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Variations on a Theorem of Abel

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To Jean-Pierre Serre

Contents

Introduction

The motivation for this series of papers is an attempt, once again, to arrive at some understanding of higher codimensional subvarieties lying on an algebraic variety.

For divisors, one first knows which homology classes are algebraic – this is the Lefschetz (1,1) theorem. Next, as proved by Picard and Lefschetz, homological and algebraic equivalent coincide. Finally, the divisors algebraically equivalent to zero modulo those rationally equivalent to zero constitute an abelian variety, the Picard variety, whose structure is reasonably well understood. For an algebraic curve this latter is the Jacobian variety which plays a decisive role in the study of the curve.

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In higher codimension the question of which homology classes are algebraic is the Hodge conjecture – it is false for torsion classes and there are precious few examples over $\mathbb{Q}$. Next, homological and algebraic equivalence may not coincide – our most optimistic guess is that the quotient forms a finitely generated Mordell-Weil type subgroup of an intermediate Jacobian. Finally, the quotient of cycles algebraically equivalent to zero modulo those rationally equivalent to zero may not even be finite-dimensional. This is due to Mumford, and will be the case for an algebraic surface S for which the geometric genus $p_g(S) \neq 0$. Roitman has refined Mumford's result to the statement that a "general" zero-cycle will be isolated in its rational equivalence class, so that the above quotient will be "as infinite-dimensional as possible." [1]

Confronted with this state of affairs it seemed a good idea to go back and have a look into just how our understanding of the beautiful codimension one theory came about. Here, almost certainly the decisive step was *Abel's theorem*. This claim is by no means intended to minimize the later works of Jacobi, Riemann, etc., but rather to maintain that it was Abel's theorem which initially got the ball rolling. His general addition theorem provided the key to unlocking the structure of an algebraic curve via its Jacobian. Moreover, once Jacobians were understood they provided the tool for the first proofs by Picard, Poincaré, and Lefschetz of the general theorems about divisors mentioned above. Indeed, these proofs were based on properties of Jacobians of a pencil of curves varying on a fixed surface. So one is confronted with the decisive role which Abel's theorem played historically in arriving at our present understanding of divisors.

Upon looking into the original papers by Abel and some of the works following it – especially Jacobi – there were several surprises. To begin with, Abel's original statement was somewhat more general than that usually presented in textbooks, although a converse to the more narrow statement is now usually provided.

More importantly, some of the original *flavor* seems to have been lost. We now think of Abel's theorem together with Jacobi inversion as stating that on an algebraic curve of genus g the effective divisors of degree g admit a rational group law, generalizing the group structure underlying the addition theorem for the classical elliptic integral. This is certainly a beautiful statement. However, as shown by Mumford's theorem, this version of Abel's theorem does not extend to higher codimension.

Now it seems to me quite possible that Abel viewed his theorem in a broader context. The formal statement of the result is of a quite general character and may be extended to higher-dimensional varieties using either the trace or residues, both

1 On a somewhat more positive note, very nice recent work by Spencer Bloch has managed to bring some order into the structure of zero-cycles modulo rational equivalence on an abelian variety A. Even though this group $C(A)$ is infinite-dimensional, he has found on it a filtration whose successive quotients have geometric meaning. For example, on an abelian surface Bloch found the pair of exact sequences

$$0 \to K \to C(A) \xrightarrow{\deg} Z \to 0,$$
$$0 \to I \to K \xrightarrow{\sigma} A \to 0$$

where "deg" is the degree map, σ is the sum in the group structure on A, and I are the zero cycles generated by complete intersections $D \cdot D'$ of divisors $D, D' \in \mathrm{Pic}^0(A)$. Some of the discussion below further suggests the special role of complete intersections (c.f. (2.26)) of which will be discussed in this paper. More importantly, the informal understanding seems to have been that the presence of *global functional relations* or *addition theorems* (loosely interpreted) was a widespread phenomenon in algebraic geometry, and one should usually expect at least some among them to yield precise insight. It is this philosophy which provides our frame of reference.

A third surprise was the very pretty fashion in which the power of Abel's theorem was illustrated by applications to elementary questions in geometry. Among the first of these was Jacobi's treatment using elliptic functions of the classical Poncelet problem concerning closed polygons inscribed in one conic and circumscribed about another. While some of these geometric applications remain, many have been forgotten – perhaps because of our preoccupation with the intrinsic birationally invariant properties of a variety as opposed to the extrinsic or projectively invariant properties. More seriously, even though we have generalized the definitions and formalism of the structure of divisors to higher codimension, these fail to yield answers to simple Poncelet type puzzles such as was possible classically by Abel's theorem.

Consequently, in this series of papers we have set about to rethink the Abel-type philosophy concerning global functional relations with emphasis on finding special cases where the necessary conditions in a geometric problem imposed by an "additional theorem" can be proved to be sufficient. It is much too soon to draw general conclusions, but we have been able to find a number of situations where inversion is possible. The higher codimensional questions characteristically appear to be nonlinear, thus explaining the absence, in general, of a group structure – although this latter will certainly be present in important special cases.

We have two purposes in this particular paper. One is to present an historical exposition of Abel's proof of his theorem together with a few of the early applications in the spirit – but not necessarily the precise form – in which they were initially given. There are several reasons for undertaking such an exposition. One is to establish the tone and frame of reference for what will be discussed later. Another is that unlike many turning points in the fairly recent history of mathematics, it seems to me that some of the original meaning of Abel's theorem has been lost – in any case it has certainly been narrowed – and on this the 150th anniversary of his original paper[2] it is hopefully worthwhile to revive his work in the general context of attempting to shed some light on higher codimension questions.

There are also some new results in this paper. One is concerned with inverting the conditions for rational equivalence of zero cycles on a surface which Mumford used, and another is a converse to a global residue theorem. These are presented in Sections II d and III a with some applications appearing from time to time. There is a principal new theorem, stated in Section II e and proved in Section III c, and which we now explain.

Let $V_n \subset \mathbb{P}^{n+r}$ be an algebraic variety of pure dimension n in a projective space of dimension $n+r$. We assume that V has no multiple components, but otherwise the singularities are arbitrary. We also assume given a rational n-form ψ on V. Denoting by $\mathbb{G}(r, n+r)$ the Grassmann variety of projective r-planes in $\mathbb{P}^{n+r}$, a

2 Memoire sur une propriéte genérâle d'une classe très étendue de fonctions transcendentes, Oeuvres de N. H. Abel, Vol. I, pp. 145–211; c.f. also, Démonstration d'une propriété genérâle d'une certainé classe de fonctions transcendentes, loc. cit., pp. 515–517

general plane $A \in \mathbb{G}(r, n+r)$ meets V in $d = degree\ V$ distinct simple points $P_\nu(A)$. We use the notation of cycles to write the intersection

$$A \cdot V = P_1(A) + \cdots + P_d(A),$$

and define the *trace* $\mathrm{Tr}(\psi)$ to be the n-form given by the formula

$$(0.1) \quad \mathrm{Tr}(\psi) = \psi(P_1(A)) + \cdots + \psi(P_d(A))$$

where $\psi(P_\nu(A))$ is the pullback of ψ under the map $A \to P_\nu(A)$. The formula (0.1) defines the trace on a dense open set in the Grassmannian, and the discussion in Section II will show among other things that $\mathrm{Tr}(\psi)$ is a rational n-form on $\mathbb{G}(r, n+r)$. We say that ψ is of *the first kind for the imbedding* $V \subset \mathbb{P}^{n+r}$ if $\mathrm{Tr}(\psi)$ is holomorphic. In case V is nonsingular this is equivalent to ψ being holomorphic on the complex manifold V. However, if, e.g., $V_n \subset \mathbb{P}^{n+1}$ is a hypersurface given in affine coordinates by an equation $f(x_1, \ldots, x_{n+1}) = 0$, then ψ may be written in the form

$$\psi = \frac{r(x)\, dx_1 \wedge \cdots \wedge dx_n}{\partial f / \partial x_{n+1}(x)}$$

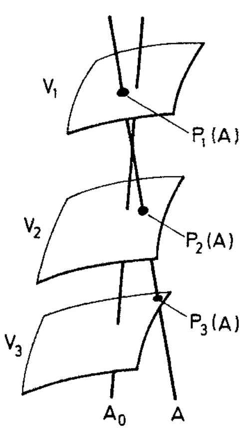
for a rational function $r(x)$, and ψ is of the first kind for this embedding if, and only if, $r(x)$ is a polynomial of degree at most $d-n-2$.

In case ψ is of the first kind the trace $\mathrm{Tr}(\psi) \equiv 0$ since there are no holomorphic forms on the Grassmannian. When written out, this relation becomes

$$(0.2) \quad \psi(P_1(A)) + \cdots + \psi(P_d(A)) \equiv 0,$$

which may be viewed as a functional relation or addition theorem globally linking together the local behavior of V around the points of intersection with a variable r-plane.

Our main theorem is a converse to (0.2). Namely, suppose we are given little pieces or germs $V_1, \ldots, V_d$ of irreducible n-dimensional complex analytic varieties in $\mathbb{P}^{n+r}$ together with meromorphic n-forms $\psi_\nu \not\equiv 0$ on V_ν. Assume that there is an r-plane A_0 meeting each V_ν once in a simple point which is not a pole of ψ_ν.



Then for A in a neighborhood U of A_0 the trace

$$\psi_1(P_1(A))+\cdots+\psi_d(P_d(A)) \quad (P_v(A)=A\cdot V_v)$$

makes sense, and we have the

Main Theorem. *Assume that the addition theorem*

$$(0.3) \quad \psi_1(P_1(A))+\cdots+\psi_d(P_d(A))\equiv 0$$

is satisfied. Then there is an algebraic variety $V_n\subset\mathbb{P}^{n+r}$ and rational n-form ψ, of the first kind relative to this embedding and such that each $V_v\subset V$ and $\psi|V_v=\psi_v$.

This then is a type of converse to the Abel relation (0.2). Intuitively, the functional equation (0.3) will allow us to *propagate* the pieces of variety V_v in much the same way that Abel's original addition theorem initially led to the construction of elliptic functions as will be discussed in Section Ib (c.f. the argument centered around Eqs. (2.18) and (2.19)).

A result of the above type for algebraic curves goes back about 100 years. Sophus Lie wanted to characterize Jacobians as being those principally polarized abelian varieties whose theta divisor was doubly of translation type. In trying to prove this he was led to the above theorem (in integrated form) in the case of plane quartic curves. His discussion of this result is lengthy and the proof appears to be rather complicated. Our first proof of the general theorem was also quite messy, and the argument we shall give below was suggested by Darboux' proof of the Sophus Lie theorem (c.f. G. Darboux, Lecons sur la théorie generale des surfaces, 2d ed., Paris (1914), Vol. I, pp. 151–161). Our main innovations are to consider the problem for n-dimensional varieties, and to observe that since what is being constructed is a variety V together with a rational form ψ on V the notion of *residue* is central to the problem. Indeed, if $V\subset\mathbb{P}^{n+1}$ is a hypersurface, then the data (V,ψ) is *equivalent* to a rational $(n+1)$-form Ψ on $\mathbb{P}^{n+1}$ whose *Poincaré residue* is ψ (c.f. Section IIIc). The construction of Ψ from the local data (V_v,ψ_v) proceeds in a natural manner, and ultimately uses the Levi-Hartogs theorem. The same result for curves in $\mathbb{P}^n$ was discussed by Wirtinger, and the Darboux proof was used by Blaschke-Bol in their study of webs (c.f. W. Blaschke and G. Bol, Geometrie der Gewebe, Springer, Berlin (1938), pp. 209–224. References to Lie and Wirtinger appear on p. 240).

On a personal note I would like to say that it was B. Saint-Donat who told me of the Sophus Lie theorem. He has thought through the Sophus Lie problem in an algebraic setting (characteristic p) and has arrived at a proof in this situation. In fact, his proof — which is of more formal and less computational character than the one we shall give here — deals with the situation where the residual family need not be linear spaces, and suggests some very interesting questions. Our main theorem may be phrased in a purely algebraic context and should also be true in characteristic p.[3]

We conclude the introduction by giving some further references and comments on style. In Part I an attempt has been made to present Abel's theorem and the subsequent inversion theorem in an historical context, viewing the questions as they appeared at the time and treated using only calculus. Our proof of Abel's theorem is essentially the original, as is the discussion of the inversion of the elliptic

3 See Notes Added in Proof

integral including the degenerate cases. These arguments are direct and elementary, and are in my opinion at least as penetrating and elegant as modern proofs. The general addition theorems in Section Ic are a synthesis of Göpel, Rösenhaim, Jacobi, and Cayley. An interesting account of the early difficulties encountered in the simultaneous inversion of several abelian integrals is given by Weierstrass in the "Anmerkungen" to Volume II of the Gesammelte Werke of Jacobi (c.f. pp. 516–521). The result evolved laboriously over a period of 25 years or so, and our proof is somewhat different and uses a counting constants argument to reduce to a situation formally analogous to the cubic case. It is worthwhile to keep in mind that the early discussions of inversion were given up after Riemann's thesis in justifiable favor of his explicit inversion using the theta function – which opens the door to the lovely and subtle theory of special divisors. Our treatment is not meant to be complete – in fact, just the opposite, it may be used to provide historical background to the recent beautiful notes, *Curves and their Jacobians*, University of Michigan Press, Ann Arbor (1975), by David Mumford. As mentioned previously, the discussion of Poncelet is gleaned from Jacobi; we have used geometric arguments in lieu of his calculations involving elliptic functions. Recently, Joe Harris and this author found an even more symmetric form of the Poncelet problem for quadric surfaces in 3-space, which will be given in a future paper.

