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PRIME GEODESIC THEOREMS

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DOCTOR OF PHILOSOPHY

By

Peter Sarnak

August 1980

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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INTRODUCTION

A lot of work has been done in recent years concerning closed geodesics on Riemannian manifolds. In the case that the manifold has a "large" fundamental group such as a surface of genus $g \geq 2$, the existence of infinitely many such geodesics is easily demonstrated by exhibiting one in each free homotopy class. The work referred to above is concerned with showing the existence of infinitely many prime geodesics on any compact, closed manifold.

It is our aim in this thesis to study the asymptotic distribution of the lengths of periodic geodesics, and consequences thereof. The results we obtain have interesting applications to number theory as well as to the behavior of eigenvalues of the Laplace Beltrami operator on Riemannian manifolds.

There are three different methods of getting hold of the closed geodesics. The first is to use the fact that these geodesics are critical points of the energy integral on a path space. Secondly, by viewing these geodesics as periodic orbits of the geodesic flow, one may use methods of topological dynamics. Finally, in the case of constant curvature, the lengths of the closed geodesics can be identified through exact trace formulas such as the Selberg Trace Formula. We will be concerned mainly with the latter two methods, as they are more useful as far as asymptotics of the lengths is concerned.

The relevant trace formulas are introduced in Chapter 1. These include the Poisson Summation Formula, Selberg Trace Formula, and what we call the Duistemaat-Guillemin summation formula. These may all be viewed as a computation of the trace of the operator $\exp(i\sqrt{\Delta}\, t)$ in two ways.

We also introduce in this chapter the notions of the geodesic flow and the different entropies associated with it.

Let cP_0 be the set of closed geodesics on M. For $\gamma \in cP_0$ let $\tau(\gamma)$ be its length. For $x > 0$ define $\pi(x) = \#\{\gamma \in cP_0 : \tau(\gamma) \leq x\}$. We are interested in the behavior of $\pi(x)$ as $x \to \infty$. We know $\pi(x) \to \infty$, as $x \to \infty$ for "almost" any manifold (Klingenberg [18]). It may happen that $\pi(x) = \infty$ for some x (e.g., the two sphere). However, in the case of a Riemannian manifold with all sectional curvatures negatives, we have the following remarkable result of Margulis [22].

Theorem: $\pi(x) \sim \frac{ce^{hx}}{x}$ for constants c and h.

Here h is the topological entropy. An asymptotic expansion for $\pi(x)$ will be called a Prime Geodesic Theorem.

In view of this prime geodesic theorem, it is of interest to study the dependence of the entropy on the geometry of the manifold. In Chapter 2, monotonicity and continuity theorems are proved for the topological entropy. If λ is the "bottom" of the spectrum of the Laplacian as it acts on the universal covering of the Riemannian manifold whose entropy we are considering, then the following is true

$$\lambda \leq \frac{h^2}{4}, \text{ where } h = \text{topological entropy of the geodesic flow.}$$

In the case of constant curvature, these are equal, but they need not be so in general. We also prove the following estimates for the "natural measure" entropy h_μ:

$$(n-1)\,\frac{1}{v}\int_M \sqrt{-k^+(x)}\;dA(x) \leq h_\mu \leq (n-1)\left(\frac{1}{v}\int_M -k^-(x)dA(x)\right)^{\frac{1}{2}}.$$

In particular for a surface of genus $g \geq 2$

$$\frac{1}{v}\int_M \sqrt{-k(x)}\;dA(x) \leq h_\mu \leq \left(\frac{4\pi(g-1)}{v}\right)^{\frac{1}{2}} \qquad k = \text{curvature}.$$

(See Theorem 2.3 for definitions of k^+, k^-.) In constant curvature all of the above are equal.

Together with $h \geq h_\mu$ the above gives us another lower bound for h, in terms of the geometry.

