_0(M) - h_1(M) \le h_0(M) \le n$; the equality if possible only if $M = E_n$. Let now $\chi(M) = n - m$, $m > 0$. Use Corollary 1.6 to obtain a curve γ in M such that $M' = R[\gamma]^+(M)$ is stable; one has $\chi(M') = n - m + 1$, so by the induction hypothesis M' can be obtained from E_n by gluing $m - 1$ ribbons. By assertion 2 of Proposition 1.2 there exist i, j and ε such that $M = G[i,j]^{+,\varepsilon}(M')$ — so, M can be obtained by gluing m ribbons. □

Let now, again, $M \in \mathrm{DBS}_n$ be glued of m ribbons: $M = G[i_m, j_m]^{\varepsilon_m, \delta'_m} \dots G[i_1, j_1]^{\varepsilon_1, \delta_1} E_n$ (we will be calling such representation a *ribbon decomposition* of M). For every ribbon, draw a diagonal joining its vertices a_{i_k} and a_{j_k}, and assign the number k to it. The union of the diagonals is a graph $\Gamma \subset M$ with m numbered edges and the marked points $a_1, \dots, a_n$ as vertices; we call it a *diagonal graph* of the ribbon decomposition. The ribbon of a ribbon decomposition containing the edge number k will be denoted $r_k \subset M$.

Let a_i be a marked point of M, $\Gamma \subset M$ be a diagonal graph of a ribbon decomposition, and let $\ell_1, \dots, \ell_k$ be the numbers of the edges of Γ having a_i as an endpoint, listed left to right according to the orientation o_i. We are going to call the sequence $\mathcal{P}(a_i) = (\ell_1, \dots, \ell_k)$ the *passport* of a_i.

Theorem 1.7. *The diagonal graph Γ has the following properties:*

1. *(embedding) Γ is embedded: its edges do not intersect one another or the boundary of M except at endpoints.*
2. *(anti-unimodality) For every vertex a_i its passport $\mathcal{P}(a_i) = (\ell_1, \dots, \ell_k)$ is anti-unimodal: there exists $p \le k$ such that $\ell_1 > \dots > \ell_p < \dots < \ell_k$.*
3. *(twisting rule) In the notation of the above call the edges $\ell_1, \dots, \ell_p$ negative at the endpoint a_i, and edges $\ell_p, \dots, \ell_k$, positive (note that ℓ_p is both). Then any twisting edge of Γ is positive at one of its endpoints and negative at the other one, and any non-twisting edge is either positive at both endpoints or negative at them.*
4. *(retraction) The graph Γ is a homotopy retract of the surface M.*

Proof. Induction by the number m of ribbons. Apparently, everything is true for $m = 0$, that is, $M = E_n$. For any m, let $M = G[i_m, j_m]^{\varepsilon_m \delta_m} M'$, and $\Gamma' \subset M' \subset M$ be the union of all the edges of Γ except the edge number m. All the assertions of the theorem are true for M' and Γ' by the induction hypothesis.

The internal points of the edge m of Γ lie in the interior of the ribbon $r_m = M \setminus M'$ and thus do not belong to Γ' nor to the boundary of M. So, assertion 1 is true.

After gluing the ribbon r_m to M', the edge m is either the leftmost or the rightmost of all the edges ending at a_{i_m}. Thus, if $\mathcal{P}(a_{i_m}) = (\ell_1, \dots, \ell_k)$ then either $\ell_1 = m$ and $\ell_2, \dots, \ell_k$ is anti-unimodal by the induction hypothesis, or $\ell_k = m$ and $\ell_1, \dots, \ell_{k-1}$ is anti-unimodal. In both cases $\ell_1, \dots, \ell_k$ is anti-uninmodal, so assertion 2 is proved.

Assertion 3 is true for the edges of $\Gamma' \subset M'$ by the induction hypothesis. Apparently, this is preserved after the ribbon r_m is glued. The edge m is the diagonal of r_m; the "long" sides of r_m lie in ∂M, and therefore the edge m is adjacent to ∂M at both its endpoints, from the right for one of them and from the left for the other. This proves assertion 3 for the edge m, too.

Let $f : M' \to \Gamma'$ be the deformation retraction (it exists by the induction hypothesis); to prove assertion 4 it is necessary to extend f to the ribbon r_m. The edge m divides the ribbon into two triangles T_1 and T_2 attached to $\partial M'$ by the "short" sides $a_i a_i'$ and $a_j a_j'$, respectively. Put an auxiliary point c on the "long" side $a_j a_i'$ of T_1 and join it with a_i by a segment dividing T_1 into two triangles, $c a_i a_i'$ and $c a_i a_j$. The image $f([a_i a_i']) \subset \Gamma'$ is a segment of an edge attached to the vertex i; it is easy to extend f to the triangle $c a_i a_i'$ sending it to the same segment so that $f([a_i c]) = a_i$. Then extend f to $c a_i a_j$ as a projection onto the edge m parallel to the side $c a_i$. This is a retraction of T_1 to Γ; the construction for T_2 is the same. □

Now turn Theorem 1.7 into a definition: let $M \in \mathrm{DBS}_n$ and let $\Gamma \subset M$ be an embedded loopless graph with the vertices at the marked points of M and the edges numbered $1, \dots, m$. We call (M, Γ) *properly embedded* if it satisfies all the assertions of Theorem 1.7: embedding, anti-unimodality, twisting rule and retraction.

Connected components of the complement $M \setminus \partial M \setminus \Gamma$ will be called *faces*; connected components of $\partial M \setminus \{a_1, \dots, a_n\}$, *external edges*, and connected components of $\Gamma \setminus \{a_1, \dots, a_n\}$, *internal edges*.

Theorem 1.8. *Vertices, edges and faces of a properly embedded graph form a cell decomposition of M (as 0-cells, 1-cells and 2-cells, respectively), such that every face is adjacent to exactly one external edge. The total number of faces is n.*

Proof. Denote by k the number of faces of the graph. A compact manifold does not retract to its boundary; so the boundary of any face cannot be a subset of the graph and must contain an external edge. Therefore the number of faces does not exceed the number of external edges: $k \leq n$. At the same time, each external edge is adjacent to exactly one face, hence $n \leq k$. So, $k = n$ and each face has exactly one external edge on its boundary.

Cover M with the following open subsets (see Fig. 1):

- The faces $f_1, \dots, f_n$. Each f_i is a connected 2-manifold (later proved to be a disk).
- Narrow strips $e_1, \dots, e_m$, each one being an open disk covering the interior of an edge except small neighbourhoods of its endpoints.
- Narrow strips $b_1, \dots, b_n$ covering interiors of the boundary segments in the same way. Each b_i is homeomorphic to a half-disk containing its diameter but not the outer boundary.
- Small neighbourhoods $v_1, \dots, v_n$ of the vertices. Each v_i is a half-disk, like b_i.

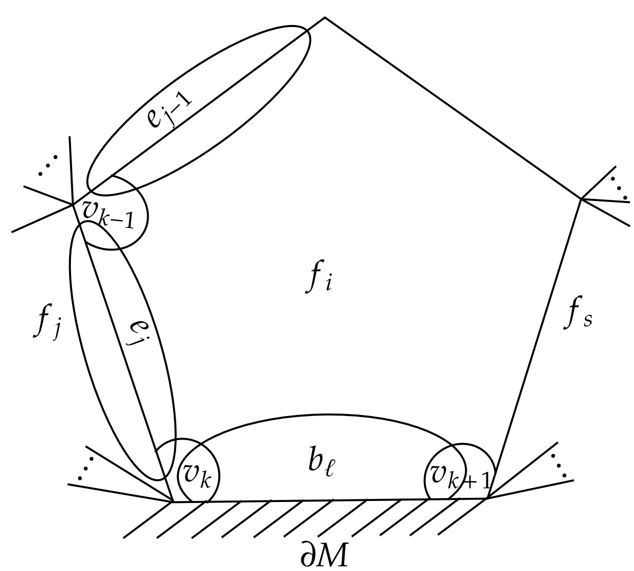

FIGURE 1. Cover of M

The intersections of the sets above are as follows:

- $f_i \cap e_j$ is an open disk if the edge is adjacent to the face, and is empty otherwise. Hence, $(\bigcup_i f_i) \cap (\bigcup_j e_j)$ is a union of $2m$ disjoint disks.
- $f_i \cap b_j$ is similar; $(\bigcup_i f_i) \cap (\bigcup_j b_j)$ is a union of n open disks.
- $f_j \cap v_j$ is an open disk if the vertex is the corner of the face, and is empty otherwise. If δ_j is the valency of (the number of edges at) the j-th vertex, then there are $\delta_j + 1$ corners adjacent to it. So the number of nonempty intersection is $\sum_j (\delta_j + 1) = n + \sum_j \delta_j$. One has $\sum_j \delta_j = 2m$ because every edge has 2 endpoints, so the intersection $(\bigcup_i f_i) \cap (\bigcup_j v_j)$ is a union of $2m + n$ open disks.
- $e_i \cap v_j$ is a disk if the vertex is an endpoint of the edge, and is empty otherwise. Hence $(\bigcup_i e_i) \cap (\bigcup_j v_j)$ is a union of $2m$ open disks.
- $b_i \cap v_j$ is similar; $(\bigcup_i b_i) \cap (\bigcup_j v_j)$ is a union of $2n$ open disks.

The remaining double intersections are empty.

The triple intersections are:

- $f_i \cap e_j \cap v_k$ is a disk if the edge is a side of the face and the vertex is an endpoint of the edge; otherwise it is empty. Hence $(\bigcup_i f_i) \cap (\bigcup_j e_j) \cap (\bigcup_k v_k)$ is a union of $4m$ open disks.
- $f_i \cap b_j \cap v_k$ is similar; $(\bigcup_i f_i) \cap (\bigcup_j b_j) \cap (\bigcup_k v_k)$ is a union of $2n$ open disks.

All the other intersections are empty (including the intersections of more than three sets).

All the sets and their intersections are disks except possibly faces. Thus the Euler characteristics of M is

$$\begin{aligned}\chi(M) &= \sum_{i=1}^{n} \chi(f_i) + m + n + n - 2m - n - (2m + n) - 2m - 2n + 4m + 2n \\ &= \sum_{i=1}^{n} \chi(f_i) - m\end{aligned}$$

On the other hand, Γ is a retract of M, so $\chi(M) = \chi(\Gamma) = n - m$, hence $\sum_{i=1}^{n} \chi(f_i) = n$.