In Part II we have taken up the trace and its relation to Abel's theorem in higher dimensions. The properties of the trace are proved from local analytical considerations and the global G.A.G.A. principle, both of which we have treated as being based on Remmert's proper mapping theorem – c.f. R. Remmert, Holomorphe und meromorphe Abbildungen komplexer Räume, Math. Annalen, Vol. 133 (1957), pp. 328–370. The use of the trace to study points on a surface has been around for some time. The earliest reference I can find is to Max Noether, Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde, Math. Annalen, Vol. 2 (1870), page 304 and Vol. 8 (1875), page 495, whose methods were taken up by Severi and then used by Mumford to draw conclusions opposite those desired by Severi (c.f. David Mumford, Rational equivalence of zero cycles on surfaces, Jour. Math. Kyoto Univ., Vol. 9 (1969), pp. 195–209). It is interesting to note that an Abel-type relation underlies this work as well as that of Bloch mentioned in footnote 0 above. Other references to zero cycles are Arthur Mattuck, Ruled surfaces and the Albanese mapping, Bull. A.M.S., Vol. 75 (1969), pp. 776–779 and A. A. Roitman, Rational equivalence of 0-cycles, Math. U.S.S.R.-Sbornik, Vol. 18 (1975), pp. 571–588. Aside from Mumford's theorem discussed in IId, the other applications of the trace and inversion of Abel-type conditions appear to be new.

Finally, in Part III we have taken up the connection between Abel's theorem and residues relative to the ambient space in which our variety is embedded. This was known for curves in $\mathbb{P}^2$ pretty much from the beginning, and may be found in Jacobi (for a precise reference see footnote 12 in Section IIIa). A recent account appears in the book by Benjamino Segre, *Some Properties of Differentiable Varieties and Transformations*, Ergebnisse der Math., Vol. 13 (1957), Springer-Verlag. Our converse to the residue theorem and use of it for surfaces other than $\mathbb{P}^2$ seems to be new, and will be taken up at greater length in the second paper in this series. It is the converse to the residue theorem which philosophically – but not as yet in a precise mathematical sense – underlies our working hypotheses concerning the inversion of Abel-type conditions which we mentioned previously. The references surrounding the Sophus Lie theorem were given above.

I. Abel's Theorem in Original Form and Applications

(a) Origins of the Theorem and Abel's Proof

We shall discuss some of the questions which led up to Abel's theorem, and then give what amounts to his proof of the result. An attempt will be made to do this from the viewpoint of the 18th and early 19th centuries.

In the applications of calculus to geometry and mechanics integrals of the kind

$$(1.1)\quad \int r(x, y(x))\, dx$$

were frequently encountered. Here $r(x, y) = p(x, y)/q(x, y)$ is a rational function of x and y, and $y(x)$ is an *algebraic function* of x – by which we mean that there is a polynomial $f(x, y)$ whose roots $y_1(x), \ldots, y_n(x)$ are multi-valued functions of x and $y = y_\nu(x)$ is one of these selected in a continuous fashion.

For example, on the circle $f(x, y) = x^2 + y^2 - 1 = 0$ the arclength is given by the integral

$$(1.2)\quad \int \sqrt{dx^2 + dy^2} = \int \frac{dx}{\sqrt{1 - x^2}},$$

since $x\,dx + y\,dy = 0$ and

$$\sqrt{dx^2 + dy^2} = \left(\sqrt{1 + \left(\frac{-x}{y}\right)^2}\right) dx$$
$$= \frac{dx}{y}$$

on this circle. Similarly, if we represent the ellipse $x^2/a^2 + y^2/b^2 = 1$ parametrically by $x = a \sin\theta$ and $y = b \sin\theta$, then the arclength integral is

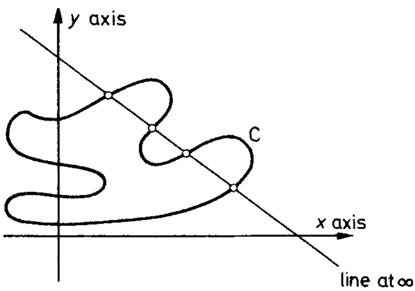
$$\int \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}\, d\theta = \int \frac{a(1 - k^2 \sin^2\theta)\, d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

where $k^2 = (a^2 - b^2)/a^2$ and we assume $a > b$. Setting $x = \sin\theta$ this integral becomes

$$(1.3)\quad a \int \frac{(1 - k^2 x^2)\, dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} \qquad (k^2 \neq 1).$$

The integrals (1.2) and (1.3) are *trigonometric* and *elliptic integrals*, respectively. It was, of course, well known that (1.2) defines the elementary function arcsin x having familiar properties, especially the addition theorem, but the understanding of elliptic and higher integrals provided a major problem during the 19th century.

In order that we may explain what Abel proved about these integrals (1.1), it is convenient to first establish some conventions and terminology. To begin with, we assume that x and y are complex numbers. At the time of Abel the utilization of complex numbers in geometry was quite recent (Poncelet, c. 1810) and he was one of the first to use them systematically. The polynomial $f(x, y)$ may be reducible but will be assumed to have no repeated factors. Thus the *algebraic curve* C defined by $f(x, y)=0$ may have several components, none of which are multiple, and arbitrary singularities otherwise. The *points at infinity* on the curve are the asymptotic directions, or equivalently, the limiting positions of the ratios $y_\nu(x)/x$ as $|x|\to\infty$. These are just the intersections of C with the line at infinity in the complex projective plane $\mathbb{P}^2$. The path of integration is given by choosing a piecewise smooth arc $x(t)$ in the complex x-plane and then continuously selecting one of the roots



$y_\nu(x(t))$ of the equation $f(x(t), y)=0$. Due account is taken of the fact that (1.1) may be an improper integral as the upper limit of integration approaches certain *singular points.* By Cauchy's theorem the integral remains invariant under continuous deformation of the path of integration, so long as we do not cross any of these singular points.

Of course, this may all be explained somewhat more satisfactorily using the *abstract Riemann surface* $\tilde{C}$ associated to C and interpreting the integrand as a meromorphic differential on $\tilde{C}$, but we want to put ourselves in the shoes of the mathematicians of that period, at least for a little while.

Integrals of the form (1.1) are called *abelian integrals* and the integrand $\omega=r(x, y(x))\,dx$ will be called an *abelian differential.* We may view ω as a rational 1-form on the algebraic curve C. The simplest case is when $r=r(x)$ is just a rational function of x. Then $r(x)$ may be expanded in partial fractions, and the result directly integrated to yield

(1.4) $$\int r(x)\,dx=R(x)+\sum_\nu a_\nu \log(x-x_\nu)$$

where $R(x)$ is again a rational function of x. Consequently, such abelian integrals are not mysterious.

For later use we wish to observe that the same is true in several variables: If $x=(x_1,\dots,x_n)$ and $\omega=\sum_{i=1}^{n} r_i(x)\,dx_i$ is a *closed* differential 1-form with rational coefficient functions, then $\int\omega$ is invariant under continuous deformation of the path of integration provided we don't cross a singularity. It is asserted that again

(1.5) $$\int\omega=R(x)+\sum_\nu \log S_\nu(x)$$

where $R(x)$ and the $S_\nu(x)$ are rational functions of x. We prove this when $n=2$ and under the harmless assumption that the origin is not a singular point. It is then possible to choose the path of integration to lie on the line $\xi=x_1/x_2=$constant. Setting $x_1=t\xi$, $x_2=t$ the integrand

$$\omega=\tilde{r}(t,\xi)\,dt$$

where $\tilde{r}(t,\xi)$ is rational in t and ξ. According to the one-variable result,

$$\int \tilde{r}(t,\xi)\,dt=\tilde{R}(t,\xi)+\sum_\nu A_\nu \log(t-t_\nu(\xi))$$

where the $t_\nu(\xi)$ are the points on the above line at which $\tilde{r}(t,\xi)$ has a pole. It is important to note that the *residues* A_ν are constants independent of ξ since $d\omega=0$. In fact, if there is branching such as $t_\nu(\xi_0)=t_\mu(\xi_0)$ for some ξ_0, then $A_\nu=A_\mu$. If we therefore collect the points $t_\mu(\xi)$ into groups each of which permutes transitively among itself as the line ξ varies, then the A_ν are constant for each group and the product $\prod_\nu (t-t_\nu(\xi))$ for such a group is a *rational* function of t and ξ. This is because the product is evidently a single-valued *meromorphic* function on $\mathbb{C}^2$ having at most polynomial growth at infinity, and is hence rational. From this we now may deduce (1.5) by collecting together the terms in $\sum_\nu A\log(t-t_\nu(\xi))$ according to the various groups and expressing each as $\log\tilde{S}(t,\xi)$ where $\tilde{S}(t,\xi)$ is a rational function.

After the rational integrals (1.4), the next simplest are the *hyperelliptic integrals* $\int r(x,\sqrt{p(x)})\,dx$ corresponding to the curve $f(x,y)=y^2-p(x)=0$, where $p(x)$ is a polynomial of degree n having distinct roots. In case $n=1$ or 2 the curve C is a plane conic, which may be rationally parametriczed by fixing a line L with linear coordinate t and point P_0 on C, and then letting $P(t)=(x(t),y(t))$ be the residual intersection of the line $\overrightarrow{P_0\,t}$ with C. Changing variables gives

$$\int r(x,\sqrt{p(x)})\,dx=\int s(t)\,dt$$

to which (1.4) may be applied.

Thus the first interesting case are the elliptic integrals corresponding to $n=3,4$ and which we encountered previously. At the time of Abel, there was the addition relation for the Legendre integral

$$\int_0^{x_1}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}+\int_0^{x_2}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\int_0^{x_3}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

where $x_3(1-k^2x_1^2x_2^2)=x_1\sqrt{(1-x_2^2)^2(1-k^2x_2^2)}+x_2\sqrt{(1-x_1^2)(1-k^2x_1^2)}$. More precisely, if we consider the abelian integral $u(P)=\int_{P_0}^{P} dx/y$ attached to the quartic curve $y^2=(1-x^2)(1-k^2x^2)$, then $u(P)+u(Q)=u(R)$ where the coordinates of R are rationally determined from those of P and Q. What *I* shall call *Abel's theorem* implies that such addition theorems are the rule rather than the exception.

To be specific, suppose that

$$u(P)=\int_{P_0}^{P} r(x,y)\,dx$$

is a general abelian integral attached to an algebraic curve $f(x, y)=0$ and abelian differential $\omega=r(x, y)\,dx$. We assume that the curve C has degree n. It is understood that $u(P)$ is defined only up to an additive constant or *period*, about which we shall say more later, and is defined for P not one of a finite number of singular points. Suppose now that $\theta(x, y)=\theta(x, y; t)$ is a polynomial whose coefficients are rational functions of some additional variables $t=(t_1, \dots, t_N)$. Actually, we could allow the coefficients to be algebraic functions of t, but this additional generality can be handled the same way. For a general value of t, the equation $\theta(x, y)=0$ defines an algebraic curve D_t of degree m. We may assume that for some one value $t=\mathrm{t}$ the curve meets C in mn finite points of which the distinct ones have distinct finite x-coordinates. As t varies, we write

$$D_t \cdot C = \sum P(t)$$

and think of the coordinates (ξ, η) of $P(t)$ as being algebraic functions of t. Now, whereas the individual abelian integral $u(P)$ may be rather mysterious, we have

Abel's Theorem. *The abelian sum*

$$(1.6)\quad u(t)=\sum u(P(t))=\sum \int_{P_0}^{P(t)} r(x, y)\,dx$$

is of the form

$$u(t)=R(t)+\sum_{\nu} \log S_\nu(t)$$

where $R(t)$, $S_\nu(t)$ are rational functions of t.

We shall give two closely related proofs of which the first is the original one due to Abel. It is purely algebraic, and is based on the following two assertions both of which are evident from elementary considerations of field extensions:

(a) If $v(x, y)$ and $w(x, y)$ are rational functions of (x, y), then so is

$$\sum_\nu v(x, y_\nu(x))\, w(x, y_1(x)) \overset{\nu}{\dots} w(x, y_n(x))$$

where the superscript "ν" means that the ν-th term is omitted from the product; and

(b) If $\psi(x)$ is a rational function of x whose coefficients are rational functions of t, then

$$\sum_\xi \psi(\xi)$$

is a rational function of t (recall that ξ runs over the x-coordinates of $D_t \cdot C$).

We follow Abel's notation and denote by δ the total differential of a function of t. Thus, if $\psi(x)$ is a function of x and ξ has the above meaning,

$$\delta(\psi(\xi))=\sum_i \left(\psi'(\xi)\frac{\partial \xi}{\partial t_i}\right) dt_i.$$

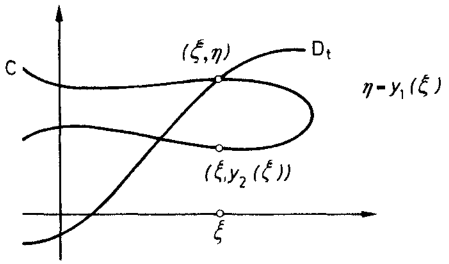
According to (1.5) it will suffice to show that $\delta u(t)$ is a rational 1-form in $dt_1, \dots, dt_N$. On the other hand, by differentiation of the integral we have

$$(1.7)\quad \delta u(t)=\sum r(\xi, \eta)\,\delta\xi,$$

the summation being over the points in $D_t \cdot C$. To calculate $\delta\xi$ set

$$\Theta(x) = \theta(x, y_1(x)) \ldots \theta(x, y_n(x))$$

so that the roots of the equation $\Theta(x) = 0$ are the x-coordinates ξ of $D_t \cdot C$.



Thus $\Theta(\xi) \equiv 0$ and so

$$\frac{\partial\Theta}{\partial x}(\xi)\,\delta\xi + \delta\Theta(\xi) = 0,$$

where $\delta\Theta$ is the t-differential of $\Theta(x)$. Substituting in (1.7)

$$\delta u = -\sum \frac{r(\xi, \eta)\,\delta\Theta(\xi)}{\dfrac{\partial\Theta}{\partial x}(\xi)}.$$

We may assume that near a particular value $t = \mathbf{t}$ and $\xi = \boldsymbol{\xi}$ that $\eta = y_{\mathbf{v}}(\xi)$ for one $\mathbf{v}$ and $\theta(\xi, y_v(\xi)) \neq 0$ for $v \neq \mathbf{v}$ (c.f. the above figure). Then

$$\delta\Theta(\xi) = \sum_v \theta(\xi, y_1(\xi)) \overset{\vee}{\ldots} \theta(\xi, y_n(\xi))\,\delta\theta(\xi, y_v(\xi)),$$

so that, taking into account that only the term $v = \mathbf{v}$ actually appears,

$$\begin{aligned} -r(\xi, \eta)\,\delta\Theta(\xi) &= +\sum_v r(\xi, y_v(\xi))\,\theta(\xi, y_n(\xi)) \overset{\vee}{\ldots} \theta(\xi, y_v(\xi))\,\delta\theta(\xi, y_v(\xi)) \\ &= \sum_i \varphi_i(\xi)\,dt_i \end{aligned}$$

where

$$\varphi_i(x) = -\sum_v r(x, y_v(x))\,\theta(x, y_1(x) \overset{\vee}{\ldots} \theta(x, y_n(x))\frac{\partial\theta}{\partial t_i}(x, y_v(x))$$

is a rational function of x by (a) above. Consequently, by (b)

$$\delta u = \sum_i \left(\sum_\xi \frac{\varphi_i(\xi)}{\dfrac{\partial\Theta}{\partial x}(\xi)} \right) dt_i$$

is rational 1-form in the dt_i's. Q.E.D.

