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Interval exchange transformations and measured foliations

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0. Introduction

In this paper we will use methods of Riemann surface theory, particularly that of Teichmüller theory, to solve a problem in topological dynamics. We are interested in the piecewise order-preserving isometries of intervals on the real axis, the so-called interval exchange maps. An interval exchange preserves Lebesgue measure and its multiples on the interval. We prove, solving a conjecture of Keane's, that *for "almost all" minimal interval exchange maps every invariant measure is a multiple of Lebesgue measure.*

Veech independently and with somewhat similar methods [18] has proved the same result.

Thurston asked a similar question about measured foliations on a C^∞ surface. We prove that for *"almost all" minimal measured foliations the transverse measure is determined up to scalar multiple by the topological structure of the foliation.*

In order to state these problems and results precisely, we begin with terminology and history of the problems. Given $\lambda \in (\lambda_1,\ldots,\lambda_n) \in \Lambda_n$ the cone of positive vectors in R^n, set $\beta_0 = 0$, $\beta_i = \sum_{j=1}^{i}\lambda_j$ and $X_i = [\beta_{i-1}, \beta_i)$. Let τ be a permutation of $\{1,\ldots,n\}$. Set $\lambda^\tau = (\lambda_{\tau^{-1}(1)},\ldots,\lambda_{\tau^{-1}(n)}) \in \Lambda_n$. Form the corresponding $\beta_i(\lambda^\tau)$ and X_i^τ. We define a map T from $I^\lambda = [0, \beta_n)$ to itself by

$$Tx = x - \beta_{i-1} + \beta^\tau_{\tau(i)-1}, \qquad x \in X_i,\ 1 \le i \le n.$$

T maps X_i isometrically onto $X^\tau_{\tau(i)}$ and is called the (λ, τ) interval exchange map. T is continuous except perhaps at $\{\beta_1,\ldots,\beta_{n-1}\}$ and it is right continuous there. If $\tau(i) + 1 = \tau(i+1)$ for some i then T is continuous at β_i and is actually an

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interval exchange on fewer than n intervals. We therefore assume $\tau(i) + 1 \neq \tau(i + 1)$ for all i.

It is obvious from the definition that T preserves Lebesgue measure on $[0, \beta_n)$. One says (λ, τ) is *uniquely ergodic* if every finite invariant measure is a multiple of Lebesgue measure.

Recall that τ is *reducible* if there exists $i < n$ such that $\tau\{1, \ldots, i\} = \{1, \ldots, i\}$. Otherwise τ is *irreducible*. If τ is reducible the I^{λ} decomposes into nonempty invariant subintervals and T cannot be uniquely ergodic. *We therefore restrict ourselves to irreducible permutations.*

Keane [8] proved that if τ is irreducible and the orbits of each β_i are infinite and distinct then (λ, τ) is *minimal*; every orbit is dense. In particular, when the components of λ are *rationally independent*, T is minimal. For $n = 2, 3$, minimality implies unique ergodicity, as the problem reduces to studying rotations of a circle where the result is well-known.

Contrary to a conjecture, Keynes and Newton [10], using work of Veech, produced a minimal nonuniquely ergodic (λ, τ) when $n = 5$ and later Keane [9] produced an example for $n = 4$ where the components are rationally independent.

THEOREM 1. *If τ is irreducible and $n \geq 4$ then for almost all $\lambda \in \Lambda_n$ (with respect to Lebesgue measure on Λ_n) the (λ, τ) interval exchange map is uniquely ergodic.*

This theorem solves a conjecture of Keane's [9]. (See also [15]). It was proved first for $n = 4$ by Veech [17] and, as mentioned above, for all n also by Veech independently.

Our approach is to place the problem in the context of measured foliations on a C^{∞} surface M, as developed in [14], and holomorphic differential forms on closed Riemann surfaces. Thurston used measured foliations to, among other things, study diffeomorphisms of closed surfaces. These are foliations with prescribed kinds of singularities and an invariant transverse measure. The set of measure equivalence classes MF has a natural piecewise linear structure and if the surface has genus $g \geq 2$, it is homeomorphic to $R^{6g-6} - \{0\}$. Measure equivalence can be thought of as the weakest equivalence generated by transverse measure preserving diffeomorphisms homotopic to the identity and Whitehead operations of collapsing saddle connections to higher order singularities.

The piecewise linear structure allows one to define a *measure* μ on MF invariant under coordinate changes.

We define a weaker *topological* equivalence on measured foliations as that generated by diffeomorphisms homotopic to the identity and Whitehead operations where the transverse measure *need not be* preserved. A measured foliation F is *uniquely ergodic* if any topologically equivalent foliation is measure equivalent to a multiple of F.

THEOREM 2. *Almost all (with respect to μ) foliations in MF are uniquely ergodic.*

COROLLARY 1. *Let X be a closed Riemann surface of genus $g \geq 2$; $H^0(X, \Omega^{\otimes 2})$ the vector space of holomorphic quadratic differentials on X. Then for almost all $q \in H^0(X, \Omega^{\otimes 2})$, the horizontal and vertical foliations of q are uniquely ergodic measured foliations.*

There is clearly a relation between the two theorems. If F is an *oriented* minimal foliation then a transverse segment has a first return map which is an interval exchange. Roughly speaking, the lengths of the intervals are local coordinates in MF which define μ. Conversely an interval exchange can be suspended to produce an oriented foliation. "Most" foliations in MF are not oriented, so Theorem 2 is in some sense Theorem 1 in a special nonorientable case.

Our final results concern a group action on a space of foliations. Let $\mathrm{Mod}(g) = \mathrm{Diff}^+ M/\mathrm{Diff}_0 M$, the *mapping class group*. It acts on MF by pull-back and therefore on $MF \times MF$ as well and preserves the measures μ and $\mu \times \mu$ respectively. Let PF be the projective space of foliations defined by $F_1 \sim F_2$ if $F_1 = \lambda F_2$. PF is the sphere S^{6g-7}. A set $E \subset PF$ has *measure zero* if the set of all its representatives has μ measure zero in MF. Two sets define the same *measure class* if their symmetric difference has measure zero.

Since $\mathrm{Mod}(g)$ is measure preserving on MF and $MF \times MF$, $\mathrm{Mod}(g)$ is measure class preserving on PF and $PF \times PF$.

THEOREM 3. $\mathrm{Mod}(g)$ *acts ergodically on $PF \times PF$.*

COROLLARY 2. $\mathrm{Mod}(g)$ *acts ergodically on PF.*

The idea behind the proofs of the first two theorems is to consider pairs of transverse measured foliations. A pair defines a *holomorphic quadratic differential* on a compact Riemann surface. If the foliations come from interval exchange maps and thus are oriented, the quadratic differential is the square of an *abelian differential* and has zeroes of orders determined by the interval exchange. On the various spaces of quadratic differentials determined by the orders of their zeroes, we define a flow coming from the *Teichmüller extremal maps* and a measure invariant under both the flow and the action of Mod(g). The measures are natural in that a set of interval exchange maps of zero (resp. nonzero) measure gives rise to a set of differentials of zero (resp. nonzero) measure. The invariance under Mod(g) gives rise to a measure preserving flow on the quotients.

We then prove these quotient measures are all *finite*. The last and main step is to show that there is a closed set of *measure zero* such that the positive orbit of any quadratic differential with a nonuniquely ergodic vertical foliation has Ω limit points in this set. We may then apply Poincaré recurrence to show that the quadratic differentials in any one of these various spaces with nonuniquely ergodic vertical foliations have measure zero.

We show that Theorem 3 is equivalent to showing the Teichmüller flow on the quotient of the quadratic differentials by Mod(g) is ergodic. To prove that, we will be able to follow Hopf's proof [5] of ergodicity for a conservative geodesic flow on a surface of constant negative curvature almost line for line.

In the first three sections we give the preliminaries on ergodic theory, the Teichmüller flow and building measured foliations from interval exchanges. Sections 4 and 5 are devoted to building the measures with the required properties and showing that the quotient measures are finite. In Section 6 we consider orbits of quadratic differentials.

We rely heavily on the theory of measured foliations as found in [14] and [6], on the former to build the measure and on the latter for the relationship between measured foliations and quadratic differentials. For the theory of Teichmüller extremal maps and quadratic differentials we refer to [1], [2], or [3]. References for the theory of the boundary of moduli space used in Section 5 can be found in [1] and [4].

I would like to thank the referees for making several valuable suggestions; particularly, how to vastly improve Section 3.

1. Convex sets of measures

We record here the ergodic theory needed later. Let (λ, τ) be an interval exchange, τ irreducible or F a measured foliation. Normalize Lebesgue measure so that the interval has unit length in the first case, or so that a transverse segment has unit length in the second. If (λ, τ) or F is *not* uniquely ergodic, the sets $\Sigma(\lambda, \tau)$ and $\Sigma(F)$ of invariant normalized measures on the interval or foliation are a *convex* set of positive dimension. Katok [7] and also Veech [16] proved that dim $\Sigma(\lambda, \tau) \leq [\frac{n}{2}]$. It is possible to prove by passing to a double cover of F and using the arguments of those papers that $\dim \Sigma(F) \leq 3g\text{-}3$ where M has genus g, although this will not be needed.

The extreme points of these convex sets are the ergodic measures and they are mutually singular. For ν ergodic, the Birkhoff ergodic theorem says if $f \in L^2[\nu]$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n(x)) = \int_{I^\lambda} f d\nu \text{ almost everywhere.}$$

The points for which this holds are called *generic* points for f and ν.

If F is an ergodic foliation, then for any f which is L^2 on a transversal J with transverse measure ν, $\lim_{N\to\infty} 1/N \sum_{n=1}^{N} f(T^n(x)) = \int_J f d\nu$ for almost all leaves L. Here the $T^n(x)$ are the successive intersections of L with J.

2. The Teichmüller flow and spaces of differentials

We let T_g be the Teichmüller space of closed Riemann surfaces of genus $g \geq 2, Q - \{0\} \to T_g$ the bundle of nonzero holomorphic quadratic differentials. The unit sub-bundle $Q_0 \subset Q - \{0\}$ is defined by the norm, $\|q\| = \int_X |q|$. For any $q \in Q$ and $t \in R$, q is the *initial* quadratic differential of the Teichmüller extremal map with dilatation

$$\mu = \frac{k\bar{q}}{|q|}; \quad \frac{1+k}{1-k} = e^{2t}; \quad -1 < k < 1.$$

Denote by q_t the terminal quadratic differential satisfying $\|q_t\| = \|q\|$. The map

$$q \to q_t$$

defines a flow on $Q - \{0\}$ called the *Teichmüller flow*. It is clearly also defined on Q_0. It is invariant under the action of Mod(g) on Q_0 and therefore defines a flow on $Q_0/\mathrm{Mod}(g)$.

For any $g \geq 2$ let $k = (k_1, \ldots, k_p)$ satisfy

i) each k_i is a positive even integer,

ii) $\sum_{i=1}^{p} k_i = 4g\text{-}4$.

Define $Q^k \subset Q$ and $Q_0^k \subset Q_0$ to be the set of q or normalized q such that

i) q is the square of an abelian differential;

ii) the distinct zeroes of q have orders k_i.

The initial differential belongs to Q_0^k if and only if the terminal differential does, so the Teichmüller flow is defined on each $Q_0^k/\mathrm{Mod}(g)$.