A face is a connected noncompact 2-manifold, so $\chi(f_i) \leq 1$. The equality $\sum_{i=1}^{n} \chi(f_i) = n$ implies now that $\chi(f_i) = 1$ for every i, so each f_i is a disk. This disk is adjacent to exactly one external edge (a segment of the boundary ∂M) and to $k_i \geq 1$ internal edges (unless the face occupies the whole connected component of the surface). So the closure of f_i is an image of the $(k_i + 1)$-gon Q_i mapping its interior homeomorphically to f_i and the sides, to the edges: one side to the external edge and one or two sides to every internal edge. These maps $Q_i \to M$ for all the faces f_i form a cell decomposition of M. □

Corollary 1.9. *Let $M \in \mathrm{DBS}_n$ be stable and $\Gamma \subset M$ be a properly embedded graph. Then Γ is the diagonal graph of a ribbon decomposition of M.*

Proof. Induction by the the number m of edges of Γ. The base: $m = 0$, that is, Γ consists of several isolated vertices — the marked points of M. The DBS M is retracted to Γ, so, its every connected component is contractible and contains exactly one marked point. Thus, $M = E_n$.

Now let $m > 0$; take the edge e_m of Γ. It joins the vertices a_i and a_j (necessarily different) and separate faces f_p and f_q (which may be the same). By the anti-unimodality, the edge e_m is adjacent to the boundary of M at both its endpoints;

in other words, if $\phi_p : Q_p \to M$ is the characteristic map of the face f_p then a side v_0v_1 of Q_p is mapped to the external edge of f_p and the adjacent side v_1v_2, to e_m. Let $v' \in v_0v_1$ be a point near the vertex v_1, $a_i' \stackrel{\text{def}}{=} (v') \in \partial M$; consider the image $T_p \stackrel{\text{def}}{=} \phi_p(v'v_1v_2) \subset M$ of the triangle $v'v_1v_2$. Then the union of T_p and T_q is a ribbon H, and the edge e_m, its diagonal.

Let Γ' be the graph Γ with the edge e_m removed. Take $\varepsilon = +$ if the boundary of M near a_i is oriented from a_i to a_i', and $\varepsilon = -$ otherwise. Then Γ' is embedded into $M' \stackrel{\text{def}}{=} R[e_m]^\varepsilon(M)$; an immediate check shows that the embedding is proper. Then by Proposition 1.2 M can be obtained by gluing the ribbon H to M', which finishes the induction. □

1.3. **Oriented case and the orientation cover.** A DBS M is called *oriented* if all the orientations o_i are consistent with a global orientation of the surface M.

It is easy to see that if $M \in \mathrm{DBS}_n$ is oriented and $\varepsilon_i = \varepsilon_j$ (the gluing $G[i,j]^{\varepsilon_i,\varepsilon_j}$ is non-twisted) then $G[i,j]^{\varepsilon_i,\varepsilon_j}(M) \in \mathrm{DBS}_n$ is oriented, too: the orientation o of M consistent with all the o_k is naturally extended to M', and this extension is consistent with all the o_k'. If the gluing is twisted, then orientability breaks: M' need not be orientable, and even if it is, the orientations o_k' of the boundary $\partial M'$ are not necessarily consistent with a global orientation.

Call a ribbon decomposition of a DBS $M = G[i_m, j_m]^{\varepsilon_m\delta_m} \dots G[i_1, j_1]^{\varepsilon_1\delta_1}(E_n)$ oriented if all the signs $\varepsilon_j, \delta_j = +$. By the remark above, M is an oriented DBS then.

Theorem 1.10. *The diagonal graph Γ of the oriented ribbon decomposition $G[i_m, j_m]^{++} \dots G[i_1, j_1]^{++}E_n$ has the following properties (in addition to those granted by Theorem 1.7):*

(1) *(vertex monotonicity) For every vertex a_i of Γ its passport $\mathcal{P}(a_i) = (\ell_1, \dots, \ell_k)$ is increasing: $\ell_1 < \dots < \ell_k$.*
(2) *(face monotonicity) For every face f_i of Γ let $\ell_1, \dots, \ell_p$ be the numbers of the internal edges on its boundary listed counterclockwise so that ℓ_1 and ℓ_p are adjacent to the (only) external edge on the boundary of f_i. Then the sequence $\ell_i, \dots, \ell_p$ is increasing: $\ell_1 < \dots < \ell_p$.*
(3) *(face separation) Every internal edge of Γ is adjacent to two different faces. In other words, a characteristic map (of Theorem 1.8) of every face is one-to-one on the interior of every edge.*
(4) *(boundary permutation) Consider a permutation $\sigma = (i_m j_m) \dots (i_1 j_1) \in S_n$; then the marked point following a_k in the positive direction of the boundary ∂M is $a_{\sigma(k)}$. In other words, the numbers of marked points read counterclockwise off the components of ∂M form a cyclic decomposition of σ.*

In Property 4 and below we denote by S_n the permutation group on the set $\{1, \dots, n\}$.

Proof. Vertex monotonicity is a particular case of anti-unimodality of Theorem 1.7.

If ℓ_j and ℓ_{j+1} are two internal edges on the boundary of f_i sharing an endpoint a then the orientation of the boundary near a is consistent with the counterclockwise orientation of f_i. Then the vertex monotonicity implies $\ell_j < \ell_{j+1}$, which proves face monotonicity. The face monotonicity implies, in its turn, the face separation:

as one moves around a face, the numbers of the internal edges seen are increasing and therefore cannot repeat.

Let a_k and a_s be neigbouring vertices on the boundary ∂M, that is, the endpoints of an external edge. By Theorem 1.8 and the face monotonicity, this is the sole external edge of a face f, its remaining sides being internal edges numbered $\ell_1 < \dots < \ell_p$, as one moves from a_k to a_s. Consider an action of S_n on the vertices of $M \in \mathrm{DBS}_n$ by permuting their numbers; in particular, the transposition $(i_t j_t)$ exchanges the numbers of the vertices joined by the t-th edge of the diagonal graph, leaving the other vertices intact. So, the transposition $(i_{\ell_1} j_{\ell_1})$ moves a_k to its neighbour at the face f; then the transposition (i_{ℓ_2}, j_{ℓ_2}) (where $\ell_2 > \ell_1$, so it is applied *after* the first one) moves it to the next vertex of the same face, etc.; eventually, $\sigma = (i_m j_m) \dots (i_1 j_1)$ moves a_k to $a_s = a_{\sigma(k)}$. □

Every manifold M has the orientation cover, uniquely defined up to an obvious isomorphism: it is an oriented manifold $\widehat{M}$ of the same dimension together with a fixed-point-free orientation-reversing smooth involution $\mathcal{T} : \widehat{M} \to \widehat{M}$ such that M is diffeomorphic to its orbit space. (The quotient map $\widehat{M} \to \widehat{M}/\mathcal{T} = M$ is a locally trivial 2-sheeted covering, hence the name.) If M is orientable then $\widehat{M}$ is a disjoint union of two copies of M with the opposite orientation; $\mathcal{T}$ exchanges their namesake points. If M is connected and not orientable then $\widehat{M}$ is connected, too.

An important property of the orientation covers of 2-manifolds with boundary (to be used later in Section 2) is

Lemma 1.11. *The orientation cover is trivial over the boundary of a 2-manifold.*

Proof. Let $\widehat{M}$ be an orientation cover of M, and $\mathcal{T} : \widehat{M} \to \widehat{M}$, the corresponding fixed-point-free involution. The boundary ∂M of a 2-manifold M is a union of circles; if its 2-cover is nontrivial then there is a component $C \subset \partial M$ of the boundary covered by a $\mathcal{T}$-invariant circle $C' \subset \partial\widehat{M}$.

A continuous map $A : S^1 \to S^1$ has at least $|\deg A - 1|$ fixed points, so the fixed-point-free map $\mathcal{T} : C' \to C'$ has degree 1 and therefore preserves orientation. Since $C' \subset \partial\widehat{M}$ it means that $\mathcal{T} : \widehat{M} \to \widehat{M}$ also preserves local orientation at every point $a \in C'$. But $\mathcal{T}$ is orientation-reversing everywhere — a contradiction. □

Let $\tau \in S_{2n}$ be a fixed-point-free involution defined as $\tau(i) = i + n \bmod (2n)$, $i = 1, \dots, 2n$. The notion of an orientation cover can be extended to decorated-boundary surfaces: an $\widehat{M} \in \mathrm{DBS}_{2n}$ with the marked points $b_1, \dots, b_{2n}$ is called the orientation cover of $M \in \mathrm{DBS}_n$ with the marked points $a_1, \dots, a_n$ if $\widehat{M}$ is oriented and there exists a fixed-point-free orientation-reversing smooth involution $\mathcal{T} : \widehat{M} \to \widehat{M}$ such that $\mathcal{T}(b_k) = b_{\tau(k)}$ for all $k = 1, \dots, 2n$, and also there exists a diffeomorphism $p : \widehat{M}/\mathcal{T} \to M$ of the orbit space and M such that $p(\{b_k, b_{\tau(k)}\}) = a_k$ for all $k = 1, \dots, n$.

Let $1 \le i \le n$ and $\varepsilon \in \{+, -\}$. Denote $i^\varepsilon = \begin{cases} i, & \varepsilon = +, \\ \tau(i), & \varepsilon = -. \end{cases}$

Theorem 1.12. *Let $M = G[i_m, j_m]^{\varepsilon_m \delta_m} \dots G[i_1, j_1]^{\varepsilon_1 \delta_1} E_n$. Then*

$$\widehat{M} = G[i_m^{\varepsilon_m} j_m^{\delta_m}]^{++} \dots G[i_1^{\varepsilon_1} j_1^{\delta_1}]^{++} G[i_1^{-\varepsilon_1} j_1^{-\delta_1}]^{++} \dots G[i_m^{-\varepsilon_m} j_m^{-\delta_m}]^{++} E_n \tag{1.1}$$

is its orientation cover. The involution $\mathcal{T} : \widehat{M} \to \widehat{M}$ maps the ribbon r_ℓ and the edge number ℓ to the ribbon $r_{2m+1-\ell}$ and the corresponding edge for all $\ell = 1, \dots, 2m$.

Proof. Let a_i be a marked point of M with $\mathcal{P}(a_i) = (\ell_1, \dots, \ell_k)$ where $\ell_1 > \dots > \ell_p < \dots < \ell_k$, and let $b_i, b_{\tau(i)} \in \widehat{M}$ be its preimages. Use the induction on m to prove the theorem showing simultaneously that $\mathcal{P}(b_i) = (m+1-\ell_1, \dots, m+1-\ell_p, m+\ell_{p+1}, \dots, m+\ell_k)$ and $\mathcal{P}(b_{\tau(i)}) = (m+1-\ell_k, \dots, m+1-\ell_{p+1}, \ell_p + m, \dots, \ell_1 + m)$.