Our second proof of Abel's theorem is again algebraic and is based on an especially symmetric formula for the variation $\delta u(t)$ of the abelian sum (1.6). To give this we write the abelian differential in the form

$$\omega=\frac{r(x,y)\,dx}{\dfrac{\partial f}{\partial y}(x,y)}, \tag{1.8}$$

an expression which will be motivated when we discuss *Poincaré residues* in Section IIIc. For simplicity we assume that t is a single variable and write as before

$$D_t\cdot C=\sum(\xi(t),\eta(t)),$$
$$u(t)=\sum\int_{(\xi_0,\eta_0)}^{(\xi(t),\eta(t))}\omega.$$

Then the variation

$$u'(t)=\sum\frac{r(\xi(t),\eta(t))\,\xi'(t)}{\dfrac{\partial f}{\partial y}(\xi(t),\eta(t))}.$$

Differentiation of the relations

$$f(\xi(t),\eta(t))\equiv 0,$$
$$\theta(\xi(t),\eta(t))\equiv 0$$

gives

$$\frac{\partial f}{\partial x}\xi'(t)+\frac{\partial f}{\partial y}\eta'(t)=0$$

$$\frac{\partial\theta}{\partial y}\xi'(t)+\frac{\partial\theta}{\partial y}\eta'(t)+\frac{\partial\theta}{\partial t}=0.$$

With the notation

$$\frac{\partial(f,g)}{\partial(x,y)}=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

for the Jacobian determinant, these linear equations may be solved to yield

$$\xi'(t)=\frac{\dfrac{\partial\theta}{\partial t}\dfrac{\partial f}{\partial y}}{\dfrac{\partial(f,\theta)}{\partial(x,y)}}.$$

Plugging this in the formula for the variation of the abelian sum gives

$$u'(t)=\sum\frac{s(\xi(t),\eta(t))}{\dfrac{\partial(f,\theta)}{\partial(x,y)}(\xi(t),\eta(t))}, \tag{1.9}$$

where

$$(1.9') \quad s(x,y)=r(x,y)\frac{\partial\theta}{\partial t}(x,y)$$

is a rational function of (x, y). Again, by an easy extension of (b) the sum in (1.9) is a rational function of t, thus proving Abel's theorem.

This argument is not essentially different from the first – the reason for presenting it is that writing the abelian differential in the form (1.8) allows us to eliminate $\xi'(t)$ from the variation of the abelian sum in a manner which will tie in with residues in Section III b.

Both these proofs are rather general in nature and fail to yield explicit formulae for the variation δu of the abelian sum (1.6), this obviously because (a) and (b) are general principles and not explicit expressions. In case D_t is the family of straight lines

$$\theta(x,y)=y-t_1 x-t_2=0$$

and the abelian differential is

$$\omega=\frac{p(x,y)\,dx}{\frac{\partial f}{\partial y}(x,y)},$$

we now compute explicitly the derivatives $\frac{\partial u}{\partial t_i}$ $(i=1,2)$ using the *Lagrange interpolation formula* (c.f. Section III b, (3.22) for a general version)

$$(1.10) \quad \sum_i \frac{g(x_i)}{h'(x_i)}=\text{constant term in}\left\{\frac{x\,g(x)}{h(x)}\right\}$$

for polynomials $g(x)$ and $h(x)$, the latter having simple zeros at $x=x_i$. For any function $a(x,y)$ set $A(x)=a(x, t_1 x+t_2)$ so that

$$(1.11) \quad A'(x)=\frac{\partial a}{\partial x}+t_1\frac{\partial a}{\partial y}.$$

The roots of $F(x)=0$ are the x-coordinates ξ of the intersection $D_t \cdot C$. Differentiation of $f(\xi(t_1,t_2),\ t_1\xi(t_1,t_2)+t_2)\equiv 0$ gives

$$\left(\frac{\partial f}{\partial x}+t_1\frac{\partial f}{\partial y}\right)\frac{\partial \xi}{\partial t_1}+\xi\frac{\partial f}{\partial y}=0,$$

$$\left(\frac{\partial f}{\partial x}+t_1\frac{\partial f}{\partial y}\right)\frac{\partial \xi}{\partial t_2}+\frac{\partial f}{\partial y}=0,$$

which, by (1.11), amounts to

$$(1.12) \quad \begin{aligned} F'(\xi)\frac{\partial \xi}{\partial t_1}+\xi\frac{\partial f}{\partial y}&=0,\\ F'(\xi)\frac{\partial \xi}{\partial t_2}+\frac{\partial f}{\partial y}&=0.\end{aligned}$$

For the abelian sum $u(t)=\sum \int^{(\xi,\eta)} \omega$,

$$\begin{aligned}\frac{\partial u}{\partial t_1}&=\sum \frac{p(\xi,\eta)\dfrac{\partial \xi}{\partial t_1}}{\dfrac{\partial f}{\partial y}(\xi,\eta)}\\ &=-\sum \frac{\xi p(\xi,\eta)}{F'(\xi)} \qquad \text{(by (1.12))}\\ &=-\sum_{\xi} \frac{\xi P(\xi)}{F'(\xi)}\end{aligned}$$

since $\eta=t_1\xi+t_2$. Making the same computation for $\dfrac{\partial u}{\partial t_2}$ and using (1.12) gives the beautiful formulae

$$\text{(1.13)}\qquad \begin{aligned}\frac{\partial u}{\partial t_1}&=\text{constant term in}\left\{\frac{-x^2 p(x,t_1x+t_2)}{f(x,t_1x+t_2)}\right\}\\ \frac{\partial u}{\partial t_2}&=\text{constant term in}\left\{\frac{-x p(x,t_1x+t_2)}{f(x,t_1x+t_2)}\right\}\end{aligned}$$

Note that the right-hand side of (1.13) are obviously rational functions of (t_1,t_2), thus again proving Abel's theorem for this special case.

More interestingly, we observe that

$$\text{(1.14)}\qquad \frac{\partial u}{\partial t_1}=\frac{\partial u}{\partial t_2}=0 \quad \textit{in case } \deg p \leqq \deg f-3.$$

To interpret this, we make the following

Definitions. (i) The abelian integral $u(P)=\int_{P_0}^{P}\omega$ is of the first kind if it has no singularities. (ii) The abelian integral u is of the first kind relative to the family of curves D_t if the abelian sum $u(t)=\sum u(P(t))$ has no singularities.

Obviously, (i) $\Rightarrow$ (ii) but not conversely. If (ii) holds, the $u(t)$ is necessarily constant. We may rephrase (1.14) by the

(1.15) **Proposition.** *If $f(x,y)=0$ defines a curve C of degree n and having no multiple components, then any abelian integral*

$$u=\int \frac{p(x,y)\,dx}{\dfrac{\partial f}{\partial y}(x,y)} \quad \textit{where } \deg p \leqq n-3$$

is of the first kind relative to the family of lines in $\mathbb{P}^2$.

When we discuss residues in Section III b, we shall prove the same statement for the abelian integral $u=\int \dfrac{p(x,y)\,dx}{\dfrac{\partial f}{\partial y}(x,y)}$ where $\deg p \leqq n-3$ and D_t is the family of all curves of degree m. This more general assertion will be used in Section Ic (c.f. (1.28)).

(b) Inversion of the Trigonometric and Elliptic Integrals

As we have just seen, Abel's first idea was that an abelian sum (1.6) associated to an abelian integral

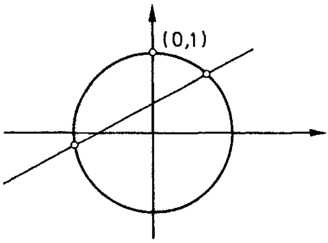
$$(1.16)\quad u=\int_{(x,y_0)}^{(x,y)} r(x,y)\,dx$$

was expressible in elementary terms despite the complexity of the integral itself. His second idea was to *invert* the integral (1.16) and consider the coordinates of a point $(x(u), y(u))$ on the curve $f(x,y)=0$ as functions of u. Later on, it was found by Göpel, Rosenhaim, and especially Jacobi, that it was necessary to invert a set $u_1, \dots, u_p$ of such integrals, and we shall discuss this later. For the moment we wish to show how Abel's theorem – or, more precisely, the explicit formulae (1.13) – may be used in a very elegant manner to invert the two integrals with which our discussion began.

We begin with some remarks concerning periods. Although not necessary, it will make the argument clearer if we now assume that the curve C is irreducible and is the image of a compact Riemann surface $\tilde{C}$ under a holomorphic mapping. At points where $\frac{\partial f}{\partial y}\neq 0$, we may use x as local coordinate on $\tilde{C}$, and similarly at the branch points $\frac{\partial f}{\partial y}=0$ of the linear projection $C\to(x\text{-axis})$ we may use y as coordinate provided $\frac{\partial f}{\partial x}\neq 0$. At the singular points P where $df=0$ it is necessary to give a neighborhood of P parametrically by a coordinate ξ on $\tilde{C}$ for which the inverse mapping is expressed by Puiseux series. The abelian differential $r(x,y)\,dx$ is then a meromorphic 1-form ω on $\tilde{C}$ having certain poles $P_1, \dots, P_N$. On the punctured Riemann surface $C^*=\tilde{C}-\{P_1,\dots,P_N\}$ the abelian integral (1.16) is, by Cauchy's theorem, well defined up to periods $\int_\gamma \omega$ where $\gamma\in H_1(C^*,\mathbb{Z})$. These periods form a subgroup Λ of $\mathbb{C}$.

The idea of inversion using Abel's theorem is this: Near a point $P_0\in C$ where ω is holomorphic and $\omega(P_0)\neq 0$ the integral (1.16) may be inverted by the ordinary inverse function theorem. Then, by use of an addition formula we may extend $(x(u), y(u))$ to entire functions. Let's first do the trigonometric case, as this apparently served as a model to Euler, Lagrange, and Abel.

Our curve is the circle $x^2+y^2=1$ and $\omega=\frac{dx}{y}=\frac{dy}{\sqrt{1-x^2}}$. We consider the family of lines $y=t_1x+t_2$ and denote by $P_1=(x_1,y_1)$ and $P_2=(x_2,y_2)$ the variable points of intersection of C with this line.



For the abelian sum

$$u(t)=\int_{(0,1)}^{(x_1,y_1)}\omega+\int_{(0,1)}^{(x_2,y_2)}\omega$$

the formulae (1.13) yield $\left(\text{note } \omega=\dfrac{2dx}{\partial f/\partial y}\right)$

$$\frac{\partial u}{\partial t_1}=\text{constant term in}\left\{\frac{-2x^2}{f(x,t_1x+t_2)}\right\}=\frac{-2}{1+t_1^2},$$
$$\frac{\partial u}{\partial t_2}=\text{constant term in}\left\{\frac{-2x}{f(x,t_1x+t_2)}\right\}=0.$$

Integration of these equations gives

$$u(t)=-2\arctan t_1=\arcsin\left(\frac{-2t_1}{1+t_1^2}\right).$$

Elementary manipulation leads to the relation

$$\frac{-2t_1}{1+t_1^2}=x_1y_2+x_2y_1,$$

so that, in classical notation, we have derived the formula

$$\text{(1.17)}\quad \int_0^{x_1}\frac{dx}{\sqrt{1-x^2}}+\int_0^{x_2}\frac{dx}{\sqrt{1-x^2}}=\int_0^{x_1y_2+y_2x_1}\frac{dx}{\sqrt{1-x^2}}.$$

If we invert the integral $u=\int\frac{dx}{y}$ in a neighborhood of (0,1) by *defining* the trigonometric functions according to the relation

$$\text{(1.18)}\quad u=\int_{(0,1)}^{(\sin u,\,\cos u)}\frac{dx}{y},\qquad |u|<\varepsilon,$$

then (1.17) becomes the familiar addition theorem

$$\sin(u_1+u_2)=\sin u_1\cos u_2+\sin u_2\cos u_1,$$

with a similar one for $\cos u$. Taking $u_1=u_2$ yields the *duplication formula*

$$\text{(1.19)}\quad \sin 2u=2\sin u\cos u.$$

Now whereas $\sin u$ was initially defined in $\Delta(\varepsilon)=\{|u|<\varepsilon\}$ by the inversion relation (1.18), we may extend its domain of definition to $\Delta(2\varepsilon)$, then to $\Delta(4\varepsilon)$, and so forth by repeatedly using (1.19). *This leads to entire analytic functions* ($\sin u$, $\cos u$) *satisfying* (1.18) *for all values of* u. It is straightforward to check that the group of periods is $\mathbb{Z}\cdot\pi$, and in this manner we have arrived at the essential properties of the trigonometric functions as defined by an algebraic integral.

Of course this was well known at the time, but our point is that Abel's method works even better (because of (1.15)) for the elliptic and higher integrals, to which we now turn.

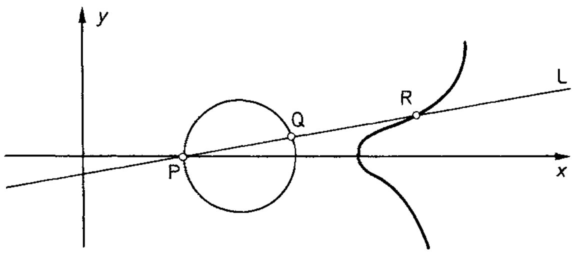
Suppose then that $C=\{f(x,y)=0\}$ is a nonsingular cubic curve and $\omega=\dfrac{dx}{\partial f/\partial y}$ with abelian integral

(1.20) $\quad u(P)=\int_{P_0}^{P}\omega.$

Eventually, we will take $f(x,y)=y^2-p_3(x)$ so that $\omega=\dfrac{dx}{2\sqrt{p(x)}}$ and u is up to a change of variables the previously defined elliptic integral. According to (1.15) the abelian integral (1.20) is of the first kind relative to the family of lines in $\mathbb{P}^2$, and since C is assumed nonsingular it follows that u is simply of the first kind. Abel's theorem in the form (1.14) gives the relation

(1.21) $\quad u(P)+u(Q)+u(R)=K$

where P, Q, R are the intersection points of C with a variable line L.