When the curvature is constant, say equal to minus 1, and we are in dimension two, we have $h = 1$ and we may seek higher order terms in the prime geodesic theorem. In the case of a compact surface (closed), Selberg and later Hejhal and Huber, determined such higher order asymptotics. The situation, in the noncompact but finite volume surface, is complicated by the presence of continuous spectrum. By methods different to the above authors we prove the prime geodesic theorem for finite volume surfaces. Our method consists of first obtaining some estimates on the spectrum of the Laplace Beltrami operator via partial differential equations, and then applying a chosen one parameter family of test functions. This method avoids the use of the Selberg Zeta function; however, that method can also be made to work, as is indicated in Selberg [27]. If $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_k < 1/4$ are the discrete eigenvalues of Δ on the surface, and if $it_m = \sqrt{\lambda_m - 1/4}$, $m = 0,\dots,k$, then the prime geodesic theorem reads

$$\pi(x) = Li(e^x) + Li(e^{(1/2 + t_1)}) + \ldots + Li(e^{(1/2 + t_k)x}) + O(x^2 e^{3/4 x})$$

where $Li(u) = \int_2^u \frac{1}{\log t} dt$.

We continue Chapter 3 by discussing the asymptotics of the lengths of prime geodesics where we impose conditions on the primes to be counted. Indeed, this part is analogous to prime ideals of algebraic number theory and their splitting in extension fields. We develop the following analogue of the Dirichlet-Chabotarev density theorem. Let S be a closed compact surface of constant curvature -1. Let W be a finite regular cover of S degree n. Let G be the group of cover transformations of W over S. Let us call a prime geodesic on any of the surfaces simply a prime. To each prime on S we associate a conjugacy class in G, depending on the cover transformation that this prime induces. For a fixed class C of G we define $\pi_C(x)$ to be the number of primes which induce C and whose length is less than x. We obtain the asymptotics of $\pi_C(x)$, (up to order $x^2 e^{3/4 x}$ as usual) in terms of the irreducible representations of G. If one view the above in terms of Selberg Zeta functions, one finds the analogue of the Artin conjectures is true.

We conclude Chapter 3 by making some eigenvalue estimates which are needed later. Specifically, let $G(\mu)$ be one of the "Hecke groups", that is, $G(\mu)$ is generated by

$$\begin{array}{l} z \to -\frac{1}{z} \\ z \to z + \mu \end{array} \qquad \mu = 2 \cos(\frac{\pi}{q}), \quad q \geq 3, \quad q \in \mathbb{Z}.$$

$G(\mu)$ is a discrete subgroup of $PSL(2,\mathbb{R})$ and acts on the upper half plane H. For the surface $S_\mu = \frac{H}{G_\mu}$ we prove that the first discrete eigenvalue λ_1 is not less than one quarter. This theorem for $q = 3$ is well known. The prime geodesic theorems of this chapter may be developed in hyperbolic n-space by similar methods.

The next chapter deals with applications to number theory. The connection comes when one tries to find the lengths of the prime geodesics for the surfaces $\frac{H}{\Gamma(N)}$ where $\Gamma(N)$ is the principal congruence subgroup of $PSL(2,\mathbb{Z})$, of level N. In these cases we have a nice interpretation in terms of number theoretic quantities.

Let $D = \{n \in \mathbb{Z},\ n > 0,\ n \equiv 0 \text{ or } 1 \bmod 4,\ n \neq m^2\}$.

For $d \in D$, let ε_d be the fundamental solution of the Pellian equation, $x^2 - dy^2 = 4$. Let $h(d)$ be the number of inequivalent classes of primitive quadratic forms of discriminant d. The lengths of the periodic geodesics in the case of the modular group are the numbers $2\log\varepsilon_d$ with multiplicity $h(d)$, $d \in D$.

The numbers $h(d)$ and $\log\varepsilon_d$ have been a mystery ever since Gauss introduced them in Section 304 of his Disquestiones Arithmetica. Gauss, seeking to understand the behavior of these class numbers $h(d)$, formed averages of h over the sets $\{1,2,\ldots,x\} \cap D$, but noticed that these have no regular behavior as a function of x. He did point out that $\phi(x) = \sum_{d \in x} h(d)\log\varepsilon_d$ does have an asymptotic expansion. This was later proved by Siegel:

$$\phi(x) = \frac{\pi^2}{18\zeta(3)}x^{3/2} + O(x\log x)\ .$$

Also through the work of Siegel we understand the behavior of $h(d) \log \varepsilon_d$ (although not effectively). It seems difficult to separate the quantities $h(d)$ and $\log \varepsilon_d$. In Chapter 4 we compute the averages of $h(d)$ over the exhausting family of sets $\{d: \varepsilon_d \leq x\}$. The results may be summarized as follows:

(i) $$\sum_{\{d:\varepsilon_d \leq x\}} h(d) = Li(x^2) + L(x^{3/2} (\log x)^2) .$$

(ii) $$\frac{1}{|\{d:\varepsilon_d \leq x\}|} \sum_{\{d:\varepsilon_d \leq x\}} h(d) = \frac{16}{35} \frac{Li(x^2)}{x} + O(x^{2/3+\varepsilon}), \quad \forall \varepsilon > 0 .$$

(i) follows from our considerations about closed geodesics, while (ii) follows from (i) and some arguments on asymptotics of the number of solutions to a diophantine inequality.

The above shows that on the average the class number is about the size of the unit. The question of the finer behavior of $h(d)$, raised by Gauss, seems as difficult today as it has been for many years.

Similar considerations for $\Gamma(p)$ where p is a prime, give rise to asymptotic averages of $h(d)$ over certain subsets of D. The precise statements may be found in Theorem 4.34. The asymptotics in these cases are closely related to the conjecture that $\lambda_1 \geq 1/4$ for the groups $\Gamma(p)$.

CHAPTER 1. TRACE FORMULAES AND OTHER PRELIMINARIES

In this chapter we will state and give outlines of proofs of various summation formulas that are basic to our later work. We begin with the simplest of these.

1.1 Poisson Summation Formula

If $S(\mathbb{R})$ denotes the Schwartz class on $\mathbb{R}$, and $f \in S(\mathbb{R})$, then

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) .$$

In the language of distributions if $S = \sum_{n \in \mathbb{Z}} \delta_n$, where δ_x = point mass at x, one has

$$S = \hat{S} .$$

There are many ways of proving 1.1. We show here how we may obtain 1.1 by computing the trace of "the wave operator".

Consider the Cauchy problem

1.2

(a) $$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial t^2}$$

(b) $$\varphi(x,0) = \delta(y)$$

(c) $$\varphi_t(x,0) = 0$$

$(x,t) \in S' \times \mathbb{R}$.

Define the operator W_t by $(W_t\varphi)(x) = \varphi(x,t)$ where $\varphi(x,0) = \varphi(x)$, $\varphi_t(x,0) = 0$ and φ is the solution to 1.2(a).

The solution $\varphi(x,y,t)$ of 1.2 is the fundamental solution of the "half" wave equation and is the distributional kernel of W_t.

We may find φ explicitly by first finding the fundamental solution to 1.2(b) on $\mathbb{R}$ and then averaging. On $\mathbb{R}$ the solution is easily constructed by waves moving to the left and right.

$$\varphi(x,y,t) = \sum_n \frac{1}{2}\{\delta(x+n-y-t) + \delta(x+n-y+t)\}.$$

Now comes the basic idea (which goes back to Selberg (1956)): One computes the trace of W_t in two ways. On the one hand,

$$\mathrm{Trace}(W_t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x,x,t)\,dx = \sum_{n\in\mathbb{Z}} \frac{1}{2}[\delta(n+t) + \delta(n-t)]$$
$$= \sum_{n\in\mathbb{Z}} \delta_n(t).$$

On the other hand, using eigenvalues and eigenfunctions of the operator $\frac{d^2}{dx^2}$ on S^1, we have

$$\varphi(x,y,t) = \sum_n \cos(2\pi nt)\, e^{2\pi inx - 2\pi iny}.$$

Therefore, $\frac{1}{2\pi}\int_0^{2\pi} \varphi(x,x,t)\,dx = \sum_{n\in\mathbb{Z}} e^{2\pi int} = \widehat{\sum_{n\in\mathbb{Z}} \delta_n(t)}$. Equating these gives 1.1.

