The Eightfold Way: The Beauty of Klein's Quartic Curve

Silvio Levy

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[Reviewed by Ruth Michler, on 07/31/2000]

The Eightfold Way: The Beauty of Klein's Quartic Curve is the 35th volume of the *MSRI Research Publications* series from Cambridge University Press. The book is unique in that it also includes an article on the aesthetic aspects of mathematics. In fact several of the papers collected here make reference to the beauty of mathematics. The book was written to commemorate the November 14, 1993 unveiling of "The Eightfold way," a marble sculpture by Helaman Ferguson inspired by the remarkable mathematical properties of the Klein Quartic curve. The German Mathematician Felix Klein first discussed this curve in his 1878 paper "Ueber die Transformationen siebenter Ordnung der elliptischen Funktionen" (see below for the exact reference).

The sculpture is permanently installed on the southeast patio of MSRI, at 1000 Centennial Drive in Berkeley CA, and is accessible to the public. This volume contains several color photographs of the sculpture as well as of some of the rather elaborate equipment used in its production. The base of the sculpture represents a regular hyperbolic tessellation of the Riemann surface, whereas the tetrahedral top part, made of white Carrara marble, represents the same surface folded over itself. The name the "Eightfold Way" makes reference to one prominent feature of the sculpture: if one traces along any of the prominent ridges or grooves on the marble part of the sculpture, making alternating left and right turns at intersections for a total of 8 turns, one arrives back at one's starting point.

The distinguishing feature of this book is that it tries to convey the intrinsic beauty of mathematics by highlighting the interrelation of art and mathematics. It was mathematics that inspired this piece of art and then art (or this Sculpture in particular) inspired mathematics.

The book consists of a preface by Silvio Levy, an article by Claire and Helaman Ferguson about the sculpture, seven mathematical articles related to the Klein Quartic curve, and finally Silvio Levy's English translation of Felix Klein's original article.

William Thurston's article is a copy of the hand-out given to the general public at the unveiling. It is a hands-on introduction on geometric group theory and is written for a general audience. In particular it explains the eightfold way of walking around the sculpture. It takes eight turns on the surface of the sculpture to get back to the starting point. The reader is challenged to explore what happens on a tetrahedron, cube or dodecahedron.

Hermann Karcher and Matthias Weber wrote a semi-expository article proving the platonicity of the tessellation. Like the Fermat surfaces, i.e. surfaces defined by equations $x^k + y^k + z^k = 0$, the Klein Quartic can be uniquely described by its platonic tessellation. A tessellation is platonic if the symmetry group acts transitively on flags of faces, edges and vertices. Moreover, the authors explicitly relate the hyperbolic description of the Riemann surface to its defining equation $x^3y + y^3z + z^3x = 0$.

Noam Elkies contributed an extremely well-written expository account of the importance of the Klein quartic in number theory. His article also contains a very elegant and elementary proof of Fermat's last theorem in the case when $n = 7$. The second part of the article addresses the relevance of Klein's Quartic curve to Kenku's recent proof of the Stark-Heegner theorem on imaginary quadratic number fields of the class number one.

Murray Macbeath's article deals with the unusually large number of automorphisms of the Klein quartic. A compact Riemann surface of genus $g > 1$ can have at most $84(g - 1)$ automorphisms. The Klein Quartic is the plane curve of lowest degree that attains this upper bound. Other curves with that property are called Hurwitz curves. A. Murray Macbeath was the first to exhibit the existence of infinitely many Hurwitz curves. In his article he gives a historical account of his own involvement with Hurwitz groups. He also points out some open questions (as of 1993).

Jeremy Gray's *Mathematical Intelligencer* article is included for its discussion of the remarkable geometric properties of the Klein quartic such as its genus, its 24 inflection points, 28 bitangents and the degree of the homogeneous polynomial defining the plane curve. The article also explains the geometry behind the projective special group of 2x2 matrixes with entries in the field with seven elements, $PSL_2(\mathbf{Z}/7\mathbf{Z})$, and its subgroups. The proof that Klein's quartic is a plane curve of genus 3 uses Euler's formula for a triangulation and is arguably simpler than using Hurwitz's formula for branched coverings. This article is essentially self-contained and might be accessible to mathematically-sophisticated undergraduates. Like Macbeath, Gray takes a historical approach. In particular, he provides evidence that Felix Klein was unaware of the non-Euclidean aspects of the Klein Quartic.

Claire and Helaman Ferguson 's article details the creative process and the technical difficulties that had to be overcome in the production of the sculpture. Since most of the carving is done by state-of-the-art computer technology, the artist might be accused of executing a hi-tech paint by numbers. However, after reading this article the reader realizes the deliberate choices required in the choice both of the mathematics to depict and of its representation.

The next two articles were contributed by Dr. Allan Adler. His first article explains Felix Klein's tricks for finding invariants of the 3-dimensional complex representation of $PSL_2(\mathbf{Z}/7\mathbf{Z})$. Klein showed that the ring of invariants is generated by four invariants, and gave geometric explanations of the algberaic zero loci of these invariants. Adler then explains how Klein's tricks can be viewed as so-called geometric constructions, a concept introduced in section 6 of this article. The main result proves that the modular curve $X(p)$ can be constructed from a special 3-tensor when p is either 11 or a prime bigger than 13.

Allan Adler's second article is more technical, dealing with the construction of a Hilbert modular surface for the real number field obtained by adjoining a square root of 7 to $\mathbf{Q}$. Adler outlines a proof that the Hilbert modular surface in question "is" the projective plane. The second half of the article provides explicit algebraic equations for Klein's generating invariants.

While the editor has exhibited excellent taste in the selection of articles, it is regretable that the recent applications to coding theory were not included. Recently efficient decoding algorithms for error-correcting codes constructed from the Klein Quartic curve have been found, providing yet another example of the use even the purest mathematics can be put to.

The intended readership of this volume is research mathematicians. Most non-Mathematicians will enjoy the articles by William Thurston and by Claire and Helaman Ferguson, and maybe also the article by Jeremy Gray. Of course, everyone will enjoy the photographs. Most non-mathematicians will be confused by the alternate mention of the Klein quartic as a *curve* and as a (Riemann) *surface*. With this sculpture Helaman Ferguson has succeeded in conveying some of the beauty of the subject to a non-mathematical audience. Helaman Ferguson is clearly doing something unique in his work. He is uniquely qualified to do so, having trained as a mathematician as well as a sculptor. His work deserves the recognition it gets in this special volume of the MSRI Research Publications.

References

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- D. Rotillon and A Thiong Ly "Decoding codes on the Klein Quartic" EuroCode '90, LNCS 514, Springer 1991.

Prof. Ruth Michler was Associate Professor at the University of North Texas until her recent tragic death in a traffic accident while visiting Northeastern University. She sent us this review last September.

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