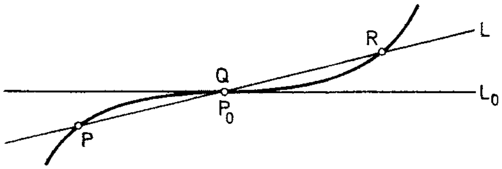
This is a picture of the real points on the curve $y^2=p(x)$ where $p(x)$ is assumed to have real coefficients.

We shall choose our base point P_0 to be a *flex* on C, which will now be explained. Since the curve is nonsingular, at each point $P\in C$ there is a unique tangent line T_P having contact of order $\geqq 2$ with the curve at P. It is elementary that there are a finite number of flexes where T_P has contact of order $\geqq 3$. In fact, if $\deg C=d$, then the number of flexes is $3d(d-2)$ (counting multiplicities), as may be proved by showing that P is a flex if, and only if,

$$F(P)=0=H_F(P),$$

where $F(x_0,x_1,x_2)=0$ is the homogeneous equation of C and $H_F=\det\left(\dfrac{\partial^2 F}{\partial x_i\,\partial x_j}\right)$ is the Hessian of F.

Suppose then that $P_0=(x_0, y_0)$ is a flex on C with flex tangent L_0.



We may assume that $\frac{\partial f}{\partial y}(P_0)\neq 0$ so that $\omega(P_0)\neq 0$. As before, we invert the elliptic integral by the relation

$$(1.22)\quad u=\int_{(x_0, y_0)}^{(x(u),\, y(u))} \omega$$

valid in a disc $\Delta(\varepsilon)=\{|u|<\varepsilon\}$. Set $P(u)=(x(u), y(u))$. For δ sufficiently small and $u_1, u_2\in\Delta(\delta)$, the line $\overrightarrow{P(u_1)\, P(u_2)}$ will be a small perturbation of L_0 and will meet C in a third point $P(u_3)$ for some $u_3\in\Delta(\varepsilon)$. According to the addition formula (1.21)

$$(1.23)\quad u_1+u_2+u_3=0.$$

Now the coordinates of $P(u_3)$ are clearly *rational* functions of the coordinates of $P(u_1)$ and $P(u_2)$, so that we may reinterpret (1.23) by saying that

$$x(-(u_1+u_2))=R(x(u_1), y(u_1), x(u_2), y(u_2)),$$
$$y(-(u_1+u_2))=S(x(u_1), y(u_1), x(u_2), y(u_2))$$

where R and S are rational functions of their variables. In particular, there is a duplication formula

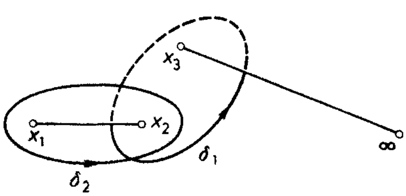
$$x(-2u)=R(x(u), y(u)),$$
$$y(-2u)=S(x(u), y(u))$$

as in the trigonometric case, and using it we may extend $(x(u), y(u))$ to *entire meromorphic functions* satisfying the equation of the curve and (1.22), or equivalently by taking the differential of that equation

$$(1.24)\quad \begin{aligned} &f(x(u), y(u))\equiv 0,\\ &x'(u)=\frac{\partial f}{\partial y}(x(u), y(u)). \end{aligned}$$

In case $f(x, y)=y^2-p(x)$ we have constructed the Weierstrass functions.

Keeping this same equation, the Riemann surface $\tilde{C}=C$ may be visualized as a 2-sheeted covering of the x-plane branched at the roots x_i $(i=1, 2, 3)$ of $p(x)$ together with the point $x=\infty$.



There are thus two generating periods $\pi_\nu = \int_{\delta_\gamma} \omega$, and we claim that they are linearly independent over $\mathbb{R}$. Indeed, if there were a linear relation $\alpha_1 \pi_1 + \alpha_2 \pi_2 = 0$ for α_ν real, then multiplying ω by $1/\pi_1$ and multiplying the above relation by $-1/\alpha_1$ gives

$$\alpha \pi_2 = 1, \qquad \alpha \text{ real.}$$

Consequently, π_2 is real and so $\eta = \omega - \bar{\omega}$ has no periods. Setting $v = \int \eta$ we have

$$\begin{aligned} 0 < \sqrt{-1} \int_C \omega \wedge \bar{\omega} &= \sqrt{-1} \int_C \eta \wedge \bar{\omega} \\ &= \sqrt{-1} \int_C d(v \wedge \bar{\omega}) \\ &= 0 \end{aligned}$$

by Stokes' theorem, which is a contradiction. If follows that Λ is a *lattice* in $\mathbb{C}$, and $P(u) = (x(u), y(u))$ gives a holomorphic mapping

(1.25) $\mathbb{C}/\Lambda \xrightarrow{\mathrm{P}} C \subset \mathbb{P}^2$.

This mapping is unramified since $P^* \omega = du \neq 0$, and thus for topological reasons is a finite covering. In fact, it is one-to-one: If $P(u_1) = P(u_2)$, then for a variable point u the line $\overrightarrow{P(u_1) \cdot P(u)} = \overrightarrow{P(u_2) \cdot P(u)}$ has intersection $P(u_i) + P(u) + P(u')$ with C where u' is a single-valued function of u near some point u_0. The relations

$$u_i + u + u'(u) \equiv 0(\Lambda)$$

for $i = 1, 2$, then imply, by subtraction, that $u_1 - u_2 \equiv 0(\Lambda)$ which is equivalent to $P(u)$ being one-to-one.

In this way, by essentially algebro-geometric considerations of an elementary nature we have arrived at the existence and basic properties of the elliptic functions. The philosophy of using an addition theorem to propagate a local analytic object into a global one is what underlies our main theorem discussed in the introduction.

(c) Singular Cubics — General Addition Theorems

In the previous section we have shown how Abel's theorem in the form (1.21) leads to a group structure on a nonsingular cubic curve. We will now discuss briefly how a generalized form of (1.21) may be similarly applied to any irreducible curve C of degree $\geqq 3$.

We begin with the case where C is a singular cubic. Any singular point P_0 cannot be worse than a double point, since a line $\overrightarrow{P_0 Q}$ $(Q \in C)$ can have only three intersections with C. Similarly, there can be only one singular point. In case P_0 is an ordinary double point we may choose coordinates such that $P_0 = (0, 0)$ and the defining polynomial has the form $f(x,) = xy + (\ldots)$ where $(\ldots)$ denotes higher-order terms. One branch of C passing through P_0 is given parametrically by $y = y(x) = x + (\ldots)$, and the other by $x = x(y) = y + (\ldots)$. Since on C

$$0 = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

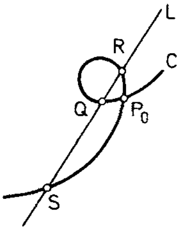
the abelian differential

$$\psi = \frac{dx}{\partial f/\partial y} = -\frac{dy}{\partial f/\partial x}$$

has on these branches the respective forms

$$\psi = (1+\cdots)\frac{dx}{x}, \quad \psi = -(1+\cdots)\frac{dy}{y}.$$

If L is any line tending to pass through P_0,



then the logarithmic singularities in the first two terms of the abelian sum

$$(1.26) \quad \int^{Q} \psi + \int^{R} \psi + \int^{S} \psi \equiv K$$

cancel out, thus explaining why $u = \int \psi$ is of the first kind with respect to the family of lines in $\mathbb{P}^2$. The addition theorem (1.26) may now be used to make $C^* = C - \{P_0\}$ into a group by repeating the argument from the previous section.

We may identify this group as follows. Choose a line L_0 with linear coordinate t and not containing P_0. The line $\overrightarrow{P_0 t}$ meets C in the cycle $2P_0 + Q(t)$, and the map $t \to Q(t)$ is a one-to-one rational transformation. We may choose t so that the tangent lines to the two branches of C passing through P_0 correspond to $t = 0, \infty$. Taking into account the expressions for ψ, or equivalently, the logarithmic nature of the integral $u(Q) = \int^{Q} \psi$ as Q tends towards P_0, we see that the pulled back differential $\psi(t)$ has first-order poles at $t = 0, \infty$. Multiplying by a constant if necessary, it follows that $\psi(t) = \frac{dt}{t}$. The addition formula (1.26) now reads

$$\int^{t(Q)} \frac{dt}{t} + \int^{t(R)} \frac{dt}{t} + \int^{t(S)} \frac{dt}{t} \equiv K,$$

so that $t(S) = (t(Q)\,t(R))^{-1}$. Choosing our base point to be $t = 1$, we obtain the addition formula

$$\int_1^{t_1} \frac{dt}{t} + \int_1^{t_2} \frac{dt}{t} \equiv \int_1^{t_1 t_2} \frac{dt}{t} \qquad (2\pi i)$$

characteristic of the logarithm function. The group in this case is $\mathbb{C}^*$.

In the remaining case where the tangents to C passing through P_0 coincide, we may use the same projection method to give an isomorphism

$$C - \{P_0\} \simeq \mathbb{C} = \mathbb{P}^1 - \{\infty\}.$$

The differential $\psi(t)$ will be holomorphic and nowhere vanishing on $\mathbb{C}$, hence after multiplying by a constant if necessary, $\psi(t)=dt$. The group structure is just the usual additive structure on $\mathbb{C}$.

Inverting the abelian integral $u=\int_{t_0}^{t(u)} \psi(t)$ in these two cases gives $t(u)=\exp u$ and $t(u)=u$, respectively.

There is nothing new in this. Our point is that Abel's original theorem applies equally to singular cubics.

Now we come to curves of higher degree. The first step is to extend (1.15) to intersections of our curve C with curves D of degree $m \geqq 1$. Assuming that $C=\{f(x,y)=0\}$ is of degree n and meets $D=\{g(x,y)=0\}$ in mn distinct finite points, we will in Section IIIb (c.f. (3.21)) derive the *Jacobi relation*

$$(1.27)\quad \sum_{\nu} \frac{h(P_\nu)}{\dfrac{\partial(f,g)}{\partial(x,y)}(P_\nu)}=0, \qquad \deg h \leqq m+n-3.$$

Comparing this with (1.9) and (1.9′) we find:

(1.28) *The abelian integral*

$$(1.28')\quad u=\int \frac{p(x,y)\,dx}{\partial f/\partial y(x,y)}, \qquad \deg p \leqq n-3$$

is of the first kind relative to the family of all curves of degree m.

Now we assume C is irreducible, but otherwise has arbitrary singularities. On the basis of (1.28) it is pretty clear that we should consider simultaneously *all* abelian integrals of the type (1.28′), although the recognition that this was so required some effort historically. (c.f. C. G. J. Jacobi, Considerationes generales de transcentibus abelianus, Gesammelte Werke, Band II, pp. 7–16 and De functionibus darum variabilium quadrupliciter periodicus, Ges. Werke, Band II, pp. 55–78 for two of the earliest treatments of the general inversion problem.) Recall that the vector space of polynomials of degree $\leqq d$ has dimension

$$\frac{(d+1)(d+2)}{2}=\frac{d(d+3)}{2}+1,$$

and choose a basis $\{p_\alpha(x,y)\}$ for those of degree $\leqq n-3$. The differentials

$$\psi_\alpha=\frac{p_\alpha(x,y)\,dx}{\dfrac{\partial f}{\partial y}(x,y)}$$

span a vector space whose dimension

$$\pi=\frac{(n-1)(n-2)}{2}$$

is called the *virtual genus* of the curve C. (Early writers used the word *deficiency*.) The associated abelian integrals are denoted by

$$u_\alpha(P) = \int_{\mathbf{P}_\alpha}^{P} \psi_\alpha$$

where $\mathbf{P}_\alpha$ is a base point to be selected later.

For a general curve D of degree m we set

$$D \cdot C = \sum_{v=1}^{mn} P_v.$$

According to (1.28) the abelian equations

$$\text{(1.29)} \quad \sum_{v=1}^{mn} u_\alpha(P_v) \equiv K \qquad (\alpha = 1, \ldots, \pi)$$

impose π constraints on the points $\{P_\alpha\}$, suggesting that $mn - \pi$ of them should rationally determine the rest as was the case with the straight lines meeting the cubic. If this can be proved, then we will be in formally analogous situation to the cubic in that we will be in possession of a general addition theorem for the π abelian integrals u_α and can use this to propagate a local inversion relation into a global one.

To carry this out we shall be interested in intersections $D \cdot C$ where $D = g(x, y) = 0$ has degree $m \gg n$. If $D' = \{g'(x, y) = 0\}$ is another curve of degree m, then D and D' cut out the same set of points on C if, and only if,

$$g(x, y) - g'(x, y) = h(x, y) f(x, y), \quad \deg h \leqq m - n.$$

Using the notation of linear systems, we denote by $|D_m|_C$ the set of all points of the form $D \cdot C$. Then $|D_m|_C$ is a projective space of dimension

$$\frac{(m+1)(m+2)}{2} - \frac{(m-n+1)(m-n+2)}{2} - 1 = mn - \pi.$$

This second appearance of the arithmetic genus π is the key. Namely, given $mn - \pi$ general points$\{Q_j\}$ on C, by what we just proved there will be a unique curve $D \in |D_m|_C$ passing through the Q_j's. Writing

$$\text{(1.30)} \quad D \cdot C = (Q_1 + \cdots + Q_{mn-\pi}) + (R_1 + \cdots + R_\pi)$$

we have shown that a general set of $mn - \pi$ points $\{Q_j\}$ *rationally* determines a residual set $\{R_\alpha\}$ according to the rule (1.30).