Let us examine the extent to which the above arguments may be carried out in a general setting. Let M be a compact C^∞ manifold without boundary, equipped with a smooth metric. The Laplace Beltrami operator on functions is given in local coordinates by

$$\Delta\varphi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g}\, g^{ij} \frac{\partial}{\partial x^i} \varphi)$$

where $ds^2 = g_{ij}dx^i dx^j$. Let $dA = \sqrt{g}\, dx' \ldots dx^n$ denote the corresponding volume element. The operator Δ is self adjoint in $L^2(M,dA)$ and has a spectral decomposition on eigenfunctions φ_n with eigenvalues λ_n, by which we mean

$$\Delta\varphi_n + \lambda_n\varphi_n = 0 .$$

Furthermore, the functions $\{\varphi_n\}$ are an orthonormal basis for $L^2(M,dA)$. The λ_n are real and non-negative. $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$ $\lambda_n \to \infty$ as $n \to \infty$.

As before, consider the Cauchy problem

1.3
$$\frac{\partial^2\varphi}{\partial t^2} = \Delta\varphi$$
$$\varphi(0,x,y) = \delta_y(x)$$
$$\varphi_t(0,x,y) = 0$$

Define $(W_t f)(x)$ to be the solution at time t to the equation $\frac{\partial^2\varphi}{\partial t^2} = \Delta\varphi$, $\varphi(0,x) = f(x)$, $\varphi_t(0,x) = 0$. Then 1.3 is the distribution kernel of W_t. We may write

$$W_t = \cos(\sqrt{\Delta}\, t) .$$

As before, one may calculate the trace of W_t in terms of eigenvalues as follows:

$$\varphi(t,x,y) = \sum_{n=0}^{\infty} \cos(\sqrt{\lambda_n}\, t)\varphi_n(x)\varphi_n(y)$$

1.4 $$\mathrm{Trace}(W_t) = \int_M \varphi(t,x,x)\,dA(x)$$

$$= \sum_{n=0}^{\infty} \cos(\sqrt{\lambda_n}\, t) .$$

Can we compute $\int_M \varphi(t,x,x)\,dA(x)$, or better still $\varphi(t,x,y)$, in some other way?

1.5 Selberg Trace Formula

If the space in question has some homogeneous properties, then $\varphi(t,x,y)$ can often be calculated explicitly. For example, this is true in a rank one symmetric space and in particular in hyperbolic n-space.

Let $H^{n+1} = \{(x,y)\colon x \in \mathbb{R}^n,\ y > 0 \in \mathbb{R}\}$. By hyperbolic n-space we mean H^{n+1} equipped with the metric

$$ds^2 = \frac{dx_1^2 + \ldots + dx_n^2 + dy^2}{y^2} .$$

This gives rise to a Riemannian space of constant sectional curvature. It is a rank one space. The Laplace Beltrami Operator

$$\Delta = y^2\left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y^2}\right) - (n-1)y\,\frac{\partial}{\partial y}$$

affords an invariant differential operator which generates the algebra of invariant operators. The space M which was considered earlier, will in this case be H^n/Γ, for some discrete group Γ of isometries of H^n, which acts discontinuously.

One may obtain the fundamental solution to the problem (notice slightly modified)

1.6
$$\frac{\partial^2\varphi}{\partial t^2} = \Delta\varphi + \left(\frac{n}{2}\right)^2 \varphi$$
$$\varphi(x,0) = 0$$
$$\varphi_t(x,0) = f(x)$$

by first obtaining the solution on H^n and then automorphizing (averaging over). One may then proceed as before to compute the trace of W_t in two ways. This is carried out for $n = 1$ in Lax-Phillips (p.236). There are no added difficulties for higher n, indeed n even is somewhat easier than n odd. For another proof (due to Selberg) of the formula which comes out of the above computation, we refer to Cohen and Sarnak [5].

We state the resulting formula in the case of the hyperbolic plane.

We assume that $M = H/\Gamma$ is compact and smooth (i.e. Γ has no fixed points in H). Also, let λ_n be eigenvalues of Δ, and let $r_n = \pm\sqrt{\lambda_n - 1/4}$, and let CP be as in the introduction.