2.2. Suppose q is a nonzero quadratic differential on a compact surface of genus ≥ 2. Then $\mathrm{Im}\, q^{1/2} dz$ and $\mathrm{Re}\, q^{1/2} dz$ define two transverse measured foliations denoted F_q and F_{-q}; F_q is the *horizontal* foliation, and F_{-q} the *vertical*. The measures are respectively, $|\mathrm{Im}\, q^{1/2} dz|$ and $|\mathrm{Re}\, q^{1/2} dz|$.

Conversely, suppose F_1 and F_2 are transverse foliations. This means their singularities coincide and are the same order, and elsewhere they are transverse in the usual sense. Together they define a complex structure X on M and a quadratic differential q such that $F_q = F_1$, $F_{-q} = F_2$. We will write

$$q = (F_1, F_2) = (F_q, F_{-q}).$$

For any F_1 the set of transverse F_2 has nonempty interior in MF, [12]. Since scalar multiplication of measures does not affect transversality, to any set E of measured foliations, let

$$\rho(E) = \{q \in Q_0 \quad \text{such that } F_{-q} \in E\}.$$

We may describe the Teichmüller flow in this context by $t, q = t,\ (F_1, F_2) \to (e^{-t}F_1, e^t F_2) = q_t$ since the Teichmüller map multiplies the distance between vertical leaves by e^t (expands on horizontal leaves) and multiplies the distance between horizontal leaves by e^{-t}.

3.

We need to show in this section that we can associate to each interval exchange (λ, τ), τ irreducible, a holomorphic 1-form $\omega(\lambda, \tau)$ such that the foliation of $\operatorname{Re}\omega$ defines (λ, τ) as Poincaré or first return map on some transverse interval. The quadratic form ω^2 lies in some $Q^k/\mathrm{Mod}(g)$ where k depends only on τ. There is no canonical way of determining a marking on the underlying Riemann surface explaining why the quotient by $\mathrm{Mod}(g)$ is necessary.

Given λ, τ, define points $V_j^{\pm}, 1 \le j < n$, by

$$V_j^+ = \beta_j + \sqrt{-1}\sum_{l=1}^{j}(\tau l - l); \qquad V_j^- = \beta_j(\lambda^\tau) - \sqrt{-1}\sum_{l=1}^{j}(\tau^{-1}l - l).$$

The irreducibility of τ (and τ^{-1}) implies $\operatorname{Im}V_j^+ > 0$, $\operatorname{Im}V_j^- < 0$, $1 \le j < n$. Also define

$$V_0^{\pm} = 0, \qquad V_n^{\pm} = \beta_n = \beta_n(\lambda^\tau).$$

Let $P(\lambda, \tau)$ be the polygon whose sides are the segments $S_j^{\pm} = V_{j-1}^{\pm}V_j^{\pm}$, $1 \le j \le n$. It is easy to check that S_j^+ and $S_{\tau j}^-$ are parallel and of the same length. Gluing them by parallel translation for $1 \le j \le n$ yields a holomorphic 1-form $\omega(\lambda, \tau)$ on a Riemann surface $M(\lambda, \tau)$. The corresponding vertical foliation given by $\operatorname{Re}\omega$ has first return (λ, τ) on $[0, \beta_n) \subset M(\lambda, \tau)$. We may now let λ vary.

The vertices $V_j^{\pm}$, $1 \le j < n$ are identified in even numbers $2m$. Since from each such vertex a vertical line emanates, if $m \ge 2$ this identified vertex is a zero of order $2m$-2 of ω^2. It is possible for $m = 1$, so only two among these are identified. Since $\tau(j+1) \neq \tau(j)+1$, this means this pair of vertices is also identified with 0 or β_n or both and the identified point is not a zero of ω. Below we list these possibilities, for some $1 \leq j < n$,

$$\begin{aligned} \tau(j) &= n \\ \tau(j+1) &= 1 \\ \tau(1) &= \tau(n)+1; \end{aligned} \tag{3.1}$$

for some $1 \leq j < n$,

$$\begin{aligned} \tau(j+1) &= 1 \\ \tau(1) &= \tau(j)+1; \end{aligned} \tag{3.2}$$

for some $1 \leq j < n$,

$$\begin{aligned} \tau(j+1) &= \tau(n)+1 \\ \tau(j) &= n. \end{aligned} \tag{3.3}$$

If 3.2 holds, 0, V_j^+, and $V_{\tau(j)}^-$ are identified and are not a zero. If 3.3 holds, the same is true of V_j^+, $V_{\tau(n)}^-$, and β_n and if 3.1 holds it is true of V_j^+, $V_{\tau(j)}^-$, 0, and β_n. Note, if none of these hold, both 0 and β_n become zeroes under the identifications.

Clearly the genus of the surface and the orders $k_1, \ldots, k_p$ of the zeroes depend only on τ and not on λ. We summarize our construction as:

PROPOSITION 3.1. *For each* (λ, τ) *there is a canonical holomorphic abelian differential* $\omega(\lambda, \tau) \in Q^k/\mathrm{Mod}(g)$ *such that* $\operatorname{Re}\omega$ *defines* (λ, τ) *on some transverse interval where* $k = (k_1, \ldots, k_p)$ *depends only on* τ. *If* τ *does not satisfy any of* 3.1–3.3, *then* $n = \Sigma_{i=1}^p (k_i/2 + 1) + 1$. *Otherwise* $n > \Sigma_{i=1}^p (k_i/2 + 1) + 1$.

Proof. Only the last two statements need be checked. If none of 3.1–3.3 are satisfied, all $V_j^{\pm}$, $1 \leq j < n$, are zeroes and they are identified in even numbers $2m$ to a zero of order $k_i = 2m - 2$. The total number is $2n - 2$ so $2n - 2 = \Sigma(k_i + 2)$ giving the first statement. In the second case, n is the same but now there are fewer zeroes so the inequality results.

LEMMA 3.2. *Suppose* τ *satisfies any one of* 3.1–3.3. *Then there is a subinterval* J *of* $[0, \beta_n)$, *so the induced first return on* J *has fewer than* n *intervals.*

Remark. This lemma was proved first in [16].

Proof. Suppose 3.1 holds. Consider the subinterval $J = [\beta_1, \beta_{n-1})$. The leaves of $\operatorname{Re}\omega$ leaving β_1 and β_{n-1} in the positive direction each encounters a zero before returning to $I^\lambda = [\beta_0, \beta_n)$, and therefore also to J. We can therefore form a new segment J' transverse to $\operatorname{Re}\omega$ with these zeroes as endpoints such that the interval exchanges induced by $\operatorname{Re}\omega$ on J and J' are the same. The interval exchange on J' cannot satisfy any of 3.1–3.3 since its endpoints are singularities. By Proposition 3.1, J' and therefore J have less than n intervals. If 3.2 holds but not 3.3, let $J = [\beta_1, \beta_n)$, and if 3.3 holds but not 3.2, let $J = [\beta_0, \beta_{n-1})$. If 3.2 and 3.3 both hold, let $J = [\beta_1, \beta_{n-2})$, finishing the proof.

Fix τ irreducible. For any set $E \subset \Lambda^n$ let $\rho(E) = \{\omega^2 \in Q_0^k$ such that $\operatorname{Re}\omega$ has a transverse interval with corresponding interval exchange in E with given $\tau\}$.

The point of our construction in this section is that $\rho(E) \neq \varnothing$. In fact, as we shall see, for each (λ, τ) there are families of ω inducing (λ, τ) other than the canonical $\omega(\lambda, \tau)$.

4. Measures on Q_0 and Q_0^k

Recall from Part 2.2 and Section 3 that to any set E in *MF* or set E of (λ, τ), we have associated a set $\rho(E) \in Q_0$ or $\rho(E) \in Q_0^k$ such that the vertical foliation of any $q \in \rho(E)$ is in E.

The object of this section is to prove

PROPOSITION 4.1. *There exist measures μ on MF, μ_0 on Q_0, and μ_k on Q_0^k such that*

i) *μ_0 and μ_k are invariant under the Teichmüller flow.*

ii) *μ_0 and μ_k are* $\operatorname{Mod}(g)$ *invariant.*

iii) *$\mu(E) = 0$ if and only if $\mu_0(\rho(E)) = 0$.*

iv) *If τ does not satisfy* 3.1–3.3, *and $E \subset \Lambda^n$ is a set of λ of zero (resp. nonzero) Lebesgue measure, then $\mu_k(\rho(E)) = 0$, (resp. > 0). If τ satisfies one of* 3.1–3.3, *and E has nonzero measure, $\mu_k(\rho(E)) > 0$.*

4.1. It seems difficult to describe measures on Q_0 and Q_0^k intrinsically. Rather we have to use the linear structure of *MF* to build these measures piece by piece. We need to recall some definitions and results from [14].

A set $\gamma_1, \ldots, \gamma_{3g-3}$ of simple closed curves on M is *admissible* if

i) $\gamma_i \cap \gamma_j = \varnothing, i \neq j$,

ii) γ_i is not homotopic to zero,

iii) γ_i is not homotopic to γ_j, $i \neq j$.

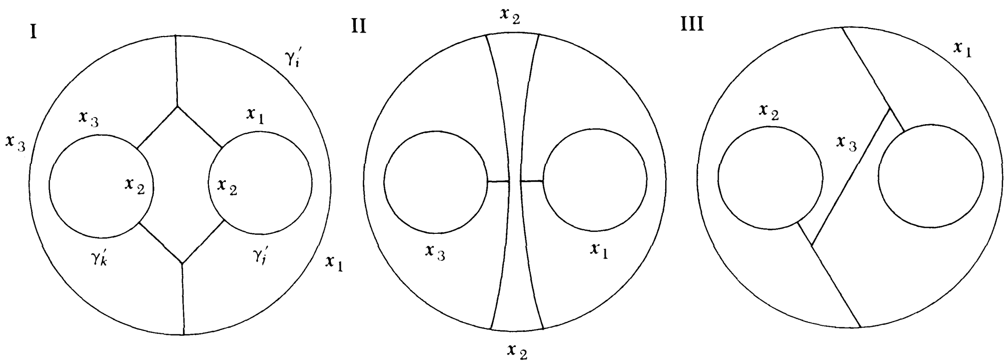
The set of γ_i bound $2g$-2, three holed spheres, "pairs of pants". We further require

iv) no γ_i occurs twice on the boundary of any pair of pants.

Any measured foliation F can be put in *normal position*, with respect to $\{\gamma_i\}$. This means there is F' measure equivalent to F and γ_i' homotopic to γ_i, the

$\{\gamma_i'\}$ still satisfying (i)–(iv), such that each γ_i' is either transverse to F' or a closed leaf possibly passing through singularities of F'.

Denote by $F(\gamma)$ the transverse length of any curve γ closed or not. We are interested in the case when all γ_i' are transverse. There are three possible cases for each pair of pants.