Referring to (1.29) the abelian equations

$$\text{(1.31)} \quad \sum_{j=1}^{mn-\pi} u_\alpha(Q_j) + \sum_{\beta=1}^{\pi} u_\alpha(R_\beta) \equiv K \qquad (\alpha = 1, \ldots, \pi)$$

are satisfied. Since the $\{p_\alpha(x, y)\}$ are linearly independent, if we choose the $\{Q_j\}$ generally (i.e., in a Zariski open set) the Equations (1.31) will be independent. Putting this together with what we proved in the previous paragraph gives the following statement which is beginning to take on the shape of an addition theorem:

(1.32) *For a general set of points* $\{Q_j\}$ $(j=1, \ldots, mn-\pi)$ *the abelian equations* (1.31) *rationally determine the residual set* $\{R_\beta\}$ $(\beta=1, \ldots, \pi)$.

In fact, there will then be a unique curve $D \in |D_m|_C$ such that (1.30) holds.

We now will distill out of (1.32) the addition theorem leading to the inversion of the π abelian integrals u_α. Choose m large relative to n and fix $mn-3\pi$ general points $\{S_k\}$ on C. We will consider zero cycles $\Gamma_P = P_1 + \cdots + P_\pi$, $\Gamma_Q = Q_1 + \cdots + Q_\pi$, etc. of degree π. Given Γ_P and Γ_Q selected generally, we may, according to (1.32), *rationally* determine Γ_R by the abelian equations

$$(1.33) \quad \sum_{\beta}^{\pi} u(P_\beta) + \sum_{\gamma=1}^{\pi} u(Q_\gamma) + \sum_{\lambda=1}^{\pi} u(R_\lambda) \equiv L,$$

where

$$u(P) = (u_1(P), \ldots, u_\pi(P))$$

and $L = (L_1, \ldots, L_\pi)$ has entries

$$L_\alpha = K_\alpha - \sum_k u_\alpha(S_k).$$

It is now possible to repeat the reasoning in the cubic case with (1.33) playing the role of (1.21). Fixing our base point cycle $\Gamma_{\mathbf{P}} = \mathbf{P}_1 + \cdots + \mathbf{P}_\pi$ such that

$$\det \frac{p_\alpha(\mathbf{P}_\beta)}{\dfrac{\partial f}{\partial y}(\mathbf{P}_\beta)} \neq 0,$$

the equations

$$(1.34) \quad u_\alpha = \sum_\beta u_\alpha(P_\beta), \qquad |u| < \varepsilon,$$

may uniquely be solved for points $P_\beta(u)$ close to $\mathbf{P}_\beta$. This is just an inverse function theorem. Set $P_\beta(u) = (x_\beta(u), y_\beta(u))$ and consider a generating set $\varphi_1, \ldots, \varphi_N$ of rational functions $\varphi_\nu(x_1, y_1, \ldots, x_\pi, y_\pi)$ which are symmetric in the (x_α, y_α). Then, setting $\varphi = (\varphi_1, \ldots, \varphi_N)$,

$$F(u) = \varphi(x_1(u), y_1(u), \ldots, x_\pi(u), y_\pi(u))$$

will be a meromorphic function of u for $|u| < \varepsilon$. According to (1.33), for u and u' satisfying $|u| < \delta$, $|u'| < \delta$,

$$F(-(u+u')) = R(F(u), F(u'))$$

will be rationally expressed in terms of the $F_\nu(u)$ and $F_\nu(u')$. In particular, there will be a duplication formula and subsequent propagation of $F(u)$ to a vector of entire meromorphic functions. We have in this way constructed an entire meromorphic mapping

$$F: \mathbb{C}^\pi / \Lambda \to \mathbb{P}^N$$

given by $F(u)=[1, F_1(u), \ldots, F_N(u)]$ and where $\Lambda \subset \mathbb{C}^\pi$ is a discrete subgroup of periods. The defining relation for F is that $F(u)$ determines points $(x_\alpha(u), y_\alpha(u))$ on the curve such that the equations

$$(1.35) \quad u_\alpha \equiv \sum_\beta \int_{(x_\beta, y_\beta)}^{(x_\beta(u), y_\beta(u))} \psi_\alpha, \qquad \text{modulo } L_\alpha$$

are satisfied. This is the general inversion theorem.

We have also made the zero cycles $\Gamma_P = P_1 + \cdots + P_\pi$ into a group variety – i.e., the group law is given rationally. In fact, this group is $\mathbb{C}^\pi/\Lambda$, and is the famous Jacobian variety (or generalized Jacobian in case C is singular).

Throughout this discussion we have been somewhat careless about using the adjective "general." For a general u the equations (1.35) uniquely determine the points $(x_\beta(u), y_\beta(u))$ – this much we have proved – but this will not be true for all u if the virtual genus $\pi \geqq 2$. The detailed analysis of what is going on here leads to the beautiful and subtle theory of *special divisors* initiated by Riemann and which is continuing to the present. The aforementioned notes Lectures on curves and their Jacobians by David Mumford give references and discussion on this question.

To conclude this discussion, we wish to make a comment concerning reducible curves. The Jacobi relation (1.27) requires neither f or g to be irreducible, only that they should meet in mn distinct points. Suppose then that $f(x, y) = l_1(x, y)\, l_2(x, y)\, l_3(x, y)$ is a product of three distinct linear factors, so that $f=0$ defines a *triangle* $T = L_1 + L_2 + L_3$. Repeating the same counting constants argument just given, we find that the intersections $|D_m|_T$ of curves D of degree m with T depend on

$$\frac{(m+1)(m+2)}{2} - \frac{(m-2)(m-1)}{2} - 1 = 3m - 1$$

parameters. Thus the $3m$ points of intersection

$$D_m \cdot T$$

cannot arbitrarily prescribed by giving m points $P_{i,1}, \ldots, P_{i,m}$ on each line L_i. The Jacobi relation (1.27) imposes one condition on these points in order that they be a complete intersection.

This is, of course, an Abel-type condition: Inversion of the abelian integral

$$u = \int \frac{dx}{\partial f/\partial y(x, y)}$$

will lead to a *disconnected* group in the same way as the irreducible cubics previously gave connected groups. Upon doing this, one finds that here the group consists of 3 disjoint copies of $\mathbb{C}^*$, corresponding to the three $\mathbb{P}^1 - \{0, \infty\}$'s obtained by deleting the vertices of the triangle. If t_i is a coordinate on L_i such that the two vertices meeting L_i correspond to $t_i = 0$ and ∞, then, after suitable normalization, the condition that the points lie on a curve D_m is

$$\prod_{i=1}^{3}\left(\prod_{\nu=1}^{m} t_i(P_{i,\nu})\right)=1.$$

In a similar manner, all of the preceding discussion may be extended to cover reducible curves.

(d) The Poncelet Theorem

As mentioned in the introduction one of the striking features about the classical Abel's theorem are the applications one can make to elementary problems in geometry. A simple one is the triangle mentioned in the preceding section; we shall now discuss the somewhat more sophisticated application to the classical Poncelet problem concerning polygons inscribed in one conic and circumscribed about another. So far as I can tell, it was Jacobi (Über die Anwendung der elliptischen Transcendenten auf ein bekanntes Problem der Elementargeometrie, Gesammelte Werke, Vol. I (1881), pp. 278–293) who first applied elliptic functions to the question, and we shall present a variant of his method here. It is interesting to note that the result we shall prove will be *equivalent* to the addition law for an elliptic integral, so that the early somewhat complicated proofs of the Poncelet theorem must have amounted to synthetic derivations of this addition formula, presumably in the same way in which the addition formula for the sine function may be derived by drawing pictures.

Before giving the Poncelet theorem some preliminary considerations are needed. We recall that in Section I a the addition formula for the elliptic integral $u=\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ was stated as being pretty much what was known before Abel. This integral is associated to the quartic curve C_0 defined by $y^2-p_4(x)=0$ where $p(x)=(1-x^2)(1-k^2x^2)$. Setting $y=1/y'$ and $x=x'/y'$, the equation of C_0 becomes

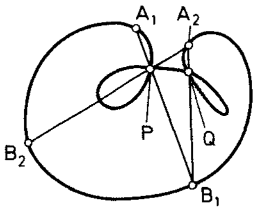
$$y'^2-\tilde{p}(x',y')=0$$

where $\tilde{p}$ is homogeneous of degree 4 in x' and y'. Thus in the projective plane the curve C_0 has a nonordinary double point P_0. The addition formula for the Legendre integral may now be derived by fixing another point R on C_0 but not on the line $y'=0$ and considering the linear system of plane quadrics Q which pass through P_0 with tangent line $y'=0$ there and also pass through R. Since the set of all quadrics depends on ∞^5 parameters, this linear system contains ∞^2 quadrics. Any such Q meets C_0 four times at P_0 and once at R; thus

$$Q\cdot C_0=4P_0+R+(A+B+C),$$

since there are eight intersections in all. We are then in a formally analogous situation to the plane cubics where A, B, C now play the role which the variable points of intersection with a line played there. In particular, the Legendre formula is just $u(A)+u(B)+u(C)=\text{constant}$.

For the Poncelet theorem we want to consider a plane quartic $C=\{f(x,y)=0\}$ having two distinct ordinary double points P and Q. Given a point $A=A_1$ on $C-\{P,Q\}$, the line AP will meet C in point $B_1\in C-\{P,Q\}$. Next, the line B_1Q will meet C in a point $A_2\in C-\{P,Q\}$, and we may keep on going in this manner.



We then pose the question: *When does $A_1=A_n$ for some n?*

To answer this we consider the abelian integral

$$u=\int\frac{l(x,y)\,dx}{\partial f/\partial y(x,y)}$$

where $l(x,y)$ is a linear polynomial. According to (1.15) this integral is of the first kind relative to the set of lines in $\mathbb{P}^2$. However, u is not necessarily of the first kind on C. To see what is going on, we may assume that P is the origin and

$$f(x,y)=x\,y+(\text{higher terms}).$$

On the branch of C with tangent line $y=0$ the curve is given parametrically by $y(x)=x+(\text{higher terms})$. Thus if $l=\alpha\,x+\beta\,y+\gamma$,

$$u=\int\frac{(\gamma+\delta\,x+\cdots)}{x}\,dx$$

is an improper integral unless $\gamma=0$. In conclusion, if $l(x,y)=0$ is the equation of the line $\overrightarrow{PQ}$. then u is of the first kind on C, and we shall take this case.

Applying now Abel's theorem to the intersections of C with the lines in our figure gives

$$\begin{aligned}
&u(A_1)+2u(P)+u(B_1)\equiv K\\
&u(B_1)+2u(Q)+u(A_2)\equiv K\\
&\quad\vdots\qquad\qquad\quad\vdots\\
&u(A_{n-1})+2u(P)+u(B_{n-1})\equiv K\\
&u(B_{n-1})+2u(Q)+u(A_n)\equiv K
\end{aligned}$$

where K is a constant and $\equiv$ means congruent modulo periods. Adding up these equations with alternating signs induces telescoping and we arrive at

$$u(A_1)+2n\,u(P)\equiv u(A_n)+2n\,u(Q).$$

Thus, a necessary condition for $A_1=A_n$ is

$$(1.36)\quad 2n(u(P)-u(Q))\equiv 0.$$

We claim that this condition is also sufficient. To see this, project C onto a line L from the double point P. Taking t to be a linear coordinate on L, this is a 2-sheeted covering with four branch points as may easily be checked. Taking one of these to be $t=\infty$, our abelian integral becomes $u=\int \frac{dt}{\sqrt{p(t)}}$ and we are back in the cubic case where it has already been proved that

$$u\colon C \to \mathbb{C}/\Lambda$$

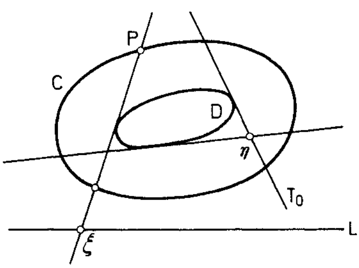
is an isomorphism. This proves the sufficiency of (1.36), and leads to the following conclusion:

(1.37) *We have $A_1=A_n$ in the above figure if, and only if $u(P)\equiv u(Q)\left(\frac{\Lambda}{2n}\right)$. This condition is independent of the initial point A_1, and imposes one constraint on the pair of points P and Q.*

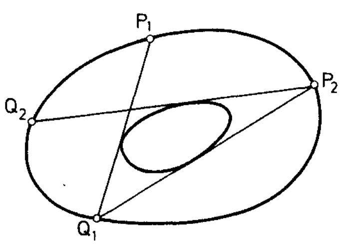
Now to the Poncelet theorem. We consider a pair of nonsingular conics C and D in the projective plane which we assume are nowhere tangent. By stereographic projection from a point $P\in C$ onto a line L with linear coordinate ξ, we may rationally parametrize C by $\xi \to (x(\xi), y(\xi))$ where $x(\xi)$ and $y(\xi)$ are quadratic functions of ξ. Similarly, if we fix a tangent line T_0 to D with linear coordinate η, then through each point η there is a unique tangent line

$$y=a(\eta)\,x+b(\eta)$$

to D other than T_0. Here $(a(\eta), b(\eta))$ are quadratic functions of η describing the dual curve $D^*\subset \mathbb{P}^{2*}$ of all tangent lines to D.



Now suppose we begin with a point $P_1\in C$, draw a tangent through P_1 to D meeting C in Q_1, then draw the other tangent to D through Q_1 meeting C in P_2, and so forth.



The Poncelet problem is when does $P_n = P_1$ for some n – i.e., *when do we obtain a closed polygon inscribed in C and circumscribed about D?* This is clearly similar to the previous problem concerning the quartic, and may be derived from it as follows:

In the product $C \times D^* \simeq \mathbb{P}^1 \times \mathbb{P}^1$ we consider the *incidence correspondence*

$$I = \{(P, T): P \in T\}.$$

This is clearly the variety relevant to our problem. It is a nonsingular curve in $\mathbb{P}^1 \times \mathbb{P}^1$ since, due to the nontangency of C and D, through each point (P, T) one of the two coordinate axes $\mathbb{P}^1 \times \{T\}$ and $\{P\} \times \mathbb{P}^1$ has two *distinct* points of intersection with I. Now $\mathbb{P}^1 \times \mathbb{P}^1$ contains $\mathbb{C} \times \mathbb{C}$ as an open set with coordinates (ξ, η), and the map $(\xi, \eta) \to [1, \xi, \eta]$ induces a rational mapping F of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$. Near a point $\infty \times \eta_0$ on $\mathbb{P}^1 \times \mathbb{P}^1$ we may use local coordinates $\left(\xi' = \frac{1}{\xi}, \eta\right)$ and then $F(\xi', \eta) = \left[1, \frac{1}{\xi'}, \eta\right] = [\xi', 1, \xi' \eta]$. Thus F blows the curve $\infty + \mathbb{P}^1$ down to the point $[0, 1, 0]$, and similarly $\mathbb{P}^1 \times \infty$ blows down to $[0, 0, 1]$. Near $\infty \times \infty$ we use local coordinates $\xi' = \frac{1}{\xi}, \eta' = \frac{1}{\eta}$ and then $F(\xi', \eta') = [\xi' \eta', \eta', \xi']$ is not defined there.