1.7 <u>Theorem (Selberg)</u>

Let $g \in C_0^\infty(\mathbb{R})$, g even, and $\hat{g}$ is Fourier transform, then

$$\sum_n \hat{g}(r_n) = \frac{\mathrm{Vol}(M)}{4\pi}\int_{-\infty}^{\infty} r\tanh(\pi r)\hat{g}(r)dr + 2\sum_{\gamma\in CP}\frac{\tau(\gamma_0)}{\sinh(\tau^{\frac{1}{2}}(\gamma))}\, g(\tau(\gamma))$$

where $\tau(\gamma_0)$ is the length of the primitive geodesic covered by γ.

The appearance of the last term above is the key to the use of the trace formula to count prime geodesics. The above may be viewed as a type of duality between the lengths of the closed geodesics and the eigenvalues of the Laplace Beltrami operator.

1.8 More General Trace Formulas

In the case of a compact negatively curved manifold, one may try to calculate the fundamental solution of the wave equation as described in the case of a homogeneous space. This could be done by calculating the fundamental solution on the universal covering M and then "automorphizing". Of course, there is no hope of an explicit solution. However, one may construct progressing wave solutions about a point. These will give approximations to the fundamental solution, to as high a degree of smoothness as desired. Having done so, one "automorphizes" and takes the trace. The result is of course not an explicit trace formula, but one which identifies singularities. A calculation along these lines was carried out by Donnelly [6]. The result is a special case of the work of Duistermaat and Guillemin.

1.9 Theorem

Let M be a compact manifold of negative curvature, then the following identity holds as distributions

$$\sum \cos(\sqrt{\lambda_n}\, t) = \sum_{\gamma \in C} \frac{\tau(\gamma_0)}{|I - P_\gamma|^{\frac{1}{2}}} \delta(t - \tau(\gamma)) + R(t) .$$

Where $R \in L^1_{loc}$ away from zero, and P_γ is the Poincare map about γ, which we describe in the next section.

When M is of arbitrary curvature, the situation is quite a lot more difficult. One needs to use the full force of the theory of Fourier Integral Operators to construct a paramatrix.

1.10 The Geodesic Flow

Let M be as usual. Denote by SM the sphere bundle to M, that is, the space of linear elements (x,θ), $x \in M$, $\theta \in T_xM$, $\|\theta\| = 1$. We denote by π the natural projection of SM on M. An element $v \in SM$ defines a geodesic $\gamma_v(t)$ where $\dot{\gamma}_v(0) = v$ (and so $\gamma_v(0) = \pi(v)$). The geodesic flow φ_t on SM is defined by

$$\varphi_t(v) = \dot{\gamma}_v(t) .$$

It is clear that for each t, φ_t is a diffeomorphism of SM. Also, the periodic orbits of φ_t correspond to the closed geodesics in M. The tangent bundle to M, TM, has a natural Riemannian structure given by the connection map K which we describe.

For $v \in TM$ and $\xi \in T_v(TM)$ let $Z\colon (-\varepsilon,\varepsilon) \to TM$ be a curve with initial vector ξ. Then $\alpha = \pi_0 Z\colon (-\varepsilon,\varepsilon) \to M$, and one defines $K(\xi) = Z'(0)$, where $'$ denotes covariant differentiation.

$$K: T(TM) \to T(M)$$

and

$$(TM)_v = \ker d\pi \oplus \ker K$$
$$= \text{vertical} \oplus \text{horizontal space};$$

see Eberlein [8].

The Riemannian structure on TM is given by $\xi, \eta \in (TM)_v$

$$\langle \xi, \eta \rangle = \langle d\pi\xi, d\pi\eta \rangle_{\pi v} + \langle K\xi, K\eta \rangle_{\pi v} .$$

We give SM the Riemannian structure induced as a submanifold of TM. Let μ denote the corresponding Riemannian volume measure. The following "Louiville Theorem" is well known.

1.20 Proposition

The flow φ_t preserves the measure μ.

We will call μ the canonical volume measure on SM.

1.21 Poincare Map

Let γ be a periodic orbit of the geodesic flow φ_t, i.e., for $x \in \gamma$ there is T such that $\varphi_T(x) = x$. Now $(\varphi_T)_*: T_x \to T_x$ and this linear map clearly has the flow direction at x as a one dimensional eigenspace. The linear Poincare map is defined to be the induced map on the normal space at x. We denote it by P_γ. This definition depends on the choice of $x \in \gamma$, but choosing another point simply gives a conjugate to P_γ. An exact expression for P_γ may be obtained by integrating a Jacobi differential equation around γ.