I occurs when the length of each γ' is less than the sum of the other two. The x_i are lengths between the critical points and satisfy

$$x_1 + x_3 = F(\gamma_i'), \qquad x_1 + x_2 = F(\gamma_j'), \qquad x_2 + x_3 = F(\gamma_k').$$

II occurs when one curve is longer than the sum of the other two (three possibilities). Here we have

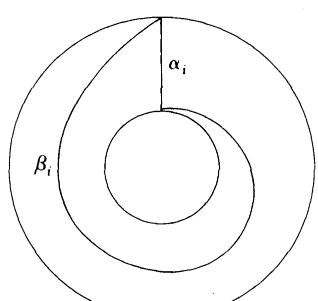
$$F(\gamma_i') = x_1 + 2x_2 + x_3, \qquad F(\gamma_j') = x_1, \qquad F(\gamma_k') = x_3.$$

III is a limiting case of I and II where one length is the sum of the other two. Here

$$F(\gamma_i') = x_1 + x_2, \qquad F(\gamma_j') = x_1, \qquad F(\gamma_k') = x_2, \qquad \text{and } x_3 = 0.$$

In all cases, the pair of pants has two singularities in its interior, in the last case a saddle connection joining them.

Canonically choose a singularity in each pair of pants. Enlarge each γ_i' to an annulus and choose a pair of curves α_i and β_i that cross the annulus joining the singularities in the two pairs of pants and differ by a twist about γ_i'.



The curves α_i and β_i can always be chosen so that they are either transverse to F' or saddle connections. Let s_i and t_i be the lengths of α_i and β_i. Then among the three numbers $F(\gamma_i')$, s_i, t_i, one is always the sum of the other two. Following the notation of [14, p. 102], whenever a triple (k_1, k_2, k_3) is such that one number is the sum of the other two, we say they are in δ $(< \nabla)$. We say a pair of pants is in δ $(< \nabla)$ if Case III holds above. The foliation F is completely determined by the 3g-3 triples $\{F(\gamma_i'), s_i, t_i\}$ in δ $(< \nabla)$.

There is a double cover $\tilde{M}_F$ of M ramified over the singularities with canonical projection Π such that Π^*F is given by a closed 1-form φ_F. Let σ: $\tilde{M}_F \to \tilde{M}_F$ be the canonical involution. The homology of $\tilde{M}$, odd with respect to σ, is generated by the lifts of curves joining the singularities, and is a free abelian group of rank 6g-6.

Suppose $U \subset MF$ is a set of foliations with the property that all γ_i are transverse for any $F \in U$ and if $F_1, F_2 \in U$ and D is any pair of pants with respect to $\{\gamma_i\}$, then D is either of Case I for both F_1 and F_2 or Case II for both. We say U is of *constant type*. MF is a disjoint union of such U together with lower dimensional sets where some γ_i is a leaf or some pair of pants is in δ $(< \nabla)$.

In any such U we may identify all odd homology groups $H_1(\tilde{M}_F, Z)^-$ and choose the φ_F in a continuous way giving rise via the 1-form φ_F to a homomorphism

$$\varphi_F\colon H_1(\tilde{M}_F, Z)^- \to R.$$

Define a 1-1 map $U \to \mathrm{Hom}(H_1(\tilde{M}_F, Z)^- \to R)$ by

$$F \to \varphi_F.$$

Let $\alpha_1, \ldots, \alpha_{6g\text{-}6}$ be a basis for $H_1(\tilde{M}, Z)^-$. For any set $A \subset U$, set $\mu(A) =$ Lebesgue measure of $\{\varphi_F(\alpha_1), \ldots, \varphi_F(\alpha_{6g\text{-}6})\} \subset R^{6g\text{-}6}$; $F \in A$. We define μ to be zero on the complement of the sets U and thus μ is defined on MF.

LEMMA 4.2. *μ is well-defined.*

Proof. On any U, if $\beta_1, \ldots, \beta_{6g\text{-}6}$ also generate $H_1(\tilde{M}, Z)^-$, $\{\beta_i\}$ and $\{\alpha_i\}$ are defined in terms of each other by integer valued matrices which are inverses of each other. They therefore have determinant either 1 or -1.

But a matrix of determinant ± 1 transforming the $\varphi_F(\alpha_i)$ to the $\varphi_F(\beta_i)$ is measure preserving.

LEMMA 4.3. *μ is preserved by* Mod(g).

Let $f \in \mathrm{Mod}(g)$. First note the image under f of a set of foliations for which $F(\gamma_i) = 0$, some i or a set for which some pair of pants is in δ ($< \nabla$) again has measure zero because the image is defined locally by a set of equations.

Consider next $f(U_1) \cap U_2$ where U_1 and U_2 have constant type. Let $\alpha_1, \ldots, \alpha_{6g-6}$ be a homology basis defined by U_1 and $\beta_1, \ldots, \beta_{6g-6}$, defined by U_2. The map f^{-1} lifts to $\tilde{f}^{-1}$ on the double covers, $\tilde{f}_*^{-1}(\beta_i) = \sum_{j=1}^{3g-3} a_{ij}\alpha_j$ where $\det(a_{ij}) = \pm 1$. Then

$$\varphi_{f^*(F)}(\beta_i) = \varphi_F\left(\tilde{f}_*^{-1}(\beta_i)\right) = \varphi_F(\Sigma a_{ij}\alpha_j) = \sum_{j=1}^{3g-3} a_{ij}\varphi_F(\alpha_j)$$

and once again the $\varphi_F(\alpha_j), \varphi_{f^*(F)}(\beta_i)$ are related by a measure preserving transformation.

Define $\mu \times \mu$ to be the measure of $MF \times MF$ and the induced measure on $Q - \{0\} \subset MF \times MF$. We define a measure μ_0 on Q_0 by

$$\mu_0(E) = \mu \times \mu\{\lambda q \mid 0 < \lambda \leq 1, q \in E\}.$$

Since $\mathrm{Mod}(g)$ preserves $\mu \times \mu$ and is norm preserving as well, $\mathrm{Mod}(g)$ preserves μ_0. This proves (ii) of Proposition 4.1.

LEMMA 4.4. *The Teichmüller flow on Q_0 preserves μ_0.*

Proof. The flow (F_q, F_{-q}), $t \to (e^{-t}F_q, e^tF_{-q})$ multiplies all lengths in the first factor by e^{-t}, all lengths in the second by e^t. It therefore preserves $\mu \times \mu$ and since it is also norm preserving the lemma is proved, giving (i) of Proposition 4.1. Finally since every foliation has a set of transverse foliations of positive measure, (iii) follows.

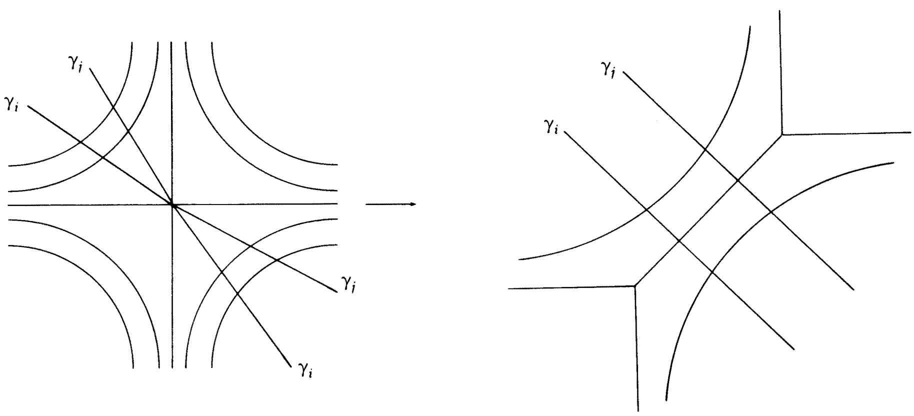
4.2. We would like to make a similar construction for Q^k. The situation however is more complicated as the set of orientable foliations in MF is horribly complicated. Instead we by-pass MF and work with Q^k directly. The advantage is that Q^k is given locally by a set of equations. The measure then will be Lebesgue measure induced by local coordinates.

Recall first [6, p. 231] that a closed curve $\gamma\colon S^1 \to M$ is *quasitransverse* for F if at every point $t \in S^1$ either $\gamma(t)$ is a singularity or γ is transversal to F near t or an inclusion in a leaf. If $\gamma(t)$ is a singularity at least one open sector on each side separates the incoming and outgoing parts of the curve.

Suppose now $q_0 = \omega_0^2 \in Q^k$ and $\{\gamma_i\}$ is an admissible system. With respect to the foliation $F_{q_0} = \mathrm{Im}\,\omega_0$, each γ_i can be represented by either a transverse or quasitransverse curve. The γ_i may intersect at a singularity, even themselves, but may not cross. We would like to write equations for $\mathrm{Im}\,\omega$, ω^2 near ω_0^2. Assume no

γ_i contains a saddle connection other than the multiple zeroes. As before, this only assumes ω_0^2 does not lie on a low dimensional set in Q^k. Then if γ_i is transverse (resp. quasitransverse), for $\operatorname{Im}\omega_0$, the same γ_i is transverse (resp. quasitransverse) for $\operatorname{Im}\omega_0$ nearby. In particular it enters and leaves the same sectors at the zeroes.

We may take a foliation equivalent to F_{q_0} obtained by expanding multiple zeroes to saddle connections so that the quasitransverse curves are now transverse. This produces a set of equations in $F_{q_0}(\gamma_i)$ and s_i. Some pairs of pants might be in δ ($< \nabla$) producing equations. Other equations come from the above procedure. If γ_i is quasitransverse to begin with, some equation in s_i, and possibly other variables, will hold. If γ_i is transverse to begin with, there will be *no* equations in s_i as there is generically no saddle connection across γ_i. We illustrate with the example below. The equation satisfied is $s_i = s_j$.



The same set of equations must hold for $\operatorname{Im}\omega$ near $\operatorname{Im}\omega_0$. We can obviously do the same for $\operatorname{Re}\omega_0$. Now pick a maximum set of independent variables ($F_q(\gamma_i)$, s_j $F_{-q}(\gamma_k)$, s_l') for these two sets of equations. There can be no other defining equation for Q^k near q_0, for we can vary these independent variables in arbitrarily small amounts and still get transverse foliations defining a point in Q^k. They form local coordinates and any other length between zeroes is a linear combination of these. We emphasize that *if* γ_i *is transverse for* $\operatorname{Im}\omega_0$, s_i *is a local coordinate*. The difficulty with defining a measure in terms of these coordinates is that (ii) of Proposition 4.1 might not be satisfied. We therefore have to choose carefully the admissible system.

4.3. Given $q_0 \in Q^k$ say an admissible system $\{\gamma_i\}$ is *canonical* for $\operatorname{Im}\omega_0$ if there exist neighborhoods U of q_0 in Q^k and U' of $\operatorname{Im}\omega_0$ in MF such that $\{\operatorname{Im}\omega, \omega^2 \in U\} = U' \cap \{F \,|\, \text{each pair of pants is in } \delta\ (< \nabla) \text{ and } s_i = n_i F(\gamma_i)$

for $n_i \in Z$ and $r = \sum_{i=1}^{p}(k_i/2 - 1)$ values of $i\}$. Note that if a pair of pants is in δ $(< \nabla)$, it contains a multiple zero.

Similarly we may define $\{\beta_i\}$ canonical for $\operatorname{Re}\omega_0$. These will, in general, not be the same set. Before proving these curves exist we will use them to define the measure and show (i), (ii), (iv) hold.