The base $m = 0$ is obvious. For $m > 0$ let $M = G[i,j]^{\varepsilon\delta} M'$ where $i \overset{\text{def}}{=} i_m$, $j \overset{\text{def}}{=} j_m$, $\varepsilon \overset{\text{def}}{=} \varepsilon_m$ and $\delta \overset{\text{def}}{=} \varepsilon_m$. If $\mathcal{P}_{M'}(a_i) = (\ell_1, \dots, \ell_k)$ where $\ell_1 > \dots > \ell_p < \dots < \ell_k$ then $\mathcal{P}_M(a_i) = (\ell_1, \dots, \ell_k, m)$ if $\varepsilon = +$ and $\mathcal{P}_M(a_i) = (m, \ell_1, \dots, \ell_k)$ if $\varepsilon = -$; the same for a_j (depending on δ instead of ε).

Denote by $\widehat{M'}$ the orientation cover of M' and define $\widehat{M}$ by (1.1). By the induction hypothesis $\widehat{M'}$ is a subset of $\widehat{M}$ (a union of all the ribbons except the first one and the last one). Extend $\mathcal{T} : \widehat{M'} \to \widehat{M'}$ to the involution $\widehat{M} \to \widehat{M}$ sending r_1 to r_{2m} and vice versa; also extend the homeomorphism $\rho : \widehat{M'}/\mathcal{T} \to M'$ to a map $\widehat{M}/\mathcal{T} \to M$ sending r_1 and r_{2m} to the m-th ribbon of M. Apparently, the extended $\mathcal{T}$ is a fixed-point-free involution and the extended ρ, a bijection continuous on the interiors of r_1 and r_m. To finish the proof we are to check that the extended $\mathcal{T}$ and ρ are continuous on the boundary of the ribbons r_1 and r_{2m}.

By the induction hypothesis, $\mathcal{P}_{\widehat{M'}}(b_i) = (m-\ell_1, \dots, m-\ell_p, \ell_{p+1}+m-1, \dots, \ell_k + m-1)$ and $\mathcal{P}_{\widehat{M'}}(b_{\tau(i)}) = (m-\ell_k, \dots, m-\ell_{p+1}, \ell_p+m-1, \dots, \ell_1+m-1)$. So, if $\varepsilon = +$ then $\mathcal{P}_{\widehat{M}}(b_i) = (m+1-\ell_1, \dots, m+1-\ell_p, \ell_{p+1}+m, \dots, \ell_k+m, 2m)$ and $\mathcal{P}_{\widehat{M'}}(b_{\tau(i)}) = (1, m+1-\ell_k, \dots, m+1-\ell_{p+1}, \ell_p+m, \dots, \ell_1+m)$, and if $\varepsilon = -$ then $\mathcal{P}_{\widehat{M}}(b_i) = (1, m+1-\ell_1, \dots, m+1-\ell_p, \ell_{p+1}+m, \dots, \ell_k+m)$ and $\mathcal{P}_{\widehat{M'}}(b_{\tau(i)}) = (m+1-\ell_k, \dots, m+1-\ell_{p+1}, \ell_p+m, \dots, \ell_1+m, 2m)$; the same for b_j and $b_{\tau(j)}$, with δ instead of ε.

Thus, if $\varepsilon = +$ then the ribbon r_{2m} is adjacent to r_{ℓ_k+m} and the ribbon r_1, to the ribbon $r_{m+1-\ell_k}$; the m-th ribbon of $M = G[i,j]^{\varepsilon\delta} M'$ is adjacent to its ribbon numbered ℓ_k. By the induction hypothesis, $\mathcal{T}$ exchanges r_{ℓ_k+m} and $r_{m+1-\ell_k}$, so the extensions of $\mathcal{T}$ and ρ are continuous on the "long" sides of r_{2m} and r_1 adjacent to the vertices b_i and $b_{\tau(i)}$, respectively. The proof in the case $\varepsilon = -$ is the same. A similar analysis of the passports of b_j and $b_{\tau(j)}$ for $\delta = +$ and $\delta = -$ shows that $\mathcal{T}$ and ρ are continuous on the other sides of r_{2m} and r_1, too. □

2. Twisted cut-and-join equation

2.1. **Algebraic preliminaries.** As usual [1], denote by B_n the group of linear operators $A : \mathbb{R}^n \to \mathbb{R}^n$ given by $A(x_1, \dots, x_n) = (\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)})$ where $\sigma \in S_n$ is a permutation and $\varepsilon_1, \dots, \varepsilon_n = \pm 1$. In other words, B_n is a semidirect product $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ where S_n acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permuting the factors.

Let $\mathbf{1} = (1, \dots, 1)$ and $\mathbf{1}_i = (1, \dots, -1, \dots, 1)$ (-1 at the i-th place); then the elements $s_{ij} \overset{\text{def}}{=} (ij) \ltimes \mathbf{1}$ and $l_i \overset{\text{def}}{=} \mathrm{id} \ltimes \mathbf{1}_i$ are obviously reflections; they generate the group.

Recall the notation $\tau \overset{\text{def}}{=} (1, n+1)(2, n+2) \dots (n, 2n) \in S_{2n}$.

Proposition 2.1. *The centralizer* $C(\tau) \overset{\text{def}}{=} \{\sigma \in S_{2n} \mid \sigma\tau = \tau\sigma\} \subset S_{2n}$ *of the element* τ *is isomorphic to* B_n.

Proof. Define maps $\lambda : S_{2n} \to S_n$ and $\varepsilon^{(i)} : S_{2n} \to \mathbb{Z}/2\mathbb{Z}$ by

$$\lambda_\sigma(i) = \begin{cases} \sigma(i), & \sigma(i) \le n, \\ \sigma(i) - n, & \sigma(i) \ge n+1 \end{cases} \quad \text{and} \quad \varepsilon_\sigma^{(i)} = \begin{cases} 1, & \sigma(i) \le n, \\ -1, & \sigma(i) \ge n+1. \end{cases}$$

An immediate check shows that if $\sigma_1, \sigma_2 \in C(\tau)$ then $\lambda_{\sigma_1\sigma_2} = \lambda_{\sigma_1}\lambda_{\sigma_2}$ and $\varepsilon^{(i)}_{\sigma_1\sigma_2} = \varepsilon^{(i)}_{\sigma_1}\varepsilon^{(\sigma_1(i))}_{\sigma_2}$. So the map $A : C(\tau) \to S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ given by $A(\sigma) = \lambda_\sigma \ltimes (\varepsilon^{(1)}_\sigma, \dots, \varepsilon^{(n)}_\sigma)$ is a group homomorphism.

If $A(\sigma) = 1$ then for every $i = 1, \dots, n$ one has $\lambda_\sigma(i) = i$, that is, $\sigma(i) = i$ or $i + n$, and $\varepsilon^{(i)}_\sigma = 1$, which implies $\sigma(i) \le n$. Hence $\sigma(i) = i$, and therefore $\sigma(i+n) = \sigma(\tau(i)) = \tau(\sigma(i)) = i + n$. So, $\sigma = \mathrm{id}$ and A is a monomorphism. One has $s_{ij} = A((ij)(i+n, j+n))$ and $l_i = A((i, i+n))$, so A is an epimorphism, too. □

So B_n is embedded into S_{2n}; we are going to denote $C(\tau) = B_n \subset S_{2n}$ for short.

Denote $C^\sim(\tau) \stackrel{\text{def}}{=} \{\sigma \in S_{2n} \mid \tau\sigma = \sigma^{-1}\tau\}$ (a "twisted centralizer" of τ).

Lemma 2.2. *Let $\sigma = c_1 \dots c_m \in C^\sim(\tau)$ where $c_1, \dots, c_m$ are independent cycles. Then for every i*

- *either there exists j such that $c_i = (u_1 \dots u_k)$ and $c_j = (u_{\tau(k)} \dots u_{\tau(1)})$;*
- *or c_i has even length $2k$ and looks like $c_i = (u_1 \dots u_k \tau(u_k) \dots \tau(u_1))$.*

In the first case we call c_i and c_j the τ-symmetric pair of cycles, and in the second case the cycle c_i is τ-self-symmetric.

Proof. Let $c_i = (u^{(i)}_1 \dots u^{(i)}_{k_i})$ for all $i = 1, \dots, m$. Then $\tau\sigma\tau^{-1} = c'_1 \dots c'_m$ where $c'_i = (\tau(u^{(i)}_1) \dots \tau(u^{(i)}_{k_i}))$. On the other side, $\sigma^{-1} = c''_1 \dots c''_m$ where $c''_i = (u^{(i)}_{k_i} \dots u^{(i)}_1)$. Once a cycle decomposition is unique, every c''_i must be equal to some c'_j. If $j \ne i$ then c_i and c_j are τ-symmetric, and if $j = i$ then c_i is τ-self-symmetric. □

Theorem 2.3. *There exists a one-to-one correspondence between the following three sets:*

1. *The quotient (the set of left cosets) S_{2n}/B_n;*
2. *The set $B^\sim_n$ of permutations $\sigma \in C^\sim(\tau)$ such that their cycle decomposition contains no τ-self-symmetric cycles.*
3. *The set of fixed-point-free involutions $\lambda \in S_{2n}$.*

The size of each set is $(2n-1)!! = 1 \times 3 \times \cdots \times (2n-1)$.

Proof. To prove the theorem we will construct injective maps $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$2 \Rightarrow 3$: Since $\tau^{-1} = \tau$, the condition $\tau\sigma\tau^{-1} = \sigma^{-1}$ is equivalent to $(\sigma\tau)^2 = \mathrm{id}$. If $\sigma = c_1c_2 \dots c_k$ then $\sigma\tau$ sends every element of the cycle c_i to an element of its τ-symmetric cycle c_j. So if $j \ne i$ for all i then the involution $\sigma\tau$ has no fixed points. The map $\sigma \mapsto \sigma\tau$ is obviously injective.

$3 \Rightarrow 1$: if λ is a fixed-point-free involution then its cycle decomposition is a product of n independent transpositions, and therefore λ belongs to the same conjugacy class in S_{2n} as τ: $\lambda = \sigma\tau\sigma^{-1}$ for some $\sigma \in S_{2n}$. Denote by $R(\lambda) \in S_{2n}/B_n$ the left coset containing σ. The equality $\sigma_1\tau\sigma_1^{-1} = \sigma_2\tau\sigma_2^{-1}$ is equivalent to $(\sigma_1\sigma_2^{-1})\tau = \tau(\sigma_1\sigma_2^{-1})$, that is, $\sigma_1\sigma_2^{-1} \in B_n$. So the left cosets containing σ_1 and σ_2 are the same and $R(\lambda) \in S_{2n}/B_n$ is well-defined. If $R(\lambda_1) = R(\lambda_2)$ where

$\lambda_i = \sigma_i \tau \sigma_i^{-1}$, $i = 1, 2$, then $\sigma_1 \sigma_2^{-1} \in B_n$ and therefore $\lambda_1 = \lambda_2$; thus, R is an injective map.