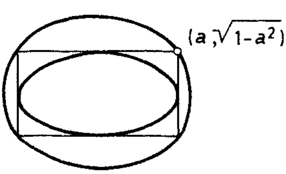
We may assume that I does not pass through $\infty \times \infty$, and then $E = F(I)$ is a plane curve having affine equation

$$x(\xi) = a(\eta)\, y(\xi) + b(\eta),$$

and is thus a quartic. E is nonsingular away from $P = [0, 1, 0]$ and $Q = [0, 0, 1]$, and these are ordinary double points since I meets $\mathbb{P}^1 \times \infty$ and $\infty \times \mathbb{P}^1$ in two distinct points. The lines through P are given by $\xi = \text{constant}$ and those through Q by $\eta = \text{constant}$; consequently, the Poncelet construction is just our previous straight line construction on the plane quartic. The conclusion (1.37) translates into the

Poncelet Theorem. *The condition that the polygon in the preceding figure be closed with n sides is independent of the initial point P_1, and its satisfaction imposes one condition on the pair of conics C and D.*

The following example was shown to me by Mark Green: If C is the circle $x^2 + y^2 = 1$, D the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a, b < 1)$ and we want a four-sided closed polygon, then by the Poncelet theorem we may begin at any point, say, $(a, \sqrt{1-a^2})$, on C and the symmetric figure



gives $b = \sqrt{1-a^2}$.

II. Abel's Theorem and the Trace

(a) Holomorphic and Meromorphic Forms on Analytic Varieties

We now change our viewpoint and lay the local analytical foundations for one of the two generalizations of Abel's theorem discussed in this paper. Our immediate purpose is to define holomorphic and meromorphic differential forms on generally singular analytic varieties. The definition of holomorphic forms is not the usual one, but rather is based on the finiteness of certain L^2-norms in much the same spirit as M. Noether's original definition of differentials of the first kind on an algebraic variety (c.f. the reference to Noether in the introduction).

We begin with an elementary lemma in several complex variables.

(2.1) **Lemma.** *Let $h(z,w)=h(z_1,\dots,z_{n-1},w)$ be a holomorphic function in the punctured polycylinder $P^*=\{(z,w):|z_i|\leqq 1, 0<|w|\leqq 1\}$. Then h extends holomorphically across the divisor $w=0\Leftrightarrow$*

$$\int_{P^*}|h|^2\,d\mu<\infty$$

where $d\mu$ is Euclidean measure.

Similarly, h has a pole along $w=0\Leftrightarrow$

$$\int_{P^*[r]}|h|^2\,d\mu=0(r^N)$$

where $P^*[r]=\left\{|z_i|\leqq 1, \frac{1}{r}\leqq|w|\leqq 1\right\}$ is an annular ring around this divisor.

Proof. We expand h in a Laurent series

$$h(z,w)=\sum_{\nu=-\infty}^{+\nu}h_\nu(z)\,w^\nu$$

and use the orthogonality relations

$$\frac{1}{2\pi}\int_{|w|=\rho}w^\nu\,\overline{w}^\mu\,d(\arg w)=\begin{cases}r^2 & \mu=\nu\\ 0 & \mu\neq\nu\end{cases}$$

to deduce that

$$\frac{1}{2\pi}\int_{P^*[r]}|h|^2\,d\mu=\sum_{\nu=-\infty}^{+\infty}\Big(\int_{|z_i|\leqq 1}|h_\nu|^2\Big)\,c_\nu(r)$$

where

$$c_\nu(r)=\begin{cases}\left(\dfrac{1}{2\nu-2}\right)\left(\dfrac{1}{r^{2\nu-2}}-1\right) & \nu\leqq-2\\ \log r & \nu=-1\\ \dfrac{r^{2\nu+2}-1}{2\nu+2} & \nu\geqq 0.\end{cases}$$

Our lemma follows from this. Q.E.D.

Suppose now that V is an analytic variety of dimension n with singular locus V_s. A *holomorphic n-form* ψ on V shall be given by a holomorphic form in the usual sense on the complex manifold $V-W=V^*$, where W is an analytic subvariety of V containing V_s but not containing any irreducible component of V, and where ψ satisfies the local L^2-estimate

$$(2.2)\quad (\sqrt{-1})^{n^2} \int_{U\cap V^*} \psi\wedge\bar{\psi}<\infty$$

in a neighborhood U of any point $P\in W$. The vector space of holomorphic n-forms is denoted by $\Omega^n(V)$. In case V is a manifold we recover the usual notion of holomorphic form by appealing to the above lemma. More precisely, suppose ψ is holomorphic outside a divisor W on V and satisfies the local L^2-estimates. Then, by the lemma, ψ extends holomorphically across the smooth points $W-W_s$ of W, is thus holomorphic outside the subvariety W_s of codimension $\geqq 2$ on V, and then by Hartogs' theorem extends holomorphically to all of V.

Similarly, a meromorphic n-form ψ is given by a holomorphic form on $V^*=V-W$ satisfying the L^2 growth condition

$$(2.3)\quad (\sqrt{-1})^{n^2} \int_{U[r]} \psi\wedge\bar{\psi}=O(r^N)$$

with $U[r]$ being the intersection of a neighborhood U of a point $P\in W$ with an annular ring of inner radius $1/r$ around W relative some local embedding of U in $\mathbb{C}^N$. The meromorphic n-forms are denoted by $\mathcal{M}^*(V)$.

For $q<n$, a holomorphic q-form ψ is given by a holomorphic form in the usual sense on $V^*=V-W$ as above, and where for any local piece of q-dimensional analytic subvariety $Z\subset V$ but $Z\not\subset W$ the restriction $\psi|Z$ is holomorphic in the previous sense. Similarly for meromorphic forms.

The usual properties of forms, such as admitting exterior products, exterior differentiation, and pulling back under holomorphic mappings may be verified for the holomorphic and meromorphic forms $\Omega^q(V)$ and $\mathcal{M}^q(V)$ as defined above. This is a little tedious to do directly from the above definitions, so here is a quick way to see what we want by using a big theorem: A *resolution of singularities* $f\colon \tilde{V}\to V$ is given by a complex manifold $\tilde{V}$ and proper, surjective holomorphic mapping f which is an isomorphism outside a proper subvariety of V. By Hironaka's theorem such resolutions exist, and any two $(\tilde{V}_i, f_i)$ $(i=1,2)$ are dominated by a third according to the diagram

$$\begin{tikzcd} & \tilde{V}_3 \arrow[dl, "g_1"'] \arrow[dr, "g_2"] \arrow[dd, "f_3"] & \\ \tilde{V}_1 \arrow[dr, "f_1"'] & & \tilde{V}_2 \arrow[dl, "f_2"] \\ & \tilde{V} & \end{tikzcd}$$

where $f_3=f_1\circ g_1=f_2\circ g_2$. It follows easily from this, together with the L^2-definitions, that

$$f^*\colon \Omega^q(V)\to\Omega^q(\tilde{V})$$

and

$$f^*: \mathcal{M}^q(V) \to \mathcal{M}^q(\tilde{V})$$

are isomorphisms for any resolution f: $\tilde{V} \to V$, and since any holomorphic mapping h: $V \to V'$ may be resolved according to a diagram

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{h}} & \tilde{V}' \\ \downarrow & & \downarrow \\ V & \xrightarrow{h} & V' \end{array}$$

the holomorphic and meromorphic forms as defined above have all the usual properties one expects of differential forms.

As any easy example, for the complex projective space,

(2.4) $\Omega^q(\mathbb{P}^n) = 0 \quad (q > 0)$.

Proof. If $0 \neq \psi \in \Omega^q(\mathbb{P}^n)$, then for some linear subspace $\mathbb{P}^q \subset \mathbb{P}^n$ the restriction $\psi | \mathbb{P}^q \neq 0$. On the affine open set $\mathbb{C}^q \subset \mathbb{P}^q$

$$\psi = h(z)\, dz_1 \wedge \cdots \wedge dz_q$$

where the holomorphic function $h(z)$ satisfies

$$\int_{\mathbb{C}^q} |h(z)|^2 \, d\mu < \infty,$$

which is impossible. Q.E.D.

(b) The Trace Mapping

The results from several complex variables which we shall require may all be deduced from

Remmert's Proper Mapping Theorem. *A proper holomorphic mapping f: $V \to W$ takes analytic subvarieties $Z \subset V$ onto analytic subvarities $f(Z) \subset W$.*

Suppose now that f: $V \to W$ is a proper, surjective holomorphic mapping where both V and W are irreducible analytic varieties of the same dimension n. We will define the *push forward mapping.*

(2.5) $f_*: \Omega^q(V) \to \Omega^q(W)$,

and similarly for meromorphic forms, going in the opposite direction from the usual pullback.[4] To do this, we note that f is *generically finite* in the sense that there is an analytic hypersurface $D \subset W$ with inverse image $D_f = f^{-1}(D)$ such that $W^* = W - D$ and $V^* = V - D_f$ are both complex manifolds and f: $V^* \to W^*$ is a finite unramified covering mapping. For sufficiently small open sets $U \subset W^*$, the inverse image $f^{-1}(U) = U_1 + \cdots + U_d$ decomposes into the *sheet-number d* disjoint

4 The mapping f_* is sometimes called the *trace*, but we shall reserve that name for the application of f_* to be zero-cycles given in the next section

open sets U_ν such that $f\colon U_\nu \to U$ is an isomorphism with inverse s_ν. Given $\psi \in \Omega^n(V)$, we define

(2.6) $f_*(\psi) = s_1^* \psi + \cdots + s_d^* \psi$

in U. This definition is clearly invariant under the particular labeling of the s_ν, and thus defines $f_*(\psi)$ on W^*. Locally on W^*.

$$s_\nu^* \psi = h_\nu(z)\, dz_1 \wedge \cdots \wedge dz_n$$

and from

$$(h_1 + \cdots + h_d)\,\overline{(h_1 + \cdots + h_d)} \leqq C(|h_1|^2 + \cdots + |h_d|^2)$$

it follows that $f_*(\psi)$ satisfies the local L^2-finiteness condition (2.2) around points of D.[5] Thus $f_*(\psi)$ is holomorphic on W, and in general we may define both

$$f_*\colon \Omega^q(V) \to \Omega^q(W)$$

and

$$f_*\colon \mathcal{M}^q(V) \to \mathcal{M}^q(W)$$

by symbolically writing (2.6) in the more suggestive form

(2.7) $(f_* \psi)(P) = \psi(P_1) + \cdots + \psi(P_d)$

which has the following meaning: For a general point $P \in W$ the inverse image $f^{-1}(P) = P_1 + \cdots + P_d$ where the P_ν vary locally holomorphically with P, and $\psi(P_\nu)$ is the pullback of ψ under the mapping $P \to P_\nu$.

The simplest case is when the sheet number is 1. Then f^* and f_* are inverse isomorphisms

$$\Omega^q(V) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \Omega^q(W).$$

To apply this, we recall that a *meromorphic mapping* $F\colon V \to W$ is given by a holomorphic mapping $\tilde{F}\colon V - Z \to W$ defined outside a subvariety Z of V such that the closure of the *graph* of $\tilde{F}$ is a subvariety G of $V \times W$ lying properly over V via the projection mapping $\pi_V\colon G \to V$. We then define

$$F^*\colon \Omega^q(W) \to \Omega^q(V)$$

by $F^* = (\pi_V)_* (\pi_W)^*$.

For example, suppose that V is compact and there is a surjective, meromorphic mapping

$$f\colon \mathbb{P}^m \to V.$$

Then $\Omega^q(V) = 0$ for $q > 0$, since $\Omega^q(\mathbb{P}^m) = 0$ for $q > 0$ and $f^*\colon \Omega^q(V) \to \Omega^q(\mathbb{P}^n)$ is, first of all, defined and secondly is injective.

5 Even in case both V and W are n-dimensional complex manifolds and $f\colon V \to W$ is a surjective holomorphic mapping, the L^2-argument provides a nice way of showing that f_* takes holomorphic forms into holomorphic forms.

Note that $f_*(d\psi) = df_*(\psi)$ but that

$$f_*(\eta \wedge \psi) \neq f_*(\eta) \wedge f_*(\psi);$$

this latter will be significant when we come to Abel's general theorem

To apply this remark, we suppose that $V \subset \mathbb{P}^N$ is a projective algebraic variety and that the linear group GL_m acts algebraically and transitively on a Zariski open set $V^* \subset V$. Then there is a surjective meromorphic mapping

$$f\colon \mathbb{P}^{m^2} \to V$$

where GL_m is embedded as Zariski open set in $\mathbb{P}^{m^2}$ via its matrix entries and f is the orbit mapping. By what we just said, $\Omega^q(V)=0$ for $q>0$.

A special case is when $V=\mathbb{G}(r, n+r)$ is the Grassmann variety of projective r-spaces in $\mathbb{P}^{n+r}$ on which GL_{n+r+1} acts transitively in the usual manner.

Perhaps more interestingly we may take

$$V^* = \{\text{set of nondegenerate curves of degree } n \text{ in } \mathbb{P}^n\}$$

and V the Zariski closure of V^* in the Chow variety of all curves of degree n in $\mathbb{P}^n$. Then every curve in V^* is a rational normal curve, and GL_{n+1} acts algebraically and transitively on V^*. Thus $\Omega^q(V)=0$ for $q>0$. This remark will be applied in the second paper in this series.

As another application of the push-forward mapping f_*, we suppose that V is a compact, complex manifold on which a finite group G operates holomorphically. Then the quotient $W=V/G$, being locally the quotient of a polycylinder by a finite group, is an analytic variety. We claim that

$$(2.8)\quad \Omega^*(W)=\Omega^*(V)^G$$

are the G-invariant forms on V.