Since all pairs of pants are in δ $(< \nabla)$ it is easy to see there are g "free" values for the lengths of γ_i. For local coordinates near q_0 we take g "free" $F_q(\gamma_i)$ and g "free" $F_{-q}(\beta_i)$ and $3g\text{-}3 - \sum_{i=1}^{p}(k_i/2 - 1)$ remaining s_i and $3g\text{-}3 - \sum_{i=1}^{p}(k_i/2 - 1)$ remaining s_i' not defined by $s_i = n_i F_q(\gamma_i)$, $s_i' = n_i' F_{-q}(\beta_i)$. We define the measure on U as Lebesgue measure on $R^{4g\text{-}3\text{-}r} \times R^{4g\text{-}3\text{-}r}$ induced by local coordinates, and then a measure μ_k on Q_0^k as before.

Proposition 4.4. *μ_k is well-defined and satisfies* (i), (ii) *and* (iv) *of Proposition* 4.1.

Proof. The proof if it is well-defined depends on the fact that all other transverse lengths between zeroes are integral linear combinations of these coordinates. When all pairs of pants are in δ $(< \nabla)$, all lengths are integral linear combinations of s_i, $F(\gamma_i)$ and in turn, any $F(\gamma_i)$ can be expressed as an integral linear combination of the "free" $F(\gamma_i)$. Now if $\{\alpha_j\}$ is any other canonical system for $\operatorname{Im}\omega_0$, we may express the coordinates of this system by an integer valued matrix of the others and vice versa. As in Lemma 4.3, this is measure preserving. The same holds for $\operatorname{Re}\omega_0$.

Property (i) is the same as above. To prove (ii), notice that if $\{\gamma_i\}$ is canonical, then the condition $s_i = n_i F(\gamma_i)$ holds if and only if there is a saddle connection joining the two zeroes across γ_i. Then $\{f(\gamma_i)\}$ is canonical for $F_{f(q)}$. If α_i' is the curve across $f(\gamma_i)$ defining s_i' then $f^{-1}(\alpha_i')$ crosses γ_i, joining the singularities in the two pairs of pants. Therefore $f^{-1}(\alpha_i') - \alpha_i$ is homologous to a sum of the boundary cycles of the two pairs of pants, where α_i defines s_i. Recall that the length of a transverse curve is the absolute value of the integral of the 1-form defining the foliation. Therefore

$$s_i' = F_{f(q)}(\alpha_i') = F_q\big(f^{-1}(\alpha_i')\big) = \pm s_i \pm \sum n F_q(\gamma)$$

where the sum is over the boundary curves and $n \in Z$. The $\pm$ depends on the situation. The matrix for f is therefore

$$\begin{bmatrix} I & * \\ 0 & J \end{bmatrix}$$

where I is the $g \times g$ identity matrix and J is a $3g\text{-}3\text{-}r \times 3g\text{-}3\text{-}r$ diagonal matrix with ± 1 as the entries on the diagonal. The matrix has determinant ± 1. Once again f preserves μ_k since obviously the same can be done for $\operatorname{Re}\omega$.

For property (iv), first note that if τ does not satisfy any of 3.1–3.3, each λ_i is a transverse length between zeroes for $\operatorname{Re}\omega$. By Proposition 3.1 the number of intervals $n = \sum_{i=1}^{p}(k_i/2 + 1) + 1$ where $\sum_{i=1}^{p}k_i/2 = 2g\text{-}2$. We have $n = 4g\text{-}3 - \sum_{i=1}^{p}(k_i/2 - 1) = \dim_R \operatorname{Re} Q^k$. Thus the independent lengths λ_i, $i = 1,\ldots,n$, must form local coordinates for $\operatorname{Re} Q^k$ (not necessarily the coordinates of a canonical system). The coordinates of a canonical system for $\operatorname{Re}\omega$ are linear combinations of these, and therefore sets of zero and nonzero measure are sent to sets of zero and nonzero measure in the coordinates $(F_{-q}(\gamma_i), s_j')$. By Fubini, the same is true in Q^k and therefore Q_0^k.

If τ satisfies 3.1, for example, and E is a set of λ's of nonzero measure, then the measure of

$$\{(\lambda_2,\ldots,\lambda_{n-1}) \mid \text{there exists } \lambda_1, \lambda_n \text{ so that } \lambda \in E\}$$

is also nonzero by Fubini. The set $\{\lambda_2,\ldots,\lambda_{n-1}\}$ are local coordinates for Q^k and we proceed as above. The other possibilities for τ are similar. This completes the proof of Proposition 4.1.

COROLLARY. *If Theorem* 1 *holds for $m < n$, it holds for τ satisfying* 3.1, 3.2 *or* 3.3.

Proof. If there were a set of (λ, τ) of nonzero measure which were not uniquely ergodic, by (iv) of Proposition 4.1 there would be a set of quadratic differentials on Q_0^k with nonuniquely ergodic vertical foliations of positive measure. Again by (iv) and Lemma 3.2 there would be a corresponding set of exchange maps on fewer than n intervals, contradicting the hypothesis.

4.4. We proceed now to show canonical curves exist for any $q_0 \in Q^k$. We will show it for $\operatorname{Im}\omega_0$. Again referring to [6], we say $\gamma(t)$ is *increasing* if $\operatorname{Im}\omega(\gamma'(t)) > 0$. In particular, $\gamma(t)$ is transverse to the foliation. We say x *leads* to y if there is an increasing curve from x to y.

LEMMA 4.5. *There are pairwise disjoint simple closed transverse curves $\gamma_1,\ldots,\gamma_l$ such that every saddle connection of $\operatorname{Im}\omega_0$ (other than multiple zeroes) is cut by a γ_i.*

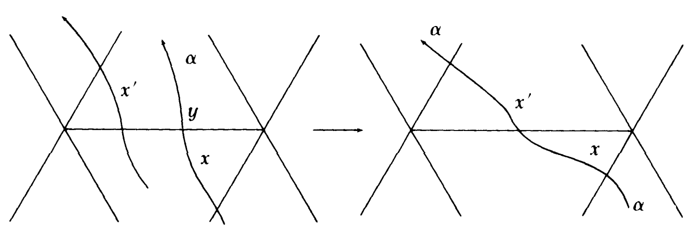
Proof. Begin by taking a saddle connection and points x and y on opposite sides so x leads to y across the connection. By [6, Prop. 2.2] every point leads to every other so y leads to x. Together the two curves form a transverse simple closed curve. If there is a second saddle connection, repeat the procedure. The new curve may intersect the first one, but by a simple connecting disconnecting trick as in Figure 4 of [6], we get a simple curve intersecting both. We continue in this manner until all saddle connections are crossed. By Proposition 2.5 of [6], two curves not intersecting the same saddle connections cannot be homotopic.

Now cut M along these transverse curves. The complement has no saddle connections and it has a transverse boundary.

LEMMA 4.6. *There are disjoint transverse curves $\gamma_{l+1},\dots,\gamma_s$ such that the complement of $\gamma_1 \cup \cdots \cup \gamma_s$ are spheres with holes, each containing only one (multiple) zero.*

Proof. In each component U of $M - \{\gamma_1,\dots,\gamma_l\}$, every noncritical point leads to every other. For just as in [6], the set of points where a given point leads is an open subset of U whose boundary is contained in the union of the boundary of U, closed leaves of F, or saddle connections. There are no saddle connections, hence no closed leaves and the open set must coincide with U. Notice that if a leaf intersects a boundary component it leads to points on both sides so the leaf is not in the boundary of the open set.

By the same argument, as in Proposition 2.6 [6], in each component the transverse curves generate homology with real coefficients so we may continue to find transverse curves until every component is a sphere with holes. If a component contains two distinct zeroes join them by a transverse curve β. This can be done by taking a sector for each, for one of which a curve leaving the zero is increasing, the other decreasing, and then joining a point in each sector and then the two points to the zeroes. Now take x and y on opposite sides of β where x leads to y across β. Join y to x by an increasing curve. The latter segment may also intersect β, but as the figure shows, we may always find a simple transverse curve intersecting β only once.

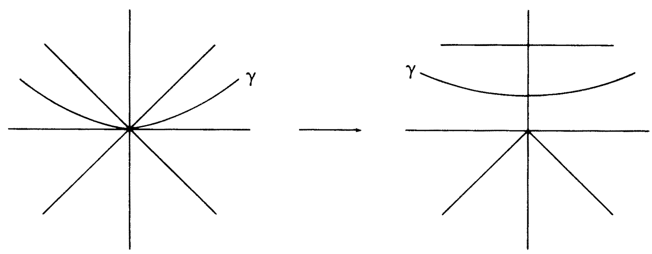


The transverse curve is dividing since the component is a sphere and the zeroes lie in different components because β crosses only once. Continuing this process proves the lemma.

LEMMA 4.7. *Canonical curves exist.*

By the last lemma, every component is now a sphere with only one multiple zero. If a zero is double, the component is a pair of pants and we may ignore it.

Suppose now a component has a zero of order $k \geq 4$. Pick two sectors separated by exactly two others. For one sector, curves leaving will be increasing, the other decreasing. We may again find a quasitransverse curve γ joining the zero to itself at those sectors. We find an equivalent foliation by expanding the zero so that this curve is now transverse. The curve separates a zero of order 2 from a zero of order $k - 2$.



We continue in this way until all zeroes are of order 2. Then every pair of pants is in δ ($< \nabla$) since the two zeroes inside coalesce to a zero of order 2. Moreover, every time the last step is performed, we get a saddle connection crossing the curve γ_i and no other. This gives $s_i = n_i F_{q_0}(\gamma_i)$ and this happens $k_i/2 - 1$ times for each zero of order k_i. This constructs a set of curves for $\operatorname{Im} \omega_0$ and as remarked before, the same set of equations holds in a neighborhood, proving the lemma.

5. The quotient measures $\mu_0(Q_0/\mathrm{Mod}(g))$ and $\mu_k(Q_0^k/\mathrm{Mod}(g))$ are finite.

The essential fact used in this section is that $T_g/\mathrm{Mod}(g)$ can be compactified by adding Riemann surfaces with nodes, and $Q_0/\mathrm{Mod}(g)$ and $Q_0^k/\mathrm{Mod}(g)$ can be compactified by adding differentials with unit norm on the surfaces with nodes, and also adding the zero differentials. The former are allowed simple poles at the nodes and only occur in the case of $Q_0 - Q_0^k$.

Let $\overline{T}_g$ be the union of T_g and its boundary spaces, $\overline{T}_g/\mathrm{Mod}(g)$, the compactified moduli space. Let $X \in \overline{T}_g/\mathrm{Mod}(g)$ be a Riemann surface with nodes pinched along $\gamma_1, \dots, \gamma_p$ completed to a set $\gamma_1, \dots, \gamma_{3g-3}$, admissible except that possibly (iv) in the definition of admissible is not satisfied. That possibility will not be important in the sequel. Let Π_0 and Π_k denote the canonical projections of $Q_0/\mathrm{Mod}(g)$ and $Q_0^k/\mathrm{Mod}(g)$ to $\overline{T}_g/\mathrm{Mod}(g)$.

Let $U_1, \dots, U_m$ be a finite covering of $\overline{T}_g/\mathrm{Mod}(g)$ by sufficiently small sets; this too is explained shortly. To show that the measures are finite we show

PROPOSITION 5.1. $\mu_0(\Pi_0^{-1}(U_i)) < \infty, \qquad \mu_k(\Pi_k^{-1}(U_i)) < \infty.$

The computations will be carried out in certain parametrized neighborhoods U_i of X. The union of these covers the compact boundary and we can pass to a subcover. This parametrization was found by Earle and Marden [4] and was used previously in [11]. In the latter paper we parametrized the space of quadratic differentials over this neighborhood of X. We reproduce that description here.