1 ⇒ 2: let $\sigma \in S_{2n}$ be an element of the coset $\lambda \in S_{2n}/B_n$; take $Q(\lambda) \stackrel{\text{def}}{=} [\sigma, \tau] \stackrel{\text{def}}{=} \sigma\tau\sigma^{-1}\tau$. Since τ is an involution, one has $\tau Q(\lambda)\tau = \tau\sigma\tau\sigma^{-1} = Q(\lambda)^{-1}$, so $Q(\lambda) \in C^{\sim}(\tau)$. Let σ' is another element of the coset λ, that is, $\sigma' = \sigma\rho$ where $\rho\tau = \tau\rho$; then $[\sigma', \tau] = \sigma\rho\tau\rho^{-1}\sigma^{-1}\tau = \sigma\tau\rho\rho^{-1}\sigma^{-1}\tau = Q(\lambda)$. If $Q(\lambda) = Q(\lambda')$ where $\lambda, \lambda' \in S_{2n}/B_n$ are represented by σ and σ', respectively, then $\sigma\tau\sigma^{-1}\tau = \sigma'\tau(\sigma')^{-1}\tau$, which is equivalent to $(\sigma')^{-1}\sigma\tau = \tau(\sigma')^{-1}\sigma$. So $(\sigma')^{-1}\sigma \in B_n$, and $\lambda = \lambda'$.

Thus, Q is a well-defined injective map from S_{2n}/B_n to $C^{\sim}(\tau)$. Prove that actually $Q(\lambda) \in B_n^{\sim} \subset C^{\sim}(\tau)$. Suppose it is not the case, that is, $Q(\lambda)$ has a τ-self-symmetric cycle $c = (u_1 \dots u_k \tau(u_k) \dots \tau(u_1))$. It means that $\tau Q(\lambda)$ has a fixed point $u = u_k$. On the other hand, $\tau Q(\lambda) = (\tau\sigma)\tau(\tau\sigma)^{-1}$ is conjugate to τ and is a product of n independent transpositions having no fixed points — a contradiction. □

Proposition 2.4. *Let $\widehat{M} \in \mathrm{DBS}_{2n}$ be the orientation cover of $M \in \mathrm{DBS}_n$, and $\sigma \in S_{2n}$ be its boundary permutation, as in assertion 4 of Theorem 1.10. Then $\sigma \in B_n^{\sim}$.*

Proof. Consider the cycle decomposition $\sigma = c_1 \dots c_k$, where $c_i = (u_1^{(i)} \dots u_{k_i}^{(i)})$, $i = 1, \dots, k$. By assertion 4 of Theorem 1.10 the i-th component of $\partial\widehat{M}$ contains the marked points numbered $u_1^{(i)}, \dots, u_{k_i}^{(i)}$, listed counterclockwise. By Lemma 1.11 the images of the points under the involution $\mathcal{T} : \widehat{M} \to \widehat{M}$ lie in the j-th component of the boundary where $j \neq i$ and have numbers $\tau(u_1^{(i)}), \dots, \tau(u_{k_i}^{(i)})$, $1 \le s \le k$, listed *clockwise* (because $\mathcal{T}$ changes orientation). Thus, one has $c_j = (\tau(u_{k_i}^{(i)}), \dots, \tau(u_1^{(i)}))$, so the cycles c_i and c_j are τ-symmetric.

Thus, σ is a product of several pairs of τ-symmetric cycles, which means $\sigma \in B_n^{\sim}$. □

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $|\lambda| = n$, and denote by $B_\lambda^{\sim} \subset B_n^{\sim}$ the set of permutation $\sigma = c_1 \dots c_{2\ell}$ whose cycle decomoisition consists of ℓ pairs of τ-symmetric cycles of lengths $\lambda_1, \dots, \lambda_\ell$. Apparently, $B_n^{\sim} = \bigsqcup_{|\lambda|=n} B_\lambda^{\sim}$.

Proposition 2.5. *$B_\lambda^{\sim}$ is a B_n-conjugacy class in S_{2n}.*

Proof. Let $\sigma = c_1 c_1' \dots c_\ell c_\ell' \in B_n^{\sim}$ be the cycle decomposition where c_i and c_i' are τ-symmetric for all i: $c_i' = \tau c_i^{-1} \tau$. Let $x \in B_n$, that is, $x\tau = \tau x$ by Proposition 2.1. Then $x\sigma x^{-1} = x c_1 x^{-1} \cdot x\tau c_1^{-1} \tau x^{-1} \cdot \dots \cdot x c_\ell x^{-1} \cdot x\tau c_\ell^{-1} \tau x^{-1}$. The permutations $\tilde{c}_i \stackrel{\text{def}}{=} x c_i x^{-1}$ and $\tilde{c}_i' = x c_i' x^{-1}$ are cycles of length λ_i, and they are τ-symmetric: $\tau\tilde{c}_i\tau = \tau x c_i x^{-1} \tau = x\tau c_i \tau x^{-1} = x(c_i')^{-1} x^{-1} = (\tilde{c}_i')^{-1}$. Thus, $x\sigma x^{-1} \in B_\lambda^{\sim}$.

On the other side, let $\tilde{\sigma} = \tilde{c}_1 \tilde{c}_1' \dots \tilde{c}_\ell \tilde{c}_\ell' \in B_\lambda^{\sim}$. Let $\tilde{c}_i = (v_1^{(i)} \dots v_{\lambda_i}^{(i)})$, so $\tilde{c}_i' = (\tau(v_{\lambda_i}^{(i)}) \dots \tau(v_1^{(i)}))$. Define an element $x \in S_{2n}$ such that $x(u_s^{(i)}) = v_s^{(i)}$ and $x(\tau(u_s^{(i)})) = \tau(v_s^{(i)})$. Then $x\sigma x^{-1} = \tilde{\sigma}$ and $x\tau = \tau x$ (that is, $x \in B_n$). □

Definition 2.6. Let λ be a partition and m, a positive integer. Then the *twisted Hurwitz numbers* $h^{\sim}_{m,\lambda}$ is defined as

$$
\begin{aligned}
h^{\sim}_{m,\lambda} \stackrel{\text{def}}{=} \frac{1}{n!}\#\{(\sigma_1,\dots,\sigma_m) \mid \sigma_s = (i_s j_s), j_s \neq \tau(i_s), s = 1,\dots,m,\\
\sigma_1\sigma_2\dots\sigma_m(\tau\sigma_m\tau)\dots(\tau\sigma_1\tau) \in B^{\sim}_{\lambda}\}.
\end{aligned} \tag{2.1}
$$

Remark 2.7. τ is an involution, so the internal τ in $(\tau\sigma_m\tau)(\tau\sigma_{m-1}\tau)\dots$ may be omitted.

For a conjugacy class $B^{\sim}_{\lambda}$ of B_n denote

$$
\mathcal{C}^{\sim}_{\lambda} \stackrel{\text{def}}{=} \sum_{\sigma \in B^{\sim}_{\lambda}} \sigma \in \mathbb{C}[B^{\sim}_n]. \tag{2.2}
$$

Being a conjugacy class sum, $\mathcal{C}^{\sim}_{\lambda}$ commutes with B_n. Also, call the set

$$
\mathcal{Z}(B^{\sim}_n) \stackrel{\text{def}}{=} \{y \in \mathbb{C}[B^{\sim}_n] \mid xyx^{-1} = y \,\forall x \in B_n\}
$$

a *twisted center* of B_n. It is clear that $\mathcal{C}^{\sim}_{\lambda} \in \mathcal{Z}(B^{\sim}_n)$ form a basis of $\mathcal{Z}(B^{\sim}_n)$.

Let now $\mathbb{C}[p]$ be a ring of polynomials where $p = (p_1, p_2, \dots)$ is an countable set of variables. The ring $\mathbb{C}[p]$ is graded by the total degree of the polynomial, where one assumes $\deg p_k = k$ for all $k = 1, 2, \dots$. It is easy to see that a linear map $\Psi : \mathcal{Z}(B^{\sim}_n) \to \mathbb{C}[p]_n$ defined as

$$
\Psi(\mathcal{C}^{\sim}_{\lambda}) = p_{\lambda} \stackrel{\text{def}}{=} p_{\lambda_1}\dots p_{\lambda_s} \tag{2.3}
$$

is an isomorphism.

Define an operator $\mathfrak{CJ}^{\sim} : \mathcal{Z}(B^{\sim}_n) \to \mathcal{Z}(B^{\sim}_n)$ by

$$
\mathfrak{CJ}^{\sim}(\sigma) = \sum_{\substack{1 \le i < j \le 2n\\ j \neq i, \tau(i)}} (ij)\sigma(\tau(i)\tau(j))
$$

Definition 2.8. The *twisted cut-and-join operator* is a linear operator $\mathcal{CJ}^{\sim} : \mathbb{C}[p]_n \to \mathbb{C}[p]_n$ making the following diagram commutative:

$$
\begin{array}{ccc}
\mathcal{Z}[B^{\sim}_n] & \xrightarrow{\mathfrak{CJ}^{\sim}} & \mathcal{Z}[B^{\sim}_n] \\
\downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\
\mathbb{C}[p]_n & \xrightarrow[\mathcal{CJ}^{\sim}]{} & \mathbb{C}[p]_n
\end{array} \tag{2.4}
$$

Let now λ, λ' be partitions such that $|\lambda| = |\lambda'| = n$. Take an element $\sigma \in B^{\sim}_{\lambda}$ and consider a set

$$
S(\sigma; \lambda') \stackrel{\text{def}}{=} \{(i,j) \mid \le i, j \le 2n, j \neq i, \tau(i), (ij)\sigma_*(\tau(i)\tau(j)) \in B^{\sim}_{\lambda'}\}.
$$

Proposition 2.9. *For every $x \in B_n$ and $\sigma \in B^{\sim}_{\lambda}$ the map $(i,j) \mapsto (x(i), x(j))$ is a bijection between $S(x\sigma x^{-1}, \lambda')$ and $S(\sigma, \lambda')$.*

Proof. If $(i,j) \in S(x\sigma x^{-1}; \lambda')$ then $(ij)x\sigma x^{-1}(\tau(i)\tau(j)) \in B^{\sim}_{\lambda'}$ and therefore $(x(i)x(j))\sigma(\tau(x(i))\tau(x(j))) = x^{-1}(ij)x\sigma x^{-1}(\tau(i)\tau(j))x \in B^{\sim}_{\lambda'}$ by Proposition 2.5. It means that $(x(i), x(j)) \in S(\sigma, \lambda')$. ☐

Corollary 2.10. *The size of the set $S(\sigma, \lambda')$ for $\sigma \in B^{\sim}_{\lambda}$ depends on λ and λ' only.*

Proof. If $\sigma' \in B^{\sim}_{\lambda}$ then by Proposition 2.5 there exists $x \in B_n$ such that $\sigma' = x\sigma x^{-1}$. ☐