We conclude this chapter by introducing the notion of entropy of a flow. This number will play a vital role in the prime geodesic theorems.

1.22 Entropy

Let φ_t be a one parameter family of smooth diffeomorphisms of a compact manifold M. Let $d(\cdot,\cdot)$ be a metric (distance function) on M. For $T > 0$ define

$$d^T(x,y) = \max_{0 \leq t \leq T} d(\varphi_t x, \varphi_t y) .$$

Let $N(T,\varepsilon)$ be the minimal number of ε balls in the d^T metric needed to cover M. The topological entropy h, of this flow, is defined by

1.23 $$h = \lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{\log N(T,\varepsilon)}{T} .$$

One can show that this definition is independent of the choice of $d(\cdot,\cdot)$. Indeed, one can give a definition purely in terms of open sets, as was done in Adler [1], where the notion of topological entropy was first introduced.

For example, one checks easily that for the geodesic flow on the sphere bundle of a standard sphere, or flat torus, one has $h = 0$. We shall see later, however, that for the geodesic flow on a surface of genus greater than or equal to two (and any Riemannian structure), $h > 0$.

There is also the notion of measure entropy h_μ. Suppose that μ is a positive probability measure on M which is φ_t invariant. In this case one has the usual definition of $h_\mu(\varphi_t)$, see e.g., Sinai [30].

There is the following relation between h_μ and h:

$$h = \sup_\mu h_\mu$$

where the sup runs over all φ_t invariant probability measures μ. See Goodwyn [10].

Finally, we would like to state a result of Pesin and Margulis, which gives an "exact" formula for h_μ when μ is a Riemannian volume measure.

1.24 Liapunov Exponents

Let φ_t be as in the beginning of 1.22. Fix a Riemannian structure on M. For $v \in T_\xi M$ define

$$\chi^+(v,\xi) = \overline{\lim_{t\to\infty}} \frac{\log\|d\varphi_t(v)\|}{t}.$$

Now it is clear that χ^+ assumes at most n values on $T_\xi M$ (where $n = \dim M$). There is a filtration

$$L_1(\xi) \subset L_2(\xi) \subset \ldots \subset L_{i(\xi)}(\xi) = T_\xi M$$

where $\chi^+(v,\xi) = \lambda_i(\xi) > 0$, say for $v \in L_i(\xi) | L_{i-1}(\xi)$. Let

$k_i(\xi) = \dim L_i(\xi) - \dim L_{i-1}(\xi)$. Finally, set $\chi(\xi) = \sum k_i(\xi)\chi_i(\xi)$.

1.25 Theorem (Pesin, Margulis)

If $\{\varphi_t\}$ is C^2 and μ is a Riemannian measure invariant under φ_t, then

$$h_\mu = \int_M \chi(\xi)\,d\mu(\xi)\ .$$

For a proof see Pesin [26].

CHAPTER 2. ENTROPY

We recall the definition of the prime geodesic counting function

$$\pi(x) = \#\{\gamma \in CP_0 : \tau(\gamma) \leq x\}$$

where CP_0 is the set of primitive closed geodesics, and $\tau(\gamma)$ is the length of γ. It is difficult, without further assumptions, to give upper bounds for the growth of $\pi(x)$. This is so since, as we saw in the introduction, $\pi(x)$ may be infinite for some finite x. However, lower bounds may be obtained for certain manifolds.

For simplicity, consider a compact closed surface M of genus $g \geq 2$ (the results here will apply equally well to manifolds which carry metrics of negative curvature in every section). We prove the following conjecture due to Sinai [31].

2.1 Proposition

For any metric on M one has

$$\varliminf_{x \to \infty} \frac{\log \pi(x)}{x} > 0 .$$

Proof

We will see later that for a metric σ_0 on M, of negative curvature one has

$$\varliminf_{x \to \infty} \frac{\log \pi_0(x)}{x} = \beta > 0 .$$