Let $Y \in \bar{T}_g - T_g$ be a marked boundary surface representing X; let U_i, V_i be neighborhoods of the ith node parametrized by $|z_i| < 1$ and $|w_i| < 1$ so that the node is $w_i = 0 = z_i$. Pick the neighborhoods of different nodes to be disjoint. Further, choose an open set $W \subset Y$ disjoint from all U_i, V_i and a basis $\nu_1, \dots, \nu_{3g\text{-}3\text{-}p}$ of Beltrami differentials supported in W for the tangent space to $\bar{T}_g - T_g$ at Y. For any $t_1, \dots, t_p, \tau_1, \dots, \tau_{3g\text{-}3\text{-}p}$ in a neighborhood V of $(0,0) \in C^p \times C^{3g\text{-}3\text{-}p}$, we get a Riemann surface $Y_{t,\tau}$ by putting the complex structure $\sum_{i=1}^{3g\text{-}3\text{-}p} \tau_i \nu_i$ on W, removing the discs $0 \le |z_i| \le |t_i|$ and $0 \le |w_i| \le |t_i|$ and gluing z_i to t_i/w_i. Let α_{t_i} be the curve $|z_i| = |w_i| = |t_i|^{1/2}$.

The marking on Y determines a marking on $Y_{t,\tau}$ up to a product of Dehn twists g_i about α_{t_i}. Therefore V parametrizes a cover of a neighborhood U of X. We will show the measures are finite in V. That will show they are finite in U.

In Section 5 of [11], we showed there exists a basis $q_1, \dots, q_p, q_{p+1}, \dots, q_{3g\text{-}3}$ for the quadratic differentials on $Y_{t,\tau}$ such that

i) $$q_i\, dz_j^2 = \left[\frac{\delta_{ij}}{z_j^2} + \frac{f_{ij}(t, \tau, z_j + w_j)}{z_j} + \frac{g_{ij}(t, \tau, z_j + w_j) w_j}{z_j^2} \right] dz_j^2;$$

$$i = 1, \dots, p;$$

ii) $$q_i\, dz_j^2 = \left[\frac{f_{ij}(t, \tau, z_j + w_j)}{z_j} + \frac{g_{ij}(t, \tau, z_j + w_j) w_j}{z_j^2} \right] dz_j^2;$$

$$i = p + 1, \dots, 3g\text{-}3,$$

where f_{ij}, g_{ij} are holomorphic in (t, τ) near $(0,0)$ and in $z_j + w_j$ in $A_j = \{z_j \,|\, |t_j| < |z_j| < 1\}$. For any set $J \subseteq \{1, \dots, p\}$, the set $q_i, q_{p+1}, \dots, q_{3g\text{-}3}, i \notin J$, forms a basis for the integrable quadratic differentials on $Y_{t,\tau}$; $t_k = 0, k \in J$. *For the rest of this section C denotes any constant not depending on t, τ, z_j, w_j.*

Normalize q_i, $i = 1, \dots, p$, so that $q_i \in Q_0$. Since f_{ij} and g_{ij} are bounded, and

(5.1) $$\int_{A_j} \frac{1}{|z_j|^2} dx\, dy = -2\pi \log |t_j| \quad \text{and}$$

$$\int_{A_j} \frac{1}{|z_j|} dx\, dy \le C, \quad \int_{A_j} \frac{|w_j|}{|z_j|^2} dx\, dy \le C,$$

the normalized q_i is given on A_j,

$$q_i\,dz_j^2 = -\frac{B_{ij}}{2\pi \log|t_i|}\left(\frac{\delta_{ij}}{z_j^2} + \frac{f_{ij}}{z_j} + \frac{w_j g_{ij}}{z_j^2}\right)dz_j^2 \tag{5.2}$$

where $|B_{ij}|$ is bounded above and below.

LEMMA 5.1. *Let $q = \sum_{i=1}^{3g-3} c_i q_i$ and $q \in Q_0$. Then c_i is bounded.*

Proof. Passing to a subsequence if necessary, assume c_j unbounded and c_i/c_j bounded for all i as $t_k \to 0$ for one or more t_k. There are two cases. If $j \geq p+1$, or $j \leq p$ and $|t_j|$ is bounded away from zero, then q/c_j converges to zero since any sequence in Q_0 has a convergent subsequence. But $q/c_j = \sum_{i \neq j} c_i/c_j q_i + q_j$. By 5.2, $q_k \to 0$ for any k such that $t_k \to 0$. Thus, the assumptions on c_i/cd_j and j imply a subsequence of $\sum c_i/c_j q_i + q_j$ converges to a nonzero linear combination of basis elements on $Y_{t,\tau}$, $t_k = 0$. This contradicts $q/c_j \to 0$. If $j \leq p$ and $t_j \to 0$, let $B_j \subset A_j$ be the annulus $|t_j|^{2/3} < |z_j| < |t_j|^{1/3}$. Then from 5.1 and 5.2,

$$1 = \int_{Y_{t,\tau}} |q| \geq \int_{B_j} |q| \geq |c_j| \left(\int_{B_j} |q_j| - \int_{B_j} \sum_{i \neq j} \left|\frac{c_i}{c_j}\right| |q_i|\right) \geq M|c_j|,$$

where M is positive and bounded away from zero. The point is that the integral of $|q_j|$ over B_j has a positive lower bound but the integral of $|q_i|$ over B_j goes to zero, $i \neq j$. This is a contradiction.

COROLLARY. *For any $q \in Q$, $\|q\| \leq 1$ lying over V,*

$$q\,dz_j^2 = \left[\frac{a_j}{z_j^2} + \frac{f_j}{z_j} + \frac{g_j w}{z_j^2}\right]dz_j^2 \tag{5.3}$$

where

$$-|a_j|\log|t_j| \leq C, \qquad |f_j| \leq C, \qquad |g_j| \leq C. \tag{5.4}$$

LEMMA 5.2. *For any q with $\|q\| < 1$, let s_j, s_j' be the "s" coordinates for γ_j with respect to F_q and F_{-q}. Then*

$$s_j, s_j' \leq -C|a_j|^{1/2}\log|t_j|, \quad j \leq p,$$

$$s_j, s_j' \leq C, \quad j \geq p+1.$$

Proof. For $j \geq p+1$, γ_j is not pinched. Since any sequence has a convergent subsequence, the s_j, s_j' lengths are bounded. For $j \leq p$ let β_j be a curve crossing γ_j twice and no other γ_i, and σ a radius of A_j. The $|q|^{1/2}$ length of β_j is bounded by twice the length of σ plus a uniform constant. We have

$$\int_\sigma |q|^{1/2} |dz| \leq \int_\sigma \frac{|a_i|^{1/2}}{|z_i|} |dz| + C\int_\sigma \left(\frac{1}{|z_i|^{1/2}} + \frac{|t_i|^{1/2}}{|z_i|^{3/2}} \right) |dz|$$
$$= -|a_i|^{1/2} \log |t_i| + C.$$

However s_i and s_i' are bounded by the $|q|^{1/2}$ length of β_i since a geodesic in the $|q|^{1/2}$ metric is quasitransverse for both the horizontal and vertical foliations. This lemma indicates that from the point of view of showing the measure is finite we need only deal with $-|a_i|^{1/2}\log|t_i| \geq 1$. We will show this later.

LEMMA 5.3. *Suppose* $-|a_i|^{1/2}\log|t_i| \geq 1$. *Then* $e^{i\theta}q$ *has closed horizontal trajectories in the homotopy class of* γ_i *for some* θ *and sufficiently small* t_i.

Proof. Let C_i be the annulus $|t_i|^{1/2} \leq |z_i| \leq 1/(\log|t_i|)^3$. Write $q\,dz_i^2 = a_i/z_i^2(1 + f_i z_i/a_i + g_i w_i/a_i)\,dz_i^2$. On C_i,

$$\left| \frac{f_i z_i}{a_i} \right| \leq \frac{C}{|a_i|\left(\log|t_i|\right)^3} \leq -\frac{C}{\log|t_i|} \tag{5.5}$$

and

$$\left| \frac{g_i w_i}{a_i} \right| \leq \frac{C|t_i|}{|z_i||a_i|} \leq \frac{C|t_i|^{1/2}}{|a_i|} \leq -C|t_i|^{1/2}\log|t_i| \tag{5.6}$$

by 5.4. Each of these terms is small for $|t_i|$ small.

Let $x_i = z_i/|t_i|^{1/2}$ and D_i be the annulus

$$1 \leq |x_i| \leq \frac{\left(-\log|t_i|\right)^{-3}}{|t_i|^{1/2}}.$$

We have

$$q\,dx_i^2 = \frac{a_i}{x_i^2} h_i(x)\,dx_i^2 = \frac{a_i}{x_i^2}\left[1 + \frac{f_i x_i |t_i|^{1/2}}{a_i} + \frac{g_i t_i}{a_i x_i |t_i|^{1/2}} \right] dx_i^2.$$

By 5.5, $|h_i - 1| \leq -C(\log|t_i|)^{-1}$ so h_i has a square root in D_i and the Laurent expansion of $h_i^{1/2}$ has constant coefficient C_0 satisfying $|C_0 - 1| \leq -C(\log|t_i|)^{-1}$. Let

$$\zeta_i = \exp\left(\int \frac{h^{1/2}(x_i)}{C_0 x_i} dx_i \right).$$

The coefficient of $1/x_j$ in the integrand is 1 so ζ_j is well-defined, and can be made arbitrarily close to x_j on fixed compact sets by taking t_j small. It is therefore univalent on this compact set and the image of D_j in the ζ_j plane contains circles. We have

$$C_0^2 a_j \frac{d\zeta_j^2}{\zeta_j^2} = C_0^2 \frac{a_j h_j(x)}{C_0^2 x_j^2} dx_j^2 = q(x_j)\, dx_j^2.$$

The circles $|\zeta_j| = \rho$ are closed horizontal trajectories for $-d\zeta_j^2/\zeta_j^2 = -q/a_j C_0^2 q = re^{i\theta} q$ for some θ and $r > 0$. Then $e^{i\theta} q$ also has closed horizontal trajectories, proving the lemma.

LEMMA 5.4. *Under the hypothesis* $-|a_j|^{1/2} \log |t_j| \geq 1$, *we have*

$$F_q(\gamma_j) = |\operatorname{Re} 2\pi C_0 a_j^{1/2}|, \qquad F_{-q}(\gamma_j) = |\operatorname{Im} 2\pi C_0 a_j^{1/2}|.$$

Proof. By Lemma 5.3, $q\, dx_j^2 = C_0^2 a_j d\zeta_j^2/\zeta_j^2$. Now

$$F_q(\gamma_j) = \int_\rho \left| \operatorname{Im} \frac{C_0 a_j^{1/2} d\zeta_j}{\zeta_j} \right|$$

where ρ is a circle. Set

$$\rho = \mathrm{re}^{i\theta}, \qquad F_q(\gamma_j) = \int_0^{2\pi} |\operatorname{Im} C_0 a_j^{1/2}|\, d\theta = 2\pi\, |\operatorname{Re} C_0 a_j^{1/2}|.$$

Similarly we compute $F_{-q}(\gamma_j)$.