We will be using "physical" notation for the matrix elements of a linear operator $\mathfrak{CJ}^{\sim} : \mathcal{Z}(B_n^{\sim}) \to \mathcal{Z}(B_n^{\sim})$ (in the basis $\mathcal{C}_\lambda^{\sim}$): $\mathfrak{CJ}^{\sim}(\mathcal{C}_\lambda^{\sim}) = \sum_{\lambda'} \langle \lambda \mid \mathfrak{CJ}^{\sim} \mid \lambda' \rangle \, \mathcal{C}_{\lambda'}^{\sim}$

Theorem 2.11. *$\langle \lambda \mid \mathfrak{CJ}^{\sim} \mid \lambda' \rangle = \frac{1}{2} \# S(\sigma, \lambda')$ for any $\sigma \in B_\lambda^{\sim}$.*

Proof. By definition,

$$\mathfrak{CJ}^{\sim}(\mathcal{C}_\lambda^{\sim}) = \sum_{\sigma \in B_\lambda^{\sim}} \mathfrak{CJ}^{\sim}(\sigma) = \sum_{\sigma \in B_\lambda^{\sim}} \sum_{\substack{1 \le i < j \le 2n \\ j \ne i, \tau(i)}} (ij)\sigma(\tau(i)\tau(j)). \tag{2.5}$$

It follows from Proposition 2.9 that (2.5) is a sum of identical summands, so it is equal to their number multiplied by each of them:

$$\mathfrak{CJ}^{\sim}(\mathcal{C}_\lambda^{\sim}) = \# B_\lambda^{\sim} \sum_{\lambda'} \sum_{\substack{1 \le i < j \le 2n \\ j \ne i, \tau(i) \\ (ij)\sigma(\tau(i)\tau(j)) \in B_{\lambda'}^{\sim}}} (ij)\sigma(\tau(i)\tau(j)).$$

for any fixed $\sigma \in B_\lambda^{\sim}$. Using Proposition 2.9 again, one obtains

$$\begin{aligned} \mathfrak{CJ}^{\sim}(\mathcal{C}_\lambda^{\sim}) &= \sum_{\lambda'} \sum_{\substack{1 \le i < j \le 2n \\ j \ne i, \tau(i) \\ (ij)\sigma(\tau(i)\tau(j)) \in B_{\lambda'}^{\sim}}} \sum_{\tau \in B_{\lambda'}^{\sim}} \tau \\ &= \frac{1}{2} \sum_{\lambda'} \#\{(i,j) \mid j \ne i, \tau(i), (ij)\sigma(\tau(i)\tau(j)) \in B_{\lambda'}^{\sim}\} \, \mathcal{C}_{\lambda'}^{\sim} . \end{aligned}$$

□

Consider the generating function $\mathcal{H}^{\sim}(\beta, p)$ of the twisted Hurwitz numbers defined as follows:

$$\mathcal{H}^{\sim}(\beta, p) = \sum_{m \ge 0} \sum_{\lambda} \frac{h_{m,\lambda}^{\sim}}{m!} p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_s} \beta^m.$$

Theorem 2.12. *$\mathcal{H}^{\sim}$ satisfies the cut-and-join equation $\frac{\partial \mathcal{H}^{\sim}}{\partial \beta} = \mathcal{CJ}^{\sim}(\mathcal{H}^{\sim})$.*

Proof. Fix a positive integer n and denote by $\mathcal{H}_n^{\sim}$ a degree n homogeneous component of $\mathcal{H}^{\sim}$. The twisted cut-and-join operator perserves the degree, so $\mathcal{H}^{\sim}$ satisfies the cut-and-join equation if and only if $\mathcal{H}_n^{\sim}$ does (for each n).

Let

$$\mathcal{G}_n \stackrel{\text{def}}{=} \sum_{m \ge 0} \sum_{\lambda : |\lambda| = n} \frac{n! h_{m,\lambda}^{\sim}}{m!} \mathcal{C}_\lambda^{\sim} \beta^m \in \mathbb{C}[S_{2n}]$$

where $\mathcal{C}_\lambda^{\sim}$ is defined by (2.2). An elementary combinatorial reasoning gives

$$\mathcal{G}_n = \sum_{m \ge 0} \frac{\beta^m}{m!} (\mathfrak{CJ}^{\sim})^m (\mathrm{e}_{2\mathrm{n}})$$

where $e_{2n} \in S_{2n}$ is the unit element. Clearly $\mathfrak{CJ}^{\sim}(\mathcal{G}_n) = \sum_{m \ge 0} \frac{\beta^m}{m!} (\mathfrak{CJ}^{\sim})^{m+1}(e_{2n}) = \sum_{m \ge 1} \frac{\beta^{m-1}}{(m-1)!} (\mathfrak{CJ}^{\sim})^m (e_{2n}) = \frac{\partial \mathcal{G}_n}{\partial \beta}$. Applying Ψ one obtains $\Psi \mathfrak{CJ}^{\sim}(\mathcal{G}_n) = \Psi(\frac{\partial \mathcal{G}_n}{\partial \beta}) = \frac{\partial}{\partial \beta} \Psi(\mathcal{G}_n)$. By (2.3), $\Psi(\mathcal{G}_n) = \mathcal{H}_n^{\sim}$, hence $\frac{\partial}{\partial \beta} \Psi(\mathcal{G}_n) = \frac{\partial \mathcal{H}_n^{\sim}}{\partial \beta}$. By the definition of the twisted cut-and-join operator, $\Psi \mathfrak{CJ}^{\sim}(\mathcal{G}_n) = \mathcal{CJ}^{\sim}(\Psi(\mathcal{G}_n)) = \mathcal{CJ}^{\sim}(\mathcal{H}_n^{\sim})$, and the equality $\frac{\partial \mathcal{H}_n^{\sim}}{\partial \beta} = \mathcal{CJ}^{\sim}(\mathcal{H}_n^{\sim})$ follows. □

Corollary 2.13. *$\mathcal{H}^{\sim}(\beta, p) = \exp(\beta \mathcal{CJ}^{\sim}) \exp(p_1)$.*

Proof. It follows from (2.1) that $h_{0,\lambda} = \frac{1}{n!}$ if $\lambda = 1^n$ and $h_{0,\lambda} = 0$ otherwise. Thus, $\mathcal{H}^{\sim}(0,p) = \exp(p_1)$, and the formula follows from Theorem 2.12. □

2.2. **Surgery on cosets and the cut-and-join equation.** To turn Corollary 2.13 into a formula for the twisted Hurwitz numbers we will need an explicit expression for the operator $\mathcal{CJ}^{\sim}$. The main result of this section is

Theorem 2.14. *The twisted cut-an-join operator is given by the formula*

$$\mathcal{CJ}^{\sim} = \sum_{i,j\geq 1} (i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + 2ijp_{i+j}\frac{\partial^2}{\partial p_i \partial p_j} + \sum_{k\geq 1} k(k-1)p_k \frac{\partial}{\partial p_k} \tag{2.6}$$

To prove it we will first calculate the matrix elements $\langle \lambda \mid \mathfrak{CJ}^{\sim} \mid \lambda' \rangle$ for all possible λ, λ' explicitly.

Let $\sigma \in S_k$ and $1 \leq i < j \leq k$, for any k. Recall that the cycle structure of the product $\sigma' = (ij)\sigma$ depends on the cycle structure of σ and the positions of i and j as follows: if i and j belong to the same cycle $(x_1, \dots, x_\ell)$ of σ, $i = x_1$, $j = x_{\ell_1+1}$, then in σ' the cycle splits into two cycles ("a cut"): $(i = x_1, \dots, x_{\ell_1})$ and $(j = x_{\ell_1+1}, \dots, x_\ell)$. If i and j are in different cycles $(i = x_1, \dots, x_{\ell_1})$ and $(j = y_1, \dots, y_{\ell_2})$ then in σ' then the cycles glue together ("a join") to the cycle $(i = x_1, \dots, x_{\ell_1}, j = y_1, \dots, y_{\ell_2})$.

Let $\sigma \in B_\lambda^{\sim} \subset B_n^{\sim}$ where $\lambda = 1^{a_1}2^{a_2}\dots n^{a_n}$; in other words, the element $\sigma \in S_{2n}$ contains a_k pairs of τ-symmetric cycles of length k for any $k = 1, \dots, n$. Let $1 \leq i < j \leq 2n$, $j \neq \tau(i)$ and $\sigma' \stackrel{\text{def}}{=} (ij)\sigma(\tau(i)\tau(j)) \in B_{\lambda'}^{\sim}$. The cyclic structure of σ' depends on the position of i, j, $\tau(i)$, $\tau(j)$ and the cycles of σ; there are three possible cases shown in Fig. 2.

The partition λ' and the matrix elements $\langle \lambda' \mid \mathfrak{CJ}^{\sim} \mid \lambda \rangle$ in these cases are as follows.

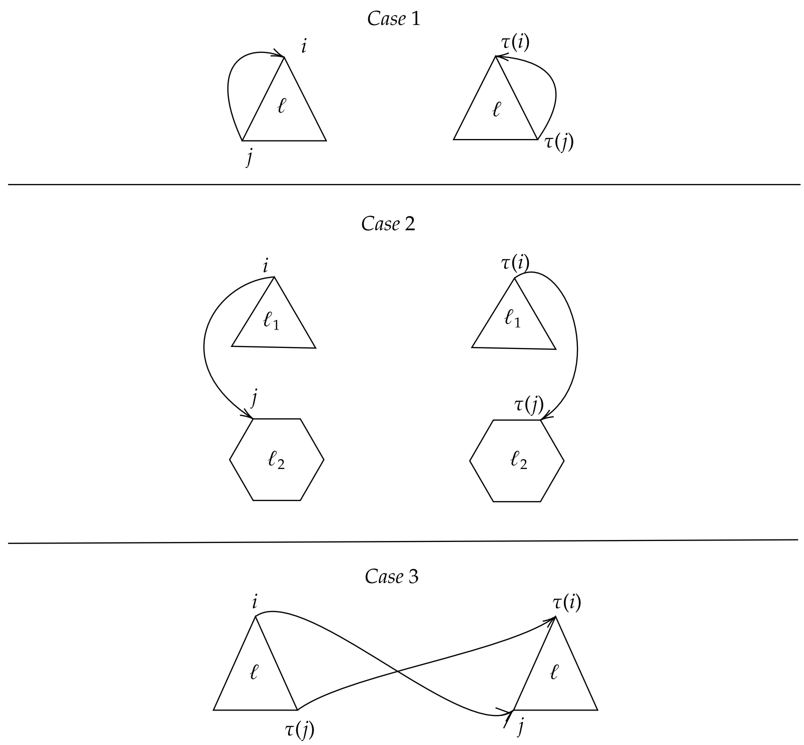

FIGURE 2. Terms of $\mathfrak{CJ}^\sim$

Case 1. Here λ' is obtained from λ by a cut:

$$\lambda' = 1^{a_1} \dots \ell_1^{a_{\ell_1}+1} \dots \ell_2^{a_{\ell_2}+1} \dots \ell^{a_\ell - 1} \dots n^{a_n}$$

where $\ell_1 + \ell_2 = \ell$ and $\ell_1 \neq \ell_2$, or

$$\lambda' = 1^{a_1} \dots \ell_1^{a_{\ell_1}+2} \dots \ell^{a_\ell - 1} \dots n^{a_n}$$

where $\ell_1 = \ell_2 = \ell/2$ (the term ℓ in λ is replaced by the two terms ℓ_1, ℓ_2 in λ'). For a fixed $\sigma \in B_\lambda^\sim$ look for i, j such that $\sigma' \stackrel{\text{def}}{=} (ij)\sigma(\tau(i)\tau(j)) \in B_{\lambda'}^\sim$. The element σ contains $2a_\ell$ cycles of length ℓ, so there are $2\ell a_\ell$ possible positions for i. In σ' the elements i and j are in different cycles; if $\ell_1 \neq \ell_2$ then ℓ_1 may be the length of either. Then for $\ell_1 \neq \ell_2$ the element j should be at the same cycle in σ as i at a distance of ℓ_1 or ℓ_2 from it; so there are two possible positions for j once i is chosen. It means that $\langle \lambda' \mid \mathfrak{CJ}^\sim \mid \lambda \rangle = \frac{1}{2} \# S(\sigma, \lambda') = 2\ell a_\ell$. If $\ell_1 = \ell_2 = \ell/2$ then the position for j is unique and $\langle \lambda' \mid \mathfrak{CJ}^\sim \mid \lambda \rangle = \ell a_\ell$.

Case 2. Here λ' is obtained from λ by a join:

$$\lambda' = 1^{a_1} \dots \ell_1^{a_{\ell_1}-1} \dots \ell_2^{a_{\ell_2}-1} \dots \ell^{a_\ell + 1} \dots n^{a_n}$$

where $\ell_1 + \ell_2 = \ell$ and $\ell_1 \neq \ell_2$ or

$$\lambda' = 1^{a_1} \dots \ell_1^{a_{\ell_1}-2} \dots \ell^{a_\ell + 1} \dots n^{a_n}$$

where $\ell_1 = \ell_2 = \ell/2$ (the terms ℓ_1, ℓ_2 in λ are replaced by the term $\ell_1 + \ell_2$ in λ'). If $\ell_1 \neq \ell_2$ then i may belong to the cycle of either length. If i belongs to the cycle of length ℓ_1 then there are $2a_{\ell_1}\ell_1$ possible positions for it (cf. Case 1) and $2a_{\ell_2}\ell_2$ positiions for j; vice versa if i belongs to the cycle of length ℓ_2. The matrix element is then $\langle \lambda' \mid \mathfrak{CJ}^{\sim} \mid \lambda\rangle = 4\ell_1\ell_2 a_{\ell_1}\ell_2 a_{\ell_2}$. If $\ell_1 = \ell_2 = \ell/2$ then i and j belong to cycles of the same length ℓ_1; the cycle containing j contains neither i nor $\tau(i)$. Hence there are $4a_{\ell_1}(a_{\ell_1} - 1)$ possibilities for choosing a pair of cycles to contain i and j and ℓ_1^2 possible positions for i and j in them, and therefore $\langle \lambda' \mid \mathfrak{CJ}^{\sim} \mid \lambda\rangle = 2a_{\ell_1}(a_{\ell_1} - 1)\ell_1^2$.

Case 3. Here $\lambda' = \lambda$. As in the previous cases we have $2\ell a_\ell$ possible positions for i and $\ell - 1$ positions for $j \neq \tau(i)$ (in the cycle τ-symmetric to the one containing i) once i is fixed. Thus, $\langle \lambda' \mid \mathfrak{CJ}^{\sim} \mid \lambda\rangle = \sum_\ell 2\ell(\ell - 1)a_\ell$.

Proof of Theorem 2.14. It follows from Theorem 2.11 and Definition 2.8 that $\mathcal{CJ}^{\sim} p_\lambda = \sum_{\lambda'} \langle \lambda \mid \mathfrak{CJ}^{\sim} \mid \lambda'\rangle p_{\lambda'}$.

For a given λ there are three types of λ' such that $\langle \lambda \mid \mathfrak{CJ}^{\sim} \mid \lambda'\rangle \neq 0$ listed above. Hence $\mathcal{CJ}^{\sim}$ is a sum of three terms.

Suppose λ' is as in Case 1 with $\ell_1 \neq \ell_2$. The monomial p_λ contains $p_{\ell_1}^{a_{\ell_1}} p_{\ell_2}^{a_{\ell_2}} p_\ell^{a_\ell}$ and the monomial $p_{\lambda'}$, $p_{\ell_1}^{a_{\ell_1}+1} p_{\ell_2}^{a_{\ell_2}+1} p_\ell^{a_\ell - 1}$; the other factors are the same. So the term in (2.6) acting on p_λ and giving $p_{\lambda'}$ is $2\ell p_{\ell_1} p_{\ell_2} \frac{\partial}{\partial p_\ell} p_\lambda = 2\ell a_\ell p_{\lambda'} = \langle \lambda' \mid \mathfrak{CJ}^{\sim} \mid \lambda\rangle p_{\lambda'}$ (actually there are two equal terms: $i = \ell_1, j = \ell_2$ or vice versa, hence the factor 2).

If λ' is as in Case 1 with $\ell_1 = \ell_2 = \ell/2$ then p_λ contains $p_{\ell/2}^{a_{\ell/2}} p_\ell^{a_\ell}$ and λ' contains $p_{\ell/2}^{a_{\ell/2}+2} p_\ell^{a_\ell - 1}$. So the only term in (2.6) acting on p_λ and giving $p_{\lambda'}$ is $\ell p_{\ell/2}^2 \frac{\partial}{\partial p_\ell} p_\lambda = \ell a_\ell p_{\lambda'} = \langle \lambda' \mid \mathfrak{CJ}^{\sim} \mid \lambda\rangle p_{\lambda'}$.

The calculations for the remaining two cases are similar. □

Corollary 2.15.

$$\mathcal{CJ}^{\sim} = \sum_{i,j \geq 1} (i + j)(p_i p_j + p_{i+j}) \frac{\partial}{\partial p_{i+j}} + 2ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \tag{2.7}$$

Proof. $k - 1$ is the number of pairs (i, j) such that $i, j \geq 1$ and $i + j = k$. So the second summand in the first term of (2.7) gives $\sum_k k(k - 1)\frac{\partial}{\partial p_k}$. The other terms in (2.7) and (2.6) are the same. □

3. Combinatorics

3.1. **Lower triangular operators.** First, remind some classical combinatorial notation and facts; see [2] for proofs and more information.

Let $\mathbb{C}[x_1, \dots, x_n]^{S_n}$ be the algebra of symmetric polynomials in n variables; let also $\pi_n : \mathbb{C}[x_1, \dots, x_n]^{S_n} \to \mathbb{C}[x_1, \dots, x_{n-1}]^{S_{n-1}}$ be the operator acting as $\pi_n(P)(x_1, \dots, x_{n-1}) = P(x_1, \dots, x_{n-1}, 0)$. Denote by $\mathbb{C}[x]^S$ the projective limit of $\mathbb{C}[x_1] \stackrel{\pi_2}{\leftarrow} \mathbb{C}[x_1, x_2]^{S_2} \stackrel{\pi_3}{\leftarrow} \mathbb{C}[x_1, x_2, x_3]^{S_3} \stackrel{\pi_4}{\leftarrow} \dots$; it is an algebra (called the algebra of symmetric polynomial of infinitely many variables). It is isomorphic to the algebra $\mathbb{C}[p]$ of polynomials of the variables $p_1, p_2, \dots$ considered in Section 2.1 above; the algebra isomorphism $\mathfrak{S} : \mathbb{C}[p] \to \mathbb{C}[x]^S$ sends p_k to $x_1^k + x_2^k + \dots$, $k = 1, 2, \dots$. To keep notation simple we will often omit $\mathfrak{S}$ in formulas, or denote $\mathfrak{S} f \stackrel{\text{def}}{=} f(x) \in \mathbb{C}[x]^S$, where $f = f(p) \in \mathbb{C}[p]$.

For a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ denote by s_λ the Schur polynomial (see [2]) and by m_λ, the monomial symmetric function: $m_\lambda \stackrel{\text{def}}{=} \sum_{1 \leq i_1, \dots, i_k} x_{i_1}^{\lambda_1} \dots x_{i_k}^{\lambda_k}$. Both s_λ and m_λ are bases in $\mathbb{C}[x]^S$, where λ runs through the set Λ of all partitions.

The set Λ is a POS with respect to the *dominance order*: $\mu = (\mu_1 \geq \dots \geq \mu_k) \preceq \lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ if $\mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i$ for every $i = 1, \dots, k$ (if one partition is shorter than the other then it is padded by zeros before comparison).

Proposition 3.1 ([2]).

$$s_\lambda = \sum_{\substack{\mu \preceq \lambda \\ |\mu| = |\lambda|}} K_{\lambda\mu} m_\mu$$

where $K_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ (called the Kostka number) is the number of ways to fill the boxes of the Young diagram of μ with λ_1 ones, λ_2 twos, etc., so that the entries were nondecreasing in each row, and striclty increasing in each column.

In particular, $K_{\lambda\lambda} = 1$ for each $\lambda \in \Lambda$; one may take formally $K_{\lambda\mu} = 0$ if $\mu \not\preceq \lambda$.

Let Λ be a finite POS with the partial order $\preceq$; a linear space V with the basis e_λ indexed by $\lambda \in \Lambda$ will be called a *space with a POS basis*. A linear operator $A : V \to V$ is called *lower triangular* if

$$Ae_\lambda = \sum_{\mu \preceq \lambda} a_{\lambda\mu} e_\mu$$

for every $\lambda \in \Lambda$ and some constants $a_{\lambda\mu}$. (If $\preceq$ is a linear order then A is lower triangular in the usual sense.) Lower triangular operators in a space with a POS basis $\{e_\lambda, \lambda \in \Lambda\}$ form a linear space and share many of the properties of the usual lower triangular matrices:

Lemma 3.2. (1) *Eigenvalues of a lower triangular operator are equal to the diagonal elements of its matrix in the basis e_λ. In particular, a lower triangular operator is invertible if and only if all its diagonal elements are nonzero.*
(2) *A composition of two lower triangular operators is lower triangular. If a lower triangular operator is invertible then its inverse is lower triangular, too. Hence the set of invertible lower triangular operators is a group.*

Proof. Assertion 1 is obvious if the order on Λ is linear. A partial order can be extended it to a linear one; it remains to notice that a lower triangular operator remains lower triangular after such extension. The proof of assertion 2 is a trivial check. □

Definition 3.3. A lower triangular operator A is called *simple* if $a_{\lambda\lambda} \neq a_{\mu\mu}$ for all $\lambda, \mu \in \Lambda$ such that $\mu \prec \lambda$. A vector $v \in V$ is called *λ-regular*, $\lambda \in \Lambda$, if

$$v = e_\lambda + \sum_{\mu \prec \lambda} b_{\mu\lambda} e_\mu$$

for some constants $b_{\lambda\mu}$.