Proof. The mapping $\pi\colon V \to W$ is proper and surjective, and there is an obvious inclusion

$$\Omega^*(W) \subseteq \Omega^*(V)^G.$$

If ψ is a G-invariant form on V, then $\pi^* \pi_*(\psi)=|G| \cdot \psi$ where $|G|$ is the order of G, and thus the inclusion is equality. Q.E.D.

For instance, take $V=M^d$ to be the Cartesian product $\underbrace{M \times \cdots \times M}_{d \text{ times}}$ of an n-dimensional compact, complex manifold M and G the permutation group. The quotient $M^d/G=M^{(d)}$ is the *d-fold symmetric product* whose points we may think of as zero-cycles

$$\Gamma = P_1 + \cdots + P_d$$

of *degree d* on M. When M is a Riemann surface, $M^{(d)}$ is smooth but not otherwise. The holomorphic forms on $M^{(d)}$ are just the symmetric ones on M^d. Since we may compute $\Omega^*(M^d)$ by Künneth, this tells us $\Omega^*(M^{(d)})$. In particular, there is an identification

$$\Omega^n(M) \simeq \Omega^n(M^d)^G$$

sending $\psi \in \Omega^n(M)$ to the G-invariant form $\sum_{\nu=1}^{d} \pi_\nu^* \psi$ on M^d where $\pi_\nu\colon M^d \to M$ is projection onto the ν-th factor. This gives the isomorphism

$$(2.9)\quad \Omega^n(M) \simeq \Omega^n(M^{(d)}),$$

which may be symbolically written as

(2.10) $\Psi(\Gamma)=\psi(P_1)+\cdots+\psi(P_d)$

where $\Gamma=P_1+\cdots+P_d$ is a point of $M^{(d)}$ written as a zero cycle. We shall generally write the isomorphism (2.9) as

$$\psi\to\Psi,\qquad \varphi\to\Phi$$

etc., where (2.10) is the defining relation.

A final variant of the push-forward mapping concerns the following situation: Suppose that M and N are compact, complex manifolds of the same dimension n and $f\colon M\to N$ is a proper, surjective holomorphic mapping. Denote by $\mathscr{K}_M^\mu$ the sheaf of holomorphic sections of the μ-th power K_M of the canonical line bundle K_M of M. Then $\mathscr{K}_M^1=\Omega_M^n$. When $\dim M\geqq 2$, it may happen that $H^0(M,\Omega^n)=0$ but $H^0(M,\mathscr{K}_M^\mu)$ is very large as $\mu\to\infty$. In this case it may be desirable to extend the push-forward mapping f_* to sections of $\mathscr{K}_M^\mu$, as will now be discussed.

First, we remark that for a μ-tuple differential $\psi\in H^0(M,\mathscr{K}_M^\mu)$, then

$$(\sqrt{-1})^{n^2}(\psi\wedge\bar\psi)^{1/\mu}=\Psi$$

is a well-defined continuous density on M. Locally, $\psi=h(z)(dz_1\wedge\cdots\wedge dz_n)^\mu$ and $\Psi=|h(z)|^{2/\mu}\,d\mu$ ($d\mu$ being Euclidean measure). Next, if ψ is defined and holomorphic outside a divisor D without most poles there, and if for each $P\in D$ and small neighborhood U of P the L^2-estimate

$$\int_{U\cap M^*}\Psi<\infty$$

is valid, then we may conclude that:

ψ has a pole of order at most $(\mu-1)$ along D.

Proof. It suffices to prove this around a simple point of D, and then we may choose coordinates $(z_1,\ldots,z_{n-1},w)$ such that D is given by $w=0$. Effectively then we are reduced to the 1-variable situation of a holomorphic function $h(z)$ defined for $0<|z|\leqq 1$, having a pole at $z=0$, and satisfying

$$\int_{0<|z|\leqq 1}|h(z)|^{2/\mu}\,dz\,d\bar z<\infty.$$

It then follows that $h(z)$ has at most a pole of order $\mu-1$ at $z=0$. Q.E.D.

Now consider $f\colon M\to N$ as above. Given $\psi\in H^0(M,\mathscr{K}_M^\mu)$, we may define the the section $f_*\psi$ of $\mathscr{K}_N^\mu$ outside the ramification divisor D of f by the formula

(2.11) $f_*(\psi)(P)=\psi(P_1)+\cdots+\psi(P_d)$

where $f^{-1}(P)=P_1+\cdots+P_d$. This is because higher differentials are contravariant tensors in the usual way. Using the estimate

$$((\alpha_1+\cdots+\alpha^d)\,(\overline{\alpha_1+\cdots+\alpha_d})^{2/\mu}\leqq C(|\alpha_1|^{2/\mu}+\cdots+|\alpha_d|^{2/\mu})$$

we may conclude that $(f_*(\psi)\wedge\overline{f_*(\psi)})^{1/\mu}$ has finite L^2-norm on N, and thus has a pole of order at most $(\mu-1)$ along D. In summary, (2.12) *Given compact, complex manifolds M and N of the same dimension n and a proper surjective holomorphic mapping $f: M \to N$ unramified outside a divisor D on N, the push-forward mapping*

$$f_*: H^0(M, \mathscr{K}_M^\mu) \to H^0(N, \mathscr{K}_N^\mu((\mu-1)\,D))$$

is well defined by the formula (2.11).

As an illustration, suppose that $f: M \to \mathbb{P}^n$ is a branched covering with ramification divisor a hypersurface of degree d. Suppose, moreover, that the canonical bundle of M is ample, so that given distinct points $Q_1, Q_2, \ldots, Q_d$ of M we may find μ and $\psi \in H^0(M, \mathscr{K}_M^\mu)$ such that $\psi(Q_1) \neq 0$, $\psi(Q_2) = \cdots = \psi(Q_{d-1})$. If

$$f^{-1}(P) = Q_1 + \cdots + Q_d,$$

then $f_*(\psi)$ is a nonidentically zero section in the group $H^0(\mathbb{P}^n, \mathscr{K}_{\mathbb{P}^n}((\mu-1)\,D))$. Using the standard notation $\mathcal{O}(k)$ for the sheaf of sections of the line bundle $[k \cdot \mathbb{P}^{n-1}]$ on $\mathbb{P}^n$, $\mathscr{K}_{\mathbb{P}^n} \simeq \mathcal{O}(-n-1)$ and consequently,

$$\mathscr{K}_{\mathbb{P}^n}((\mu-1)\,D) \simeq \mathcal{O}(\mu(d-n-1)-d).$$

Since $H^0(\mathbb{P}^n, \mathcal{O}(k)) = 0$ if $k < 0$, we conclude that

$$d \geqq n+2.$$

This estimate places a lower bound on the size of the ramification divisor of f, one which may be proved to be sharp. For example, in case V is a compact Riemann surface of genus $g \geqq 2$ this gives that any covering mapping $f: V \to \mathbb{P}^1$ branches at 3 or more points, a well-known consequence of the Riemann-Hurwitz formula.

(c) The G.A.G.A. Principle and Abel's Theorem in General Form

We wish to apply the above *local analytic considerations* to *global algebraic varieties*. Underlying this is the famous G.A.G.A. principle[6] which we shall briefly discuss.

The basic results are that an *analytic subvariety* as defined locally by analytic equations in projective space is an *algebraic variety* defined globally by polynomial equations (*Chow's theorem*), and that a meromorphic function on an algebraic variety is a rational function. We will sketch proofs of these two statements based on Remmert's proper mapping theorem. and the topological fact that every linear r-plane meets an *analytic* subvariety $V_n \subset \mathbb{P}^{n+r}$ the same number of times (counting multiplicities), provided of course that the intersection is discrete. Our reasons for doing this are that these theorems are not so complicated to prove as commonly believed, and, more importantly, that due to the partly expository nature of this paper it seems a good idea to keep our perspective by properly emphasizing the *mechanics* of the transition from local analytic to global algebraic considerations.

6 So named after the paper by J. P. Serre, Géométrie algébrique et géometrie analytiquè, Ann. Inst. Fourier, Vol. 6 (1955–56), pp. 1–42

Suppose then that $V_n \subset \mathbb{P}^{n+r}$ is an analytic variety. The set of $r-2$ planes in $\mathbb{P}^{n+r}$ which meet V forms, in the obvious way, an analytic subvariety W of the product $V \times \mathbb{G}(r-2, n+r)$. The fiber of the projection $W \to V$ is the Grassmannian $G(r-2, n+r)$ of $(r-2)$-planes through the origin in $\mathbb{C}^{n+r}$ and has dimension $(r-2)(n+2)$. There is a tautological proper mapping $f\colon W \to \mathbb{P}^{n+r}$, and an easy count shows that the image is an analytic subvariety of dimension less than $n+r$ (here we are using Remmert's theorem). Consequently, a general $r-2$ plane $\mathbb{P}^{r-2}$ fails to meet V and the standard linear projection

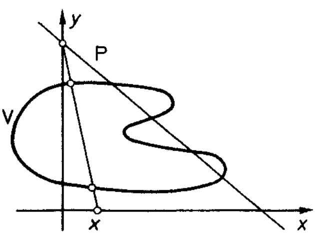
$$\begin{array}{ccc} V & \subset & \mathbb{P}^{n+r} - \mathbb{P}^{r-2} \\ \downarrow & & \downarrow \\ \pi(V) & \subset & \mathbb{P}^{n+1} \end{array}$$

is a proper mapping onto an analytic hypersurface in $\mathbb{P}^{n+1}$. In fact, it is easy to see that π is bimeromorphic. To find polynomial equations which define $V \subset \mathbb{P}^{n+r}$ it will suffice to do this for $\pi(V) \subset \mathbb{P}^{n+1}$ and vary the center of projection $\mathbb{P}^{r-2}$.

Assuming now that $V \subset \mathbb{P}^{n+1}$ is a hypersurface, choose a general point $P \in \mathbb{P}^{n+1} - V$ and go to the projection

$$\begin{array}{ccc} V & \subset & \mathbb{P}^{n+1} - \{P\} \\ \downarrow & & \downarrow \\ \mathbb{P}^n & = & \mathbb{P}^n \end{array}$$

given by intersecting the line PQ $(Q \in \mathbb{P}^{n+1} - \{P\})$ with a fixed $\mathbb{P}^n$. Since V meets each line the same number d of times, this represents V as a d-sheeted branched covering of $\mathbb{P}^n$. Choosing affine coordinates $(x, y) = (x_1, \ldots, x_n, y)$ such that $\pi(x, y) = x$, for each $x \in \mathbb{C}^n \subset \mathbb{P}^n$ the line $P\acute{x}$ meets V in points $y_1(x), \ldots, y_d(x)$.



The symmetric functions of the $y_\nu(x)$ are single-valued and holomorphic outside the branch locus and are locally bounded. Hence they are holomorphic for all x by the Riemann extension theorem. In particular,

$$f(x, y) = \prod_{\nu=1}^{d} (y - y_\nu(x))$$

is a polynomial in y with coefficients $\sigma_\mu(x)$ holomorphic functions of x, and $f(x, y) = 0$ on V. As $|x| \to \infty$, it is easy to see that $\max |y_\nu(x)| = 0(|x|)$ (c.f. the above picture) so that $|\sigma_\mu(x)| = 0(|x|)^d)$ and $f(x, y)$ is a polynomial in all variables. It follows that V is algebraic.

Next, suppose that h is a meromorphic function on $\mathbb{P}^n$ with homogeneous coordinates $X = [X_0, \ldots, X_n]$ corresponding to the projection $\pi\colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$.

The polar set of h is, by Chow's theorem, an algebraic subvariety of $\mathbb{P}^n$, which we take to have homogeneous equation $Q(X)=0$ of degree d. Then $P(X)=Q(X)\,h(X)$ is a *holomorphic* function on $\mathbb{C}^{n+1}-\{0\}$, and hence by Hartog's theorem extends to an entire analytic function. Since $P(\lambda X)=\lambda^d P(X)$, it follows that $P(X)$ is a homogeneous polynomial of degree d, and $h(X)=P(X)/Q(X)$ is a rational function. If we denote by $\mathscr{K}(V)$ and $\mathscr{M}(V)$ the fields of rational and meromorphic functions on an algebraic variety V, then we have proved that $\mathscr{K}(\mathbb{P}^n)=\mathscr{M}(\mathbb{P}^n)$.

We now prove the equality in general. For this we consider the finite branched covering $\pi\colon V\to\mathbb{P}^n$ given by $\pi(x,y)=x$ as above. Given a meromorphic function h on V, any symmetric polynomial $\sigma(h(x,y_1(x)),\dots,h(x,y_d(x))$ is a meromorphic, and hence rational, function on $\mathbb{P}^n$. It follows that the field extension

$$[\mathscr{K}(\mathbb{P}^n);\mathscr{M}(V)]\leqq d$$

is a finite algebraic extension of degree $\leqq d$. On the other hand,

$$[\mathscr{K}(\mathbb{P}^n);\mathscr{K}(V)]\geqq d$$

since we may find a polynomial $p(x,y)$ taking distinct values at a suitable set of distinct points $(x,y_1(x)),\dots,(x,y_d(x))$. Since $\mathscr{K}(V)\subseteq\mathscr{M}(V)$ we have equality, and this completes our discussion of the G.A.G.A. principle.

Easy consequences of what we have just proved are: On a projective algebraic variety every meromorphic differential form is rational; a meromorphic mapping between two such varieties is rational, etc.

Now we come to a general form of Abel's theorem. This deals with an *algebraic family* $\{\Gamma(t)\}_{t\in T}$ *of zero cycles* on a projective variety V. More precisely, we should be given in addition to V a parameter variety T and subvariety $I\subset T\times V$ such that for a general point $t\in T$ the intersection

$$I\cdot(\{t\}\times V)=\{t\}\times\Gamma(t)$$

where $\Gamma(t)=P_1(t)+\cdots+P_d(t)$ is a zero cycle on V. Equivalently, the projection $I\to T$ should be surjective and generically finite. This definition is the same as giving a rational mapping

$$f\colon T\to V^{(d)} \qquad (2.13)$$

into the d-fold symmetric product – here

$$f(t)=\Gamma(t)$$

for a general $t\in T$. Denoting by π_T and π_V the projections of I onto T and V, here is our main

Definition. Given a rational q-form ψ on V, the *trace* $\mathrm{Tr}_I(\psi)$ is the rational q-form on T defined by

$$\mathrm{Tr}_I(\psi)=(\pi_T)_*(\pi_V)^*(\psi).$$

This makes sense provided we are not in the *degenerate case* where $\pi_V(I)$ lies entirely in the polar locus of ψ – this will be excluded. Observe that the local analytic considerations of Sections II a and b and the preceding G.A.G.A. discussion are the ingredients which insure that the definition has meaning.