We finally prove Proposition 5.1. For any set of equations in $F_q(\gamma_j)$, $F_{-q}(\gamma_j)$, s_j, s_j' that define Q_0^k locally, we take local coordinates, as in Part 4.2, that define this set. The Lebesgue measure as computed with respect to these is at least as great as the measure μ_k (or $\mu \times \mu$) since as usual these coordinates are integral linear combinations of canonical coordinates. It is enough to show then that the measure computed in any set of $F_q(\gamma_j)$, $F_{-q}(\gamma_j)$, s_j, s_j' is finite. Now $F_q(\gamma_j)$ and $F_{-q}(\gamma_j)$ are bounded for all j and s_j, s_j' bounded for $j \geq p+1$ by Lemma 5.2. By Lemma 5.2, again for $j < p+1$, we may assume $-|a_j|^{1/2} \log |t_j| \geq 1$. Now by Lemma 5.3, γ_j is transversal for both F_q and F_{-q} and therefore s_j and s_j' are local coordinates. (Recall §4.2.)

The idea is now that although s_j, s_j' may be large, for given $F_q(\gamma_j)$, $F_{-q}(\gamma_j)$, s_j, the possible error in s_j' is small.

Let

$$D = \{\zeta_j \mid r_1 < |\zeta_j| < r_2\}$$

be the largest annulus swept out by closed trajectories given by Lemma 5.3. The circles $|\zeta_i| = r_i$ contain zeroes of q and s_i, s_i' are the transverse lengths of a straight line ρ joining zeroes on opposite sides. Set

$$u_i = |\operatorname{Im} C_0 a_i^{1/2}|, \qquad u_i' = |\operatorname{Re} C_0 a_i^{1/2}|.$$

We divide the set of q into the overlapping sets $u_i/u_i' \leq \frac{2}{3}$ and $u_i/u_i' \geq \frac{1}{3}$ and prove the measure of each is finite. Let σ be a radius of D going from one zero to the other side. We compare s_i and s_i' to $F_q(\sigma)$, $F_{-q}(\sigma)$. Since σ and ρ differ up to homotopy by a piece l of the boundary of D and σ, ρ and l are all transverse, the triangle inequality gives

$$F_q(\sigma) - F_q(l) \leq s_i \leq F_q(\sigma) + F_q(l). \tag{5.7}$$

with similar inequalities for s_i' and F_{-q}. Now a computation shows

$$F_q(\sigma) = u_i \log\frac{r_2}{r_1}, \qquad F_{-q}(\sigma) = u_i' \log\frac{r_2}{r_1}, \tag{5.8}$$

and by Lemma 5.4,

$$F_q(l), F_{-q}(l) \leq 2\pi |C_0 a_i|^{1/2}. \tag{5.9}$$

Now suppose $u_i/u_i' \leq \frac{2}{3}$. Then for given $F_q(\gamma_i) = 2\pi u_i'$, $F_{-q}(\gamma_i) = 2\pi u_i$, and s_i, we have

$$\begin{aligned} \left| s_i' - \frac{u_i s_i}{u_i'} \right| &\leq \left| s_i' - u_i \log\frac{r_2}{r_1} \right| + \left| \frac{u_i' u_i}{u_i'} \log\frac{r_2}{r_1} - \frac{u_i}{u_i'} s_i \right| \\ &\leq 2\pi |C_0| |a_i|^{1/2} + \frac{u_i}{u_i'} 2\pi |C_0| |a_i|^{1/2} \leq \frac{10}{3}\pi |C_0| |a_i|^{1/2}. \end{aligned}$$

Recalling that C_0 is certainly bounded, we have

$$\left| s_i' - \frac{u_i s_i}{u_i'} \right| \leq C |a_i|^{1/2}. \tag{5.10}$$

This says s_i' varies little for given u_i, u_i', s_i. We wish to use iterated integration to find the area A. First *fix* all variables $F_q(\gamma_i) = 2\pi u_i'$, $F_{-q}(\gamma_i) = 2\pi u_i$ and denote by $A_{u_i, u_i'}$ the integration with respect to s_i and s_i'. We are assuming $u_i/u_i' \leq \frac{2}{3}$. By Lemma 5.2 and inequalities 5.4 and 5.10, we have

$$A_{u_i, u_i'} \leq \int_0^{-C|a_i|^{1/2}\log|t_i|} C |a_i|^{1/2}\, ds_i = -C |a_i| \log |t_i| \leq C.$$

To find A we will then need to integrate with respect to those u_i, u_i' which are local coordinates, but they are bounded in any case.

Note that even if a u_i or u_i' is not a local coordinate, the ones that are determine all the rest. Thus if these are done last in the integration, we can assume all u_i, u_i' fixed.

Finally, the set where $u_j/u_j' \geq \frac{1}{3}$ is dealt with by computing the error in s_j with given $F_q(\gamma_j)$, $F_{-q}(\gamma_j)$, s_j'. This completes the proof of Proposition 5.1.

6. Proof of Theorems 1 and 2

We start with the following version of Birkhoff's ergodic theorem.

PROPOSITION 6.1. *Suppose F is an ergodic foliation with transverse foliation G, β a transverse segment of F and $0 = a_0 < a_1 < \cdots < a_n$ fixed numbers. For any finite leaf L of F, partition L into segments L_k of length $1/a_n(a_k - a_{k-1})\,|L|$, $k = 1,\ldots,n$, where $|L|$ is the total length of L. Let the interval exchange map on β have intervals X_j and let $\#(\cdot \cap \cdot)$ denote the number of intersections of two arcs. Then*

$$\lim_{|L| \to \infty} \frac{\#(X_j \cap L_k)}{\#(\beta \cap L_k)} = \frac{F(X_j)}{F(\beta)}$$

for almost all leaves L with initial points on β.

Proof. Apply the ergodic theorem successively to $L^k = \cup_{j=1}^k L_j$. Then note that $\#(X_j \cap L_k) = \#(X_j \cap L^k) - \#(X_j \cap L^{k-1})$ and that $\#(X_j \cap L_i)/\#(X_j \cap L_k)$ is bounded above and below away from zero because the *numbers a_i are fixed*. The rest is a routine ε argument. The main result now is

PROPOSITION 6.2. *Let $q \in Q_0$ or Q_0^k have nonuniquely ergodic minimal vertical foliation F_{-q} and $q_0 \in Q_0/\mathrm{Mod}(g)$ or $q_0 \in Q_0^k/\mathrm{Mod}(g)$ on the surface X_0 be an Ω limit point of the orbit $q_t/\mathrm{Mod}(g)$, $t > 0$. Then there exist at least two disjoint submanifolds X_1, X_2 of X_0 with boundary a common dividing curve γ satisfying $F_{q_0}(\gamma) = 0$. Further, X_i, $i = 1,2$, contains a closed curve γ_i satisfying $F_{-q_0}(\gamma_i) = 0$.*

Proof. Let F_1 and F_2 be two distinct mutually singular ergodic foliations topologically equivalent to F_{-q}. They are each transverse to F_q. We normalize so that F_q and F_1 and F_q and F_2 each determine a quadratic differential in Q_0. There exists a segment β of a leaf of F_q such that $F_1(\beta) \neq F_2(\beta)$, for otherwise $F_1 = F_2$. Let X_j be the intervals for the exchange map on β.

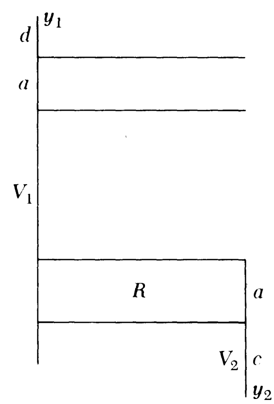
Let y_1 and y_2 be generic points of F_1 and F_2 respectively, that is, so that Proposition 6.1 holds with respect to the intervals of the exchange map and for any partition. Note that if F_{-q} is not orientable then β induces a return where some rectangles "come back on the same side". Proposition 6.1 is equally valid.

Let l_j be the height with respect to F_q of the jth rectangle and $F_i(X_j)$ the length with respect to F_i. Now if L^i, $i = 1,2$, is a leaf through y_i and $\{a_i\}$ is any set of numbers as before,

$$(6.1) \quad \lim_{|L^i|\to\infty} \frac{|L_k^i|}{\#(\beta \cap L_k^i)} = \lim_{|L^i|\to\infty} \sum_j \frac{l_j\#(X_j \cap L_k^i)}{\#(\beta \cap L_k^i)} = \frac{\sum_j l_j F_i(X_j)}{F_i(\beta)} = \frac{1}{F_i(\beta)}$$

by Proposition 6.1 and the fact that the areas of the quadratic differentials are one.

Let $q_{t_n} \to q_0$. Via the Teichmüller map, y_i can be thought of as a point on X_n, the carrier surface of q_{t_n}, and by taking subsequences we may assume it converges to a point y_i on X_0. Let V_i be the vertical trajectory of q_0 through y_i. We claim no horizontal trajectory intersecting V_1 can also intersect V_2. To prove the claim, consider any segment of V_1 with y_1 as initial point and the resulting return map on that segment. Suppose a segment of V_2 intersects and therefore crosses a rectangle R of width a of this return. Let c be the length of V_2 from y_2 to the beginning of the segment and d be the length of V_1 from y_1 to the first point of R. The figure below illustrates with nonorientable q_0.



For any $\varepsilon > 0$ consider the partitions

i) $0 < d < d + \varepsilon < d + a - \varepsilon < d + a$ and

ii) $0 < c < c + \varepsilon < c + a - \varepsilon < c + a$,

each with five division points counting endpoints.

The segment of V_1 of length $a + d$ and of V_2 of length $a + c$ are limits of segments V_{1n} and V_{2n} of q_{t_n}. These correspond under the Teichmüller map to vertical segments again written V_{1n}, V_{2n} of q of length $K_n^{1/2}(a + d)$ and $K_n^{1/2}(a + c)$ where $t_n = \log K_n^{1/2}$. Let V_{in}^a be the subsegment of V_{in} of length $K_n^{1/2}a$ determined by the partitions (i) and (ii) using division points 2 and 5, and V_{in}^ε be the subsegments of length $K_n^{1/2}(a - 2\varepsilon)$ formed by the division points 3 and 4. We also will consider these as segments of length $a - 2\varepsilon$ on q_{t_n} via the Teichmüller map.