Theorem 3.4. *A simple lower triangular operator has, for any $\lambda \in \Lambda$, a unique λ-regular eigenvector v_λ. The eigenvectors v_λ, $\lambda \in \Lambda$, form a basis in V.*

Proof. Let $v_\lambda = e_\lambda + \sum_{\mu\prec\lambda} b_{\mu\lambda} e_\mu$. Then

$$\begin{aligned} Av_\lambda &= Ae_\lambda + \sum_{\mu\prec\lambda} b_{\mu\lambda} Ae_\mu = a_{\lambda\lambda} e_\lambda + \sum_{\nu\prec\lambda} a_{\nu\lambda} e_\nu + \sum_{\mu\prec\lambda} b_{\mu\lambda} \sum_{\nu\preceq\mu} a_{\mu\nu} e_\nu \\ &= a_{\lambda\lambda} e_\lambda + \sum_{\nu\prec\lambda} \big(a_{\lambda\nu} + \sum_{\mu:\nu\preceq\mu\prec\lambda} b_{\mu\lambda} a_{\mu\nu}\big) e_\nu \end{aligned}$$

By Property 1 of Lemma 3.2, the eigenvector v_λ satisfies the equation $Av_\lambda = a_{\lambda\lambda} v_\lambda$, which is equivalent to

$$(a_{\nu\nu} - a_{\lambda\lambda}) b_{\nu\lambda} = -a_{\lambda\nu} - \sum_{\mu:\nu\prec\mu\prec\lambda} a_{\nu\mu} b_{\mu\lambda} \tag{3.1}$$

for all $\nu \prec \lambda$.

Use now the induction by $\nu \preceq \lambda$. (3.1) is a linear equation for $b_{\nu\lambda}$ with the coefficient $a_{\nu\nu} - a_{\lambda\lambda} \neq 0$ by assumption, and the right-hand side containing only $b_{\mu\lambda}$ with $\nu \prec \mu \prec \lambda$ which are supposed to be unique by the induction bypothesis. So, $b_{\nu\lambda}$ is unique, too.

The transfer matrix from the standard basis e_λ to v_λ is lower triangular; all its diagonal elements are equal to 1. So it is invertible by Property 1 in Lemma 3.2, and v_λ, $\lambda \in \Lambda$, is a basis. □

3.2. **Twisted Schur polynomials.** By Theorem 2.14, $\mathcal{CJ}^\sim = \mathcal{CJ}_0 + \mathcal{R}$ where

$$\mathcal{CJ}_0 = \sum_{i,j\geq 1} (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j}$$

is the classical cut-and-join, and

$$\mathcal{R} = \sum_{i,j\geq 1} p_{i+j} ((i+j) \frac{\partial}{\partial p_{i+j}} + ij \frac{\partial^2}{\partial p_i \partial p_j}).$$

Both summands are diagonalizable. For the cut-and-join it is a classical statement:

Proposition 3.5 ([3])**.** *The eigenfunctions of the classical cut-and-join $\mathcal{CJ}_0$ are Schur polynomials s_λ, and the eigenvalue associated to $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ is*

$$\phi(\lambda) \stackrel{def}{=} \sum_{i=1}^{k} \lambda_i(\lambda_i - 2i + 1). \tag{3.2}$$

Recall that $\mathfrak{S} : \mathbb{C}[p] \to \mathbb{C}[x]^S$ is an algebra isomorphism sending p_i to $x_1^i + x_2^i + \dots$ for all $i = 1, 2, \dots$. Then for $\mathcal{R}$ there is the following proposition:

Proposition 3.6. *The operator $\mathfrak{S}\mathcal{R}\mathfrak{S}^{-1} : \mathbb{C}[x]^S \to \mathbb{C}[x]^S$ is the restriction of the operator $x_1^2 \frac{\partial^2}{\partial x_1^2} + x_2^2 \frac{\partial^2}{\partial x_2^2} + \dots$ to $\mathbb{C}[x]^S$. Its eigenfunctions are the monomial symmetric functions m_λ, and the eigenvalue associated to $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ is*

$$\psi(\lambda) \stackrel{def}{=} \sum_{i=1}^{k} \lambda_i(\lambda_i - 1). \tag{3.3}$$

Proof. If $p_i = \sum_{\ell=1}^\infty x_\ell^i$ then $\frac{\partial p_i}{\partial x_\ell} = i x_\ell^{i-1}$. So if $F \in \mathbb{C}[x]^S$ then $\frac{\partial F}{\partial x_\ell} = \sum_{i\geq 1} \frac{\partial F}{\partial p_i} \cdot i x_\ell^{i-1}$, and therefore $x_\ell^2 \frac{\partial^2 F}{\partial x_\ell^2} = \sum_{i\geq 1} \frac{\partial F}{\partial p_i} \cdot i(i-1) x_\ell^i + \sum_{i,j\geq 1} \frac{\partial^2 F}{\partial p_i \partial p_j} ij x_\ell^{i+j}$. Summation over $\ell \ge 1$ gives $\sum_{i\ge 1} i(i-1)\frac{\partial F}{\partial p_i}\sum_{\ell\ge 1} x_\ell^i + \sum_{i,j\ge 1} ij\frac{\partial^2 F}{\partial p_i \partial p_j}\sum_{\ell \ge 1} x_\ell^{i+j}. = \sum_{i\ge 1} i(i-1)p_i\frac{\partial F}{\partial p_i} + \sum_{i,j\ge 1} ijp_{i+j}\frac{\partial^2 F}{\partial p_i \partial p_j} = \mathcal{R}F$.

The proof of the other assertions is a simple verification. □

Let $\mathbb{C}[p]_n \subset \mathbb{C}[p]$ be the homogeneous component of degree n. Is is a space with a POS basis of Schur polynomials s_λ, $\lambda \in \Lambda_n$ where $\Lambda_n \subset \Lambda$ is the set of partitions $\lambda = (\Lambda_1 \ge \dots \ge \lambda_k)$ with $|\lambda| \stackrel{\text{def}}{=} \lambda_1 + \dots + \lambda_k = n$, with the dominance order.

Proposition 3.7. *The operator $\mathcal{CJ}^\sim : \mathbb{C}[p]_n \to \mathbb{C}[p]_n$ is lower trinangular. The eigenvalues of $\mathcal{CJ}^\sim$ are equal to*

$$\xi(\lambda) \stackrel{def}{=} \phi(\lambda) + \psi(\lambda) = 2\sum_{i=1}^k \lambda_i(\lambda_i - i). \tag{3.4}$$

Proof. Lower triangular operators form a vector space, so for the first assertion it suffices to prove that $\mathcal{CJ}_0$ and $\mathcal{R}$ are lower triangular. $\mathcal{CJ}_0$ is diagonal, hence lower triangular, in the basis s_λ. $\mathcal{R}$ is diagonal in the basis m_λ, so its matrix in the basis s_λ is $K^{-1}\operatorname{diag}(\psi(\lambda))K$ where $K = (K_{\lambda\mu})$ is the matrix of Kostka numbers. K is lower triangular [2] with respect to the dominance order, so $\mathcal{R}$ is lower triangular by assertion 2 of Lemma 3.2.

By Propositions 3.5 and 3.6 and assertion 1 of Lemma 3.2 the diagonal elements of $\mathcal{CJ}_0$ and $\mathcal{R}$ are equal to $\phi(\lambda)$ and $\psi(\lambda)$, respectively. Thus, the diagonal elements of $\mathcal{CJ}^\sim$ are $\phi(\lambda)+\psi(\lambda) = \xi(\lambda)$; by the same assertion 1 of Lemma 3.2 these are the eigenvalues of $\mathcal{CJ}^\sim$, too. □

In the rest of this subsection we are going to study eigenvectors of the operator $\mathcal{CJ}^\sim$.

Theorem 3.8. *The operator $\mathcal{CJ}^\sim$ is simple in the sense of Definition 3.3.*

Corollary 3.9 (of Theorem 3.8 and Theorem 3.4)**.** *For any partition λ the twisted cut-and-join operator has a λ-regular eigenvector $\widetilde{s}_\lambda \in \mathbb{C}[p]_n$, called twisted Schur polynomial, with the eigenvalue $\xi(\lambda)$ given by* (3.4)*. The twisted Schur polynomials form a basis in $\mathbb{C}[p]$.*

To prove Theorem 3.8 we need two lemmas.

Let $\lambda = (\lambda_1 \ge \dots \ge \lambda_k)$; let p, q be integers such that $1 \le p < q \le k$, $\lambda_p \ge \lambda_{p+1} + 1$ and $\lambda_q \le \lambda_{q-1} - 1$. Consider a partition $\mu = (\mu_1 \ge \dots \ge \mu_k)$ such that $\mu_p = \lambda_p - 1$, $\mu_q = \lambda_q + 1$ and $\mu_i = \lambda_i$ for all $i \ne p, q$; call the operation $\lambda \mapsto \mu$ *a (p,q)-move.* Apparently, a (p,q)-move preserves $|\lambda|$.

Lemma 3.10. *Let $|\mu| = |\lambda|$. Then $\mu \prec \lambda$ if and only if there esists a sequence of (p,q)-moves converting λ to μ.*

Proof. If μ is obtained from λ by a (p,q)-move then, obviously, $\mu \prec \lambda$; this proves the "if" part.

Notice now that if $\mu = (\mu_1 \ge \dots \ge \mu_k)$, $\lambda = (\lambda_1 \ge \dots \ge \lambda_k)$, and $\mu_1 = \lambda_1$ then $\mu \prec \lambda$ if and only if $\mu' \prec \lambda'$ where $\mu' = (\mu_2 \ge \dots \ge \mu_k)$ and $\lambda' = (\lambda_2 \ge \dots \ge \lambda_k)$. Thus, to prove the "only if" part it suffices to find a sequence of (p,q)-moves converting μ to a partition ν such that $\nu \prec \lambda$ and $\nu_1 = \lambda_1$; the rest of the proof is done by obvious induction.