Take N large enough so that by Equation 6.1 for all $n \geq N$,

$$(6.2)\quad \left|\frac{K_n^{1/2}a}{\#(\beta \cap V_{in}^a)} - \frac{1}{F_i(\beta)}\right| < \varepsilon \quad \text{and} \quad \left|\frac{K_n^{1/2}(a-2\varepsilon)}{\#(\beta \cap V_{in}^\varepsilon)} - \frac{1}{F_i(\beta)}\right| < \varepsilon,$$
$$i = 1,2.$$

Furthermore, by the same reasoning with division points 2, 3, 4, and 5 and Equation 6.1

$$(6.3)\quad \left|\frac{2\varepsilon K_n^{1/2}}{\#(\beta \cap V_{2n}^a) - \#(\beta \cap V_{2n}^\varepsilon)} - \frac{1}{F_2(\beta)}\right| < \varepsilon.$$

Since $\lim_{n\to\infty} V_{in} = V_i$, for n large enough, every horizontal trajectory of q_{t_n} intersecting V_{1n}^ε intersects V_{2n}^a and every trajectory intersecting V_{2n}^ε intersects V_{1n}^a. We have

$$(6.4)\quad \#(\beta \cap V_{1n}^\varepsilon) < \#(\beta \cap V_{2n}^a), \qquad \#(\beta \cap V_{2n}^\varepsilon) < \#(\beta \cap V_{1n}^a).$$

Then

$$\begin{aligned}\#(\beta \cap V_{2n}^a) &< \frac{F_2(\beta)2\varepsilon K_n^{1/2}}{1 - F_2(\beta)\varepsilon} + \#(\beta \cap V_{2n}^\varepsilon) \\ &< \#(\beta \cap V_{1n}^a) + \frac{F_2(\beta)2\varepsilon K_n^{1/2}}{1 - \varepsilon F_2(\beta)} \\ &< \#(\beta \cap V_{1n}^a) + \frac{F_2(\beta)2\varepsilon}{1 - \varepsilon F_2(\beta)}\left(\varepsilon + \frac{1}{F_1(\beta)}\right)\frac{\#(\beta \cap V_{1n}^a)}{a}\end{aligned}$$

by 6.2, 6.3 and 6.4. Therefore

$$\frac{\#(\beta \cap V_{2n}^a)}{\#(\beta \cap V_{1n}^a)} - 1 < \frac{2\varepsilon F_2(\beta)}{a(1 - \varepsilon F_2(\beta))}\left(\varepsilon + \frac{1}{F_1(\beta)}\right).$$

Similarly,

$$\frac{\#(\beta \cap V_{1n}^a)}{\#(\beta \cap V_{2n}^a)} - 1 < \frac{2\varepsilon F_1(\beta)}{a(1 - \varepsilon F_1(\beta))}\left(\varepsilon + \frac{1}{F_2(\beta)}\right).$$

Since $\#(\beta \cap V_{1n}^a)$ and $\#(\beta \cap V_{2n}^a)$ do not depend on ε and ε is arbitrary, the last inequalities give

$$\overline{\lim_{n\to\infty}} \frac{\#(\beta \cap V_{1n}^a)}{\#(\beta \cap V_{2n}^a)} - 1 \leq 0 \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{\#(\beta \cap V_{2n}^a)}{\#(\beta \cap V_{1n}^a)} - 1 \leq 0$$

so

$$\lim_{n \to \infty} \frac{\#(\beta \cap V_{1n}^a)}{\#(\beta \cap V_{2n}^a)} = 1.$$

Since V_{1n}^a and V_{2n}^a have the same length, by 6.1,

$$\lim_{n \to \infty} \frac{\#(\beta \cap V_{1n}^a)}{\#(\beta \cap V_{2n}^a)} = \frac{F_1(\beta)}{F_2(\beta)} \neq 1$$

which is a contradiction, proving the claim.

Therefore no horizontal trajectory intersecting V_1 intersects V_2. Therefore no horizontal trajectory is dense so X_0 is divided into two or more domains separated by a dividing curve γ with $F_{q_0}(\gamma) = 0$. Any horizontal trajectory is contained in one domain. Further if V_1 and V_2 intersected the same horizontal domain, a horizontal trajectory in that domain would intersect both if it were dense. If the horizontal trajectory is not dense in the domain, there is an annulus of closed trajectories; V_2 could be extended so a closed leaf would intersect each, a contradiction in either case. Therefore V_1 and V_2 are entirely contained in different horizontal domains X_1 and X_2. It follows then that the vertical domains of V_1 and V_2 are also contained in X_1 and X_2 respectively, since V_i is either dense in a vertical domain or the domain is an annulus. These vertical domains are bounded by closed curves β_1 and β_2 satisfying $F_{-q_0}(\beta_i) = 0$ and $\beta_i \subset X_i$, $i = 1, 2$, completing the proof.

Remark. The process that takes place in the proposition is that some rectangles determined by a vertical segment of q_{t_n} of bounded length have heights approaching zero. The limiting situation of q_0 has fewer rectangles and not every horizontal trajectory of q_0 is contained in these rectangles; that is, the rectangles for the vertical segment V_1 do not partition the entire surface. The vertical trajectories V_{2n} are contained in the "disappearing" rectangles of the exchange map on V_{1n}.

Denote by W_k (resp. W_0) the set of $q \in Q_0^k$ (resp. $q \in Q_0$) such that there exist β_1, β_2 disjoint curves with $F_q(\beta_1) = F_{-q}(\beta_2) = 0$. If q satisfies the conclusion of Proposition 6.2, $q \in W_k$ or W_0.

LEMMA 6.3. *W_k (resp. W_0) is closed in Q_0^k (resp. Q_0) and has measure zero.*

Proof. That the measure is zero is clear because in Q_0 or Q_0^k the differentials with horizontal (or vertical) foliation assigning zero to some closed curve have measure zero.

We will now prove W_0 is closed in Q_0. The proof for W_k is exactly the same. Suppose $q_n \in W_0$ and $\lim_{n \to \infty} q_n = q$, $F_{q_n}(\gamma_n) = 0$, $F_{-q_n}(\beta_n) = 0$ and

$\beta_n \cap \gamma_n = \varnothing$. We let x_n, y_n be critical points of q_n on γ_n and β_n, and we may assume $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} y_n = y_0$ critical points of q. Then γ_n converges to a leaf γ of q at x_0 and β_n converges to a vertical leaf β at y_0. Suppose γ is dense. Then it intersects β and for large $n, \gamma_n \cap \beta_n \neq \varnothing$. Therefore γ and similarly β are not dense and are also disjoint. They must be contained in disjoint spiral domains, one for γ, the other for β. The boundaries of these domains contain the desired curves γ_0, β_0.

Remark. If there are sufficiently many ergodic measures, for instance $3g$-3, we can prove no q_0 can exist with many vertical spiral domains, each contained in a horizontal one as required by Proposition 6.2. In that case q_t has no Ω limit points in $Q_0/\mathrm{Mod}(g)$ and all Ω limit points are on the boundary.

Proof of Theorems 1 and 2. By Proposition 4.1 we need only show the set of $q \in Q_0^k$ (resp. Q_0) with nonuniquely ergodic vertical foliation has measure zero. Clearly the sets W_k and W_0 of Lemma 6.3 are invariant under $\mathrm{Mod}(g)$ so $W_k/\mathrm{Mod}(g)$ and $W_0/\mathrm{Mod}(g)$ are closed of measure zero. Now suppose E is a set of positive measure in Q_0^k (resp. Q_0) consisting of differentials with nonuniquely ergodic vertical foliations. We may assume the closure of E is disjoint from $W_k/\mathrm{Mod}(g)$ (resp. $W_0/\mathrm{Mod}(g)$). By Poincaré recurrence, for almost all $q \in E, q_t$, $t > 0$, returns for arbitrarily large t, to E and therefore q_t has Ω limit points in E. By Proposition 6.2 all such limit points are in $W_k/\mathrm{Mod}(g)$ (resp. $W_0/\mathrm{Mod}(g)$). This is a contradiction.

Proof of Corollary 1. By the main theorem of [6], the map $H^0(X, \Omega^{\otimes 2}) \to MF$ given by $q \to \operatorname{Im} q^{1/2}\,dz$ is a homeomorphism. Near q with simple zeroes it is easily seen to be C^1 in local coordinates. Therefore sets of measure zero correspond and the corollary follows from Theorem 2.

7. Proof of Theorem 3

7.1 We will prove the following result and show it is equivalent to Theorem 3.

THEOREM 4. *The Teichmüller flow on* $Q_0/\mathrm{Mod}(g)$ *is ergodic.*

COROLLARY. *Almost all orbits in* $Q_0/\mathrm{Mod}(g)$ *are dense. That there are any dense orbits was shown in* [12].

A foliation is *arational* if it has no saddle connection. This implies all leaves are dense.

We now record a basic property of uniquely ergodic foliations. If F is uniquely ergodic, then the limit in Birkhoff's ergodic theorem in Section 1 holds *uniformly for all* $x \in \beta$ and continuous f:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n(x)) = \int_J f\,d\nu \text{ uniformly.} \tag{7.1}$$

PROPOSITION 7.1. *Suppose* F_1 *and* F_2 *are uniquely ergodic arational projectively inequivalent foliations. Then* F_1 *and* F_2 *are transverse.*

Remark: S. Kerckhoff and W. Thurston indicated (oral communication), that a stronger theorem can be proved using the techniques of geodesic laminations. The proposition of course means that F_1, F_2 are equivalent to transverse foliations.

Proof. We begin by constructing a sequence γ_n of simple closed curves converging to F_1 in the topology of PF. Let x be any point on a transverse β and L a leaf through x in either direction. For some sequence n_i there will be no return point $T^j(x)$ between x and $T^{n_i}(x)$; $j < n_i$. Let γ_{n_i} be formed of the part of the leaf L from x to $T^{n_i}(x)$ followed by the segment of β between $T^{n_i}(x)$ and x.

To show $\gamma_n \to F_1$ in PF we need to show for any two simple closed curves ρ_1 and ρ_2,

$$\lim_{n\to\infty} \frac{i(\gamma_n, \rho_1)}{i(\gamma_n, \rho_2)} = \frac{F_1(\rho_1)}{F_1(\rho_2)} \tag{7.2}$$

where $i(\cdot, \cdot)$ is the geometric intersection number. Any ρ can be represented as a finite union of segments joining singularities. The number of intersections of any such segment α with L is the same as the number of intersections of L with an appropriate segment σ of β where $F(\sigma) = F(\alpha)$. By 7.1 we have

$$\lim_{|L|\to\infty} \frac{\#(\sigma \cap L)}{\#(\beta \cap L)} = F_1(\sigma) = F_1(\alpha).$$

Now 7.2 follows. We may similarly construct $\delta_n \to F_2$ in PF.

Now let G_1 be any foliation transverse to F_1. Represent δ_n with respect to the quadratic differential by a geodesic which is a finite union of line segments each making a constant angle with the leaves of F_1. Let θ_n be the minimum angle. We claim $\underline{\lim}\,\theta_n > 0$. Suppose otherwise that there is a subsequence again denoted δ_n with $\lim_{n\to\infty}\theta_n = 0$. The segment $\bar{\delta}_n \subset \delta_n$ with angle θ_n has length going to infinity. Since the leaves of F_1 are dense, $\bar{\delta}_n$ has the property that for any open set V, $\bar{\delta}_n \cap V \neq \varnothing$ for n large enough. Since δ_n is simple this implies all intersection angles of δ_n have limit zero as n goes to infinity.

Now let $\varepsilon > 0$. In the following estimates ε_i depends only on ε, ρ_1, ρ_2 and will be chosen later. By 7.1 there is an L_0 such that

$$\left| \frac{\#(L \cap \rho_1)}{\#(L \cap \rho_2)} - \frac{F_1(\rho_1)}{F_1(\rho_2)} \right| < \varepsilon_1 \tag{7.3}$$

for any segment L with $|L| \geq L_0$. Also take a neighborhood W of the zeroes small enough and L_0 large so that

$$\frac{\#(L \cap \rho_j \cap W)}{\#(L \cap \rho_j)} < \varepsilon_2, \quad j = 1,2 \tag{7.4}$$

for any L with $|L| \geq L_0$.

Consider the segments $B_1, \ldots, B_s$ each of length L_0, one on each critical leaf of F_1, each with a critical point as one endpoint. Let $\bar{\delta}_n$ be a segment of δ_n of length L_0 not intersecting any B_i and initial point x. Then since the angles δ_n makes with leaves goes to zero, for n sufficiently large, there is a leaf segment L starting at x of length L_0 intersecting ρ_j the same number of times $\bar{\delta}_n$ intersects ρ_j except possibly counting the intersections each has with $\rho_j \cap W$. By 7.4 these are very few and we can take n large enough and ε_2 small enough so that

$$\left| \frac{\#(L \cap \rho_j)}{\#(\bar{\delta}_n \cap \rho_j)} - 1 \right| < \varepsilon/4, \qquad j = 1,2. \tag{7.5}$$

Now again since the angle goes to zero so that δ_n crosses B_j rarely, we may for large n write δ_n as a disjoint union

$$\delta_n^1 \cup \cdots \cup \delta_n^{r_1(n)} \cup \cdots \cup \delta_n^{r_2(n)}$$

such that

i) $$|\delta_n^i| = L_0, \quad \delta_n^i \cap B_j = \varnothing;\ 1 \leq i \leq r_1(n), \qquad 1 \leq j \leq s;$$

ii) $$\frac{\displaystyle\sum_{i=r_1(n)+1}^{r_2(n)} \#(\delta_n^i \cap \rho_j)}{\displaystyle\sum_{i=1}^{r_1(n)} \#(\delta_n^i \cap \rho_j)} < \varepsilon_3, \qquad j = 1,2.$$

Then we find L^i satisfying 7.5 with respect to δ_n^i, $i \leq r_1(n)$ and also satisfying 7.3. By picking ε_1 small enough we conclude from 7.3 and 7.5

$$\left| \frac{\#(\delta_n^i \cap \rho_1)}{\#(\delta_n^i \cap \rho_2)} - \frac{F_1(\rho_1)}{F_1(\rho_2)} \right| < \varepsilon/2, \qquad i \leq r_1(n)$$

and therefore

$$\left| \frac{\sum_{i=1}^{r_1(n)} \#(\delta_n^i \cap \rho_1)}{\sum_{i=1}^{r_1(n)} \#(\delta_n^i \cap \rho_2)} - \frac{F_1(\rho_1)}{F_1(\rho_2)} \right| < \varepsilon/2.$$

Finally, by taking ε_3 small and n large by (ii), we have

$$\left| \frac{\#(\delta_n \cap \rho_1)}{\#(\delta_n \cap \rho_2)} - \frac{F_1(\rho_1)}{F_1(\rho_2)} \right| < \varepsilon.$$

This implies that $\delta_n \to F_1$ in *PF* which is a contradiction proving the claim.

We now finish the proof of the proposition. Eventually all components in the complement of $\delta_n \cup \gamma_n$ are simply connected. For suppose $i(\beta_n, \gamma_n) = 0$ for some β_n. The angle β_n makes with leaves goes to zero, and so β_n converges to F_1 in *PF*. This clearly means $i(\delta_n, \beta_n) \neq 0$. We may therefore find transverse foliations F_{1n}, F_{2n} such that F_{1n} has closed leaves homotopic to γ_n, F_{2n} closed leaves homotopic to δ_n ([12], Prop. 2.4). We give the cylinder of F_{1n} height $h_n = 1/\#(\gamma_n \cap \beta)$ and the cylinder of F_{2n} height $1/F_1(\delta_n)$. Since F_{1n}, F_{2n} converge in *PF* to F_1 and F_2, we need to show $q_n = (F_{1n}, F_{2n})$ has a convergent subsequence in Q.

The norm of q_n is $i(\gamma_n, \delta_n)h_n/F_1(\delta_n)$. Since the slope of δ_n is bounded away from zero, $i(\gamma_n, \delta_n)\ h_n/F_1(\delta_n)$ is bounded above and below. We therefore need only show that the carrier Riemann surfaces X_n of q_n lie in a compact set in T_g. If not, there exist curves ρ_n such that the extremal length of ρ_n on X_n converges to zero. Since the metric $|q_n|^{1/2}$ is only one competing metric in extremal length and $\|q_n\|$ is bounded above, we have

$$\lim_{n\to\infty} \left(\int_{\rho_n} |q_n|^{1/2} |dz| \right)^2 = 0.$$

This implies both

$$\lim_{n\to\infty} F_{1n}(\rho_n) = \lim_{n\to\infty} i(\rho_n, \gamma_n)h_n = 0 \quad \text{and} \quad \lim_{n\to\infty} F_{2n}(\rho_n) = \lim_{n\to\infty} \frac{i(\rho_n, \delta_n)}{F_1(\delta_n)} = 0.$$

The first means the slope of ρ_n with respect to (F_1, G_1) approaches zero, which means ρ_n converges to F_1 in *PF*. But then, $i(\rho_n, \delta_n)/F_1(\delta_n)$ is unbounded and we have a contradiction. This completes the proof.

7.2 *Proof of Theorem* 4. We follow the outline of Hopf's proof [5]. We first note that since $\mu_0(Q_0/\mathrm{Mod}(g)) < \infty$, the flow on $Q_0/\mathrm{Mod}(g)$ is conservative: for almost all q, q_t returns infinitely often to a compact set. Now for any $f \in L^1(\mu_0)$ consider

$$f^*(P) = \lim_{T \to \infty} \frac{1}{T} \int_o^T f(q_t)\,dt \text{ and}$$

$$f^*(P) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(q_{-t})\,dt.$$

Since $\mu(Q_0/\mathrm{Mod}(g)) < \infty$, the Birkhoff ergodic theorem says $f^*(P)$ and $f_*(P)$ exist almost everywhere $[\mu_0]$, and are invariant under the flow and

$$(7.6) \qquad \int_{Q_0/\mathrm{Mod}(g)} f^* h\, d\mu_0 = \int_{Q_0/\mathrm{Mod}(g)} fh\, d\mu_0 = \int_{Q_0/\mathrm{Mod}(g)} f_* h\, d\mu_0$$

for any bounded invariant h. Taking $h = \mathrm{sign}(f^* - f_*)$ we have $f^* = f_*$ almost everywhere. To show the flow is ergodic, we need to show $f^* = f_* =$ constant almost everywhere. This will follow from two lemmas. Let d be any metric on $Q_0/\mathrm{Mod}(g)$ compatible with the topology.

LEMMA 7.2. *Let f be continuous on $Q_0/\mathrm{Mod}(g)$ with compact support K. Suppose $P, P' \in Q_0/\mathrm{Mod}(g)$ satisfy $\lim_{t \to \infty} d(P_t, P'_{t+a}) = 0$ for some a. Further suppose $f^*(P)$ exists. Then so does $f^*(P')$ and $f^*(P) = f^*(P')$ with similar statements about f_* and $t \to -\infty$.*

Proof. Since f^* is flow invariant we may assume P, P' satisfy $\lim_{t \to \infty} d(P_t, P'_t) = 0$. Given $\varepsilon > 0$, let $\delta > 0$ be so that $d(P_t, P'_t) < \delta$ implies $|f(P_t) - f(P'_t)| < \varepsilon$. Let T_0 be large enough so that for $t \geq T_0$, $d(P_t, P'_t) < \delta$. Let $B = \max_{x \in K} |f(x)|$. Then

$$\begin{aligned}\left| \frac{1}{T} \int_0^T f(P_t) - f(P'_t)\,dt \right| &\leq \frac{1}{T} \int_0^{T_0} |f(P_t) - f(P'_t)|\,dt \\ &\quad + \frac{1}{T} \int_{T_0}^T |f(P_t) - f(P'_t)|\,dt \\ &\leq \frac{2BT_0}{T} + \frac{\varepsilon(T - T_0)}{T}.\end{aligned}$$

The limit of the right side is ε as $T \to \infty$. Since ε is arbitrary we have $f^*(P') = f^*(P)$.

LEMMA 7.3. *Suppose f is continuous with compact support $K \subset Q_0/\mathrm{Mod}(g)$. For any $k > 0$ let*

$$E = \{P \mid f^*(P) \geq k\}, F = \{P \mid f_*(P) \geq k\}.$$

Then

$$\mu_0(E) = \mu_0(F) = 0 \text{ or } \mu_0(Q_0/\mathrm{Mod}(g) - E) = \mu_0(Q_0/\mathrm{Mod}(g) - F) = 0.$$

Proof. Since f^* and f_* are flow invariant we may consider E, F as sets of lines and therefore as subsets of $PF \times PF/\mathrm{Mod}(g)$. Since almost all foliations are uniquely ergodic by Theorem 2, and almost all foliations are arational we may use Proposition 7.1 and restrict ourselves to the set of projective arational uniquely ergodic foliations which we write *PUE*. All pairs in $PUE \times PUE$ off the diagonal are transverse. In [13] we proved that if $P, P' \in Q$ have the *same uniquely ergodic vertical foliation*, then $\lim_{t\to\infty}\tau(X_{t+a}, X'_t) = 0$ for some a where τ is the Teichmüller metric and X_{t+a} and X'_t carry P_{t+a} and P'_t respectively. Equivalently we have the same result if $P, P' \in Q_0$ have the same *projective* uniquely ergodic vertical foliation. In this latter statement, since P_{t+a} and P'_t have unit norm and their vertical foliations are projectively the same we must also have

$$\lim_{t\to\infty} d(P_{t+a}, P'_t) = 0$$

for any sequence t such that $P_{t+a}, P'_t \in K$. This follows because $\lim_{t\to\infty}\tau(X_{t+a}, X'_t) = 0$ and the theorem in [6] which says that on any Riemann surface there is a unique normalized quadratic differential with vertical foliation in a given projective class. By Lemma 7.2, $f^*(P) = f^*(P')$ when either exists. Therefore we may write $E = PUE \times E_1$ for some $E_1 \subset PUE$.

Similarly $F = F_1 \times PUE$ for some F_1. Since $f^*(P) = f_*(P)$ almost everywhere, we have $m(E - F) = m(F - E) = 0$ where m is measure (class) on $PF \times PF$. This gives $m(F_1^c \times E_1) = m(F_1 \times E_1^c) = 0$. Therefore we either have $m(F_1^c) = m(E_1^c) = 0$ or $m(E_1) = m(F_1) = 0$ where now m is measure (class) on PF. This proves the lemma.

To finish the proof of the theorem we note following Hopf that the linear operator $f \to f^*$ is bounded in L^1. This follows from 7.6 by taking $h = \operatorname{sign} f^*$ giving $\int |f^*|\, d\mu \le \int |f|\, d\mu$. Therefore it is enough to show f^* constant almost everywhere for a dense set of $f \in L^1(\mu)$, namely those f continuous with compact support. By Lemma 7.3 for any such f, $\{P \mid f^*(P) \ge k\}$ has measure zero or its complement does for any k, proving f^* constant almost everywhere.

Proof of Theorem 3. Suppose $E \subset PF \times PF$ is invariant. Then so is $E \cap (PUE \times PUE)$ and they have the same measure class by Theorem 2. The latter set gives rise to an invariant set of lines in Q_0 which by Theorem 4 has measure zero or its complement does.

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