Suppose that $\lambda_1 > \mu_1$. By assumption, $\mu_1 + \dots + \mu_i \le \lambda_1 + \dots + \lambda_i$ for all i and $\mu_1 + \dots + \mu_k = \lambda_1 + \dots + \lambda_k$, so there exists a $j \le k$ such that $\mu_j > \lambda_j$. Then the

$(1, j)$-move can be applied to μ; it increases μ_1. Repeating this several times, one obtains the required partition ν. □

Lemma 3.11. *The mapping $\xi : \Lambda \to \mathbb{Z}$ (where Λ is the set of partitions) is strictly monotonic: if $\mu \prec \lambda$ then $\xi(\mu) < \xi(\lambda)$.*

Proof. By Lemma 3.10 it suffices to prove that $\xi(\mu) > \xi(\lambda)$ if μ is obtained from λ by a (p, q)-move. One has $\mu_p = \lambda_p + 1$, $\mu_q = \lambda_q - 1$ and $\mu_i = \lambda_i$ for all other i, so

$$\begin{aligned}\xi(\mu) - \xi(\lambda) &= (\lambda_p + 1)(\lambda_p + 1 - p) - \lambda_p(\lambda_p - p) + (\lambda_q - 1)(\lambda_q - 1 - q) - \lambda_q(\lambda_q - q)\\ &= 2\lambda_p - 2\lambda_q - p + q + 2 > 0.\end{aligned}$$

□

Theorem 3.8 obviously follows from Lemma 3.11.

Remark 3.12. A similar calculations show that the eigenvalue functions ϕ and ψ of the classical cut-and-join and of the operator $\mathcal{R}$, respectively, are strictly monotonic, too. It proves that both operators are simple and therefore, diagonalizable, but it is actually a known fact — see Propositions 3.5 and 3.6 above.

By Corollary 3.9 twisted Schur polynomials $\widetilde{s}_\lambda$, $\lambda \in \Lambda_n$, form a basis in $\mathbb{C}[p]_n$. This, for all $n \in \mathbb{Z}_{\geq 0}$ together, implies that there exist coefficients ε_λ, $\lambda \in \Lambda$, such that $\exp(p_1) = \sum_{\lambda\in\Lambda} \varepsilon_\lambda \widetilde{s}_\lambda$.

Corollary 3.13 (of Corollary 2.13 and Corollary 3.9)**.**

$$\mathcal{H}^\sim(\beta, p) = \sum_{\lambda\in\Lambda} \varepsilon_\lambda \exp(\beta\xi(\lambda))\widetilde{s}_\lambda.$$

No explicit formula for the coefficients ε_λ is known yet; finding one is a challenging task for the future research.

Similar to Proposition 3.7 one proves that the twisted cut-and-join operator $\mathcal{CJ}^\sim$ is lower triangular in the basis of monomials symmetric functions, too: $\mathcal{CJ}^\sim(m_\lambda) = \sum_{\substack{\mu\preceq\lambda\\ |\mu|=|\lambda|}} \widetilde{c}_{\lambda\mu} m_\mu$ for all partitions λ. The matrix of $\mathcal{CJ}^\sim$ in this basis appears to be sparser than in the basis of Schur polynomials — many coefficients are indeed zeros:

Proposition 3.14. $\widetilde{c}_{\lambda\mu} = 0$ *if* $\ell(\lambda) \geq \ell(\mu) + 2$*, where ℓ means the number of parts in a partition.*

Proof. It is enough to prove that the operators $\frac{\partial}{\partial p_k}$ and $\frac{\partial^2}{\partial p_i \partial p_j}$ applied to the monomial symmetric polynomial m_λ will give a sum of monomial symmetric polynomials m_μ such that $\ell(\mu) < \ell(\lambda)$. By [4, Theorem 1 and Example 1],

$$m_\lambda = \sum_{\substack{\lambda\preceq\mu\\ \ell(\lambda)\geq\ell(\mu)\\ |\mu|=|\lambda|}} Q_{\lambda\mu} p_\mu.$$

Furthermore, $\frac{\partial p_\mu}{\partial p_k} = 0$ if k is not a part of the partition μ, otherwise $\frac{\partial p_\mu}{\partial p_k} = c \cdot p_\mu$ where $\ell(\nu) = \ell(\mu) - 1$. Thus,

$$\frac{\partial}{\partial p_k} \sum_{\substack{\lambda\preceq\mu\\ \ell(\lambda)\geq\ell(\mu)\\ |\mu|=|\lambda|}} Q_{\lambda\mu} p_\mu = \sum_{\substack{\lambda\preceq\nu\\ \ell(\nu)\leq\ell(\lambda)-1\\ |\nu|=|\lambda|}} Q'_{\lambda,\nu} p_\nu$$

for some $Q'_{\lambda,\nu}$. Finally by [5], $p_\nu = \sum_{\nu \preceq \nu'} R_{\nu'\nu} m_{\nu'}$, and by the interpretation of the coefficients $R_{\nu'\nu}$ in terms of Young diagrams [6], one obtains $R_{\nu'\nu} = 0$ if $\ell(\nu) > \ell(\nu')$. Thus, $\frac{\partial m_\lambda}{\partial p_k}$ is a sum of monomial symmetric functions $m_{\nu'}$ such that $\ell(\nu') \le \ell(\mu) \le \ell(\lambda) - 1$. Applying the same reasoning for $\frac{\partial^2}{\partial p_i \partial p_j}$ proves the proposition. □

The authors are planning to write a separate paper on the combinatorics of the twisted Schur polynomials and their parametric generalizations described below.

3.3. **Parametric Schur functions.** Consider a linear operator $\mathcal{CJ}_t \stackrel{\text{def}}{=} \mathcal{CJ}_0 + t\mathcal{R}$ where $t \in \mathbb{C}$; in particular, $\mathcal{CJ}_0$ is the classical cut-and-join and $\mathcal{CJ}_1 = \mathcal{CJ}^\sim$, the twisted cut-and-join.

Proposition 3.15. *The operator $\mathcal{CJ}_t : \mathbb{C}[p]_n \to \mathbb{C}[p]_n$ is lower trinangular. The eigenvalues of $\mathcal{CJ}_t$ are equal to $\phi(\lambda) + t\psi(\lambda)$ where $\phi(\lambda)$ and $\psi(\lambda)$ are defined by* (3.2) *and* (3.3)*, respectively.*

The proof is identical to that of Proposition 3.7 above.

The value $t \in \mathbb{C}$ is called *generic* if $\mathcal{CJ}_t$ is a simple operator. By Proposition 3.15 the set of generic t is given by a finite number of inequalities between linear functions; it is nonempty because $t = 1$ is generic by Theorem 3.8. Hence the set of non-generic $t \in \mathbb{C}$ is finite.

The eigenpolynomials of $\mathcal{CJ}_t$, for t generic, will be called *parametric Schur functions* and denoted by $\widetilde{s}_\lambda(t, p)$. They are polynomial in p and rational functions of the parameter t. We will sometimes omit the argument p in notation.

Example 3.16. $\widetilde{s}_{1^n}(t) = e_n$ (the elementary symmetric function) for any t. Indeed, $e_n = m_{1^n}$ (the monomial symmetric function), so it is an eigenfunction of $\mathcal{R}$. At the same time, $e_n = s_{1^n}$ (the Schur polynomial), so it is an eigenfunction of $\mathcal{CJ}_0$ and is 1^n-regular with respect to the basis of Schur polynomials (1^n is minimal with respect to the dominance order). So, e_n is an eigenfunction of $\mathcal{CJ}_t = \mathcal{CJ}_0 + t\mathcal{R}$.

Example 3.17. Let $\lambda = (\lambda_1 \ge \dots \ge \lambda_k) \in \Lambda$. For $t = -1$ the parametric Schur function $\widetilde{s}_\lambda(-1)$ is equal to $e_\lambda \stackrel{\text{def}}{=} e_{\lambda_1} \dots e_{\lambda_k}$. To prove it notice that $\mathcal{CJ}_{-1} = \mathcal{CJ}_0 - \mathcal{R}$ is a first order differential operator (i.e. a vector field). Therefore it satisfies the Leibnitz rule $\mathcal{CJ}_{-1}(fg) = f\mathcal{CJ}_{-1}(g) + \mathcal{CJ}_{-1}(f)g$. The polynomial e_n is an eigenfunction of $\mathcal{CJ}_{-1}$ by Example 3.16, so the Leibniz rule implies that e_λ is an eigenfunction, too. It is easy to see that e_λ is λ-regular with respect to the basis of Schur polynomials, so $e_\lambda = \widetilde{s}_\lambda(-1)$.

Thus, parametric Schur functions is a family containing classical Schur polynomials (at $t = 0$), twisted Schur polynomials (at $t = 1$), elementary symmetric functions (at $t = -1$) and monomial symmetric functions (at $t = \infty$). General structure of parametric Schur functions is not clear yet; we formulate a conjecture based on numerical experiments.

Express parametric Schur functions as linear conbinations of classical Schur polynomials:

$$\widetilde{s}_\lambda(t) = \sum_{\substack{\mu \preceq \lambda \\ |\mu| = |\lambda|}} \widetilde{a}_{\lambda\mu}(t) s_\mu.$$

Conjecture 3.18. *$a_{\lambda\mu}(t)$ are rational functions with integer coefficients. If $\lambda = (\lambda_1, ..., \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ then the common denominator of all $a_{\lambda\mu}$, $\mu \preceq \lambda$, is equal to the product*

$$\prod_{\substack{1 \leq q \leq k \\ 0 \leq p \leq \lambda_q - 1}} ((p-1)t + b_{p,q}) \tag{3.5}$$

where $b_{p,q} = p + 1 + \#\{j > q : \lambda_j \geq \lambda_q - p\}$.

In other words, the product is taken over the set of cells of the Young diagram of λ, and $b_{p,q}$ is the length of a hook starting at the cell (p, q) and going up to the first (longest) row and then right until the column becomes shorter than $\lambda_q - p$.

Example 3.19. $\widetilde{s}_{3^1} = s_{3^1} - \frac{2t}{2t+3} s_{1^1 2^1} + \frac{t(2t+1)}{(t+2)(2t+3)} s_{1^3}$; the common denominator is $(t + 2)(2t + 3)$.

Remark 3.20. The notion of a common denominator is defined up to a nultiplicative constant, so the constant terms with $p = 1$ in (3.5) should not be taken into account.

Remark 3.21. We cannot yet make any sensible conjecture about numerators of $a_{\lambda\mu}(t)$. In particular, they are not always decomposable into linear factors with integer coefficients. The minimal counterexample is the numerator of $a_{4^1 2^1, 2^2 1^2}(t)$ containing a quadratic factor irreducible over $\mathbb{Z}$.

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Higher School of Economics, Moscow, Russia, and Independent University of Moscow.

Email address: burman@mccme.ru

Higher School of Economics, Moscow, Russia

Email address: raphael.fesler@gmail.com