If our family of zero cycles is given by (2.13), then the trace

(2.14) $\mathrm{Tr}_I(\psi)=f^*(\Psi)$

where Ψ is the rational q-form on $V^{(d)}$ given by (2.10). The most suggestive notation is writing the zero cycle as

$$\Gamma(t)=P_1(t)+\cdots+P_d(t)$$

and then according to (2.7)

(2.15) $\mathrm{Tr}_I(\psi)=\psi(P_1(t))+\cdots+\psi(P_d(t))$.

This is the formula which most clearly exhibits the trace and which ties in with the classical Abel theorem, as will now be explained.

Return to the situation in Section Ia of an algebraic curve C having affine equation $f(x,y)=0$ and on which we are given an abelian integral

$$u=\int r(x,y)\,dx.$$

Suppose that D_t is a family of curves given by $\theta(x,y)=0$ and set

$$D_t\cdot C=\sum_\nu P_\nu(t)$$

where $P_\nu(t)=(x_\nu(t), y_\nu(t))$. The abelian sum

$$u(t)=\sum_\nu \int^{P(t)}\psi \qquad (\psi=r(x,y)\,dx)$$

has derivatives

$$\frac{\partial u}{\partial t_i}=\sum_\nu r(x_\nu(t), y_\nu(t))\frac{\partial x_\nu}{\partial t_i},$$

so that according to (2.15):

(2.16) *The total differential*

$$du(t)=\mathrm{Tr}_I(\psi)$$

is the trace of ψ relative to the family of zero cycles $D_t\cdot C$.

In particular, Abel's theorem (1.6) follows from our local analytic discussions and the G.A.G.A. principle. We may say that the existence of the trace mapping gives a general form of Abel's theorem, one which will be interesting according to the applications which can be found. This brings us to our second main

Definition. On a projective variety V let ψ be a rational differential form and $\{\Gamma(t)\}_{t\in T}$ an algebraic family of zero cycles. Then ψ is said to be *of the first kind with respect to this family* if $\mathrm{Tr}_I(\psi)$ is holomorphic T.

This will certainly be the case if ψ is holomorphic on V, but not conversely. If ψ is a q-form of the first kind relative to the family $\{\Gamma(t)\}_{t\in T}$ and if $\Omega^q(T)=0$, then

$$\mathrm{Tr}_I(\psi)\equiv 0.$$

When written out explicitly, this condition

(2.17) $\psi(P_1(t))+\cdots+\psi(P_d(t))\equiv 0$

has the appearance of a general addition theorem, and we shall explain this in the coming sections.

(d) Holomorphic Differentials and Rational Equivalence

We want to now discuss the relation between the $\mathrm{Tr}_I(\psi)=0$ condition and the rational equivalence of points on a smooth projective algebraic variety V.

In this section it is convenient to define a *zero cycle* to be a formal finite sum $\Gamma=\sum_\nu n_\nu P_\nu$ of points on V with arbitrary integer coefficients. An *effective zero cycle* is a zero cycle Γ with nonnegative coefficients. As usual, the *degree* of Γ is

$$\deg\Gamma=\sum_\nu n_\nu.$$

Two effective zero cycles Γ and Γ' of the same degree are defined to be *rationally equivalent in the strict sense*, written $\Gamma\equiv\Gamma'$, if there is a rational mapping

(2.18) $f\colon \mathbb{P}^m\to V^{(d)}$

such that $f(t)=\Gamma$ and $f(t')=\Gamma'$. We then say that two zero cycles Γ and Γ' are rationally equivalent if there is a zero cycle Γ'' such that $\Gamma+\Gamma''$ and $\Gamma'+\Gamma''$ are both effective and $\Gamma+\Gamma''\equiv\Gamma'+\Gamma''$. This is an equivalence relation, and the group of all zero cycles modulo equivalence constitutes the *Chow group* $C(V)$.

The first question is whether or not $C(V)$ is *finite dimensional*. Intuitively, this should mean that for large d an effective zero cycle $\Gamma\in V^{(d)}$ should depend on only a finite number of parameters modulo rational equivalence. This in turn will be the case when the *codimension* of maximal images of rational maps (2.18) through a general point $\Gamma\in V^{(d)}$ remains bounded as $d\to\infty$. The way in which Abel's theorem comes into the picture is via the observation:

(2.19) *For a rational mapping* (2.18) *and* $\psi\in\Omega^q(V)$,

$$f^*\Psi\equiv 0.$$

This is equivalent to saying that

$$\sum_\nu\psi(P_\nu(t))\equiv 0$$

for a rational family $\Gamma(t)=\sum_\nu P_\nu(t)$ $(t\in\mathbb{P}^m)$ *of zero cycles on* V.

It is convenient to rephrase (2.19) in the language of differential systems. Recall that a *differential system* Σ on a variety W is given by a collection $\{\Psi\}$ of holomorphic differential forms. An *integral variety* is a subvariety $Z\subset W$ such that the restriction $\Psi|Z\equiv 0$ for all $\Psi\in\Sigma$. The condition (2.19) may be re-interpreted by the statement:

(2.20) *The image* $f(\mathbb{P}^m)\subset V^{(d)}$ *is an integral variety for the differential system* $\Omega^*(V^{(d)})$.

Because of this it is interesting to estimate the maximal dimensions of *local* integral varieties for the differential system $\Omega^*(V^{(d)})$.

We first consider the familiar case of curves from this point of view. By what was said in Section II b (c.f. (2.9)) the differential system $\Omega^*(V^{(d)})$ is generated by $\Omega^1(V^{(d)})$, and, moreover,

$$\Omega^1(V) \simeq \Omega^1(V^{(d)})$$

under the mapping

$$\psi \to \Psi$$

defined by the relation (c.f. (2.10))

$$\Psi(\Gamma) = \psi(P_1) + \cdots + \psi(P_d)$$

where $\Gamma = P_1 + \cdots + P_d$ is a variable point in $V^{(d)}$. The forms Ψ are all closed, and at a general point $\Gamma \in V^{(d)}$ the rank of the differential system $\Omega^1(V^{(d)})$ is equal to the genus π of V. Consequently, by the *Frobenius theorem* the differential system has passing through a general point $\Gamma \in V^{(d)}$ local integral varieties of codimension π. This again suggests that $C(V)$ should have finite dimension π. Note that this argument is in a certain sense parallel to the discussion of inversion for plane curves given in Section I c.

To complete the argument in this context, instead of counting constants as was done previously we use *Riemann's inequality*

$$l(\Gamma) \geqq d - \pi$$

where $l(\Gamma)+1$ is the dimension of the vector space of meromorphic functions φ on V whose divisor $(\varphi) \geqq -\Gamma$. It follows that through each point $\Gamma \in V^{(d)}$ there is a $\mathbb{P}^{l(\Gamma)}$, and so $\dim C(V) \leqq \pi$. On the other hand, we have $\dim C(V) \geqq \pi$ by the preceding remark based on Abel's theorem. Continuing on with this reasoning, one may easily prove that the Jacobian variety $C(V)$ is a π-dimensional group variety.

We now come to surfaces and Mumford's theorem, which is based on the following

Linear Algebra Lemma. *Let E be a complex vector space of dimension $2n$ and $\Phi \in \Lambda^2 E^*$ a nondegenerate 2-form. If $F \subset E$ is a linear subspace with $\Phi | F \equiv 0$, then* $\dim F \leqq n$.

Proof. Let $f_1, \ldots, f_k$ be a basis for F and complete it to a basis $\{f_1, \ldots, f_k; e_1, \ldots, e_{2n-k}\}$ for E. Then, setting $\Phi^n = \Phi \wedge \cdots \wedge \Phi$ (n times)

$$\langle \Phi^n, f_1 \wedge \cdots \wedge f_k \wedge e_1 \wedge \cdots \wedge e_{2n-k} \rangle \neq 0.$$

On the other hand, since $\langle \Phi, f_i \wedge f_j \rangle = 0$ for all i, it is clear that the right-hand side must be zero if $k \geqq n+1$. Q.E.D.

If now V is a smooth surface and $0 \neq \psi \in \Omega^2(V)$, then at a general point $\Gamma \in V^{(d)}$ the 2-form Ψ given by (2.10) is nondegenerate. Indeed, if z_v, w_v are local coordinates centered around P_v such that $\psi = h_v\, dz_v \wedge dw_v$, then upon assuming that P_v are distinct

$$\Psi = \sum_{\nu=1}^{d} h_\nu \, dz_\nu \wedge dw_\nu$$

is nondegenerate provided all $h_\nu(0) \neq 0$. Applying the linear algebra lemma gives

Mumford's Theorem. *If V is an algebraic surface with $p_g(V) = \dim \Omega^2(V) \neq 0$, then the local integral varieties of the differential system $\Omega^*(V^d)$ through a general point $\Gamma \in V^{(d)}$ have dimension $\leqq d$. In particular, if the mapping* (2.18) *is nondegenerate and has image passing through a general point $\Gamma \in V^{(d)}$, then $m \leqq d$.*[7]

This implies that $\dim C(V) = +\infty$. Roitman has refined Mumford's theorem by proving:[8]

If $p_g(V) \neq 0$, then for $d \geqq d_0$ and Γ a generic point of $V^{(d)}$ the orbit through Γ of cycles rationally equivalent to Γ has dimension zero.

Here, generic means outside a countable union of proper subvarieties of $V^{(d)}$; in any case, Γ is usually isolated in its linear equivalence class, and, consequently $C(V)$ is as infinite dimensional as possible.

Now this series of papers is devoted to finding higher codimensional situations where Abel-type conditions such as (2.17) can be inverted. As noted above, this is classically the case for curves, and we now wish to observe that:

If V is a surface with $p_g(V) \neq 0$ then the estimates provided by Mumford's theorem cannot in general be improved.

To explain precisely what this means, it is convenient to use some notations and results from surface theory.[9] If $L \to V$ is a line bundle, then $|L|$ denotes the *complete linear system* $\mathbb{P}(H^0(V, \mathcal{O}(L)))$ of effective curves C with $[C] \simeq L$. The *intersection number* of line bundles L and L' is defined by

$$L \cdot L' = c_1(L) \cdot c_1(L') [V].$$

If L is positive so that $H^q(V, \mathcal{O}(L^\mu)) = 0$ for $q \geqq 1$ and $\mu \geqq \mu_0$, the *Riemann-Roch formula* is

$$\text{(2.21)} \quad \dim |L^\mu| = \mu^2 \frac{L \cdot L}{2} - \mu \frac{L \cdot K}{2} + p_g - q \qquad (\mu \geqq \mu_0)$$

where $q = h^1(\mathcal{O})$ is the *irregularity* and K is the *canonical bundle.*

Suppose now that L is positive and $C, C' \in |L^\mu|$ have no common component and intersect at distinct points where all $\psi(P_\nu) \neq 0$ for some $\psi \in \Omega^2(V)$. For large μ

7 We are not claiming to have given a complete demonstration that the Chow group $C(V)$ is infinite-dimensional; Mumford's proof of this requires one additional step dealing with "chains" of $\mathbb{P}^1$'s connecting two effective zero cycles, and uses the relation $\Omega^*(W \times \mathbb{P}^1) \simeq \Omega^*(W)$ rather than (2.10)

8 The references to Mumford's and Roitman's papers were given in the introduction. A convenient overall discussion of cycles is given in R. Hartshorne, Equivalence relations on algebraic cycles and subvarieties of small codimension, Proc. Sym. Pure Math., Vol. XXIX, Amer. Math. Soc. (1975), pp. 129–165

9 A convenient reference for what we shall need about surfaces is E. Bombieri and D. Husemaller, Classification and embeddings of surfaces, Proc. Symp. Pure Math., Vol. XXIX, Amer. Math. Soc. (1975), pp. 329–421

this will be generally true. Then

$$\Gamma = C \cdot C' \in V^{(d)}$$

where the degree

$$d = \mu^2 L \cdot L. \tag{2.22}$$

By Mumford's theorem this d is an *upper bound* on the dimension of projective spaces in $V^{(d)}$ passing rationally through Γ. A lower bound may be obtained from the Riemann-Roch formula. Namely, we may consider Γ as the *base locus* of the pencil $|C + t C'|$ in $|L^\mu|$. The set of all such pencils is the Grassmannian of lines in the projective space $|L^\mu|$, and has dimension $\delta = 2 \dim |L^\mu| - 2$, which, by (2.21), is equal to

$$\delta = \mu^2 L \cdot L - \mu L \cdot K + 2 p_g - 2q - 2. \tag{2.23}$$

Comparing (2.22) and (2.23) we see that

$$d \sim \delta$$

as $\mu \to \infty$, so that Mumford's estimate is best possible in this asymptotic sense.

From the point of view of the Riemann-Roch and duality theorems the simplest surfaces are those for which the canonical bundle $K = 0$. If $p_g(V) \neq 0$, then such a V is either a $K3$ or abelian surface. For these we shall now show that Mumford's estimate is sharp.

In the $K3$ case, $p_g = 1$ and $q = 0$ so that $d = \delta$ for any positive line bundle.

The abelian surface is slightly more intricate. The first step is to derive a linear algebra lemma relevant to the situation. Suppose then that F is a complex vector space on which we have a nondegenerate 2-form Ω. We may choose linear coordinates $(z_1, \ldots, z_n; w_1, \ldots, w_n)$ such that

$$\Omega = \sum_{i=1}^{n} dz_i \wedge dw_i$$

where $\dim F = 2n$. A quadratic form Q and linear automorphism $J: F \to F$ are defined by having respective matrices

$$Q = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \qquad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Then, for all vectors $e, f \in F$

$$\Omega(e, f) = Q(e, Jf). \tag{2.24}$$

Using the isomorphism of F with its dual F^* given by Q we may think of J as acting on F^*.

Second Linear Algebra Lemma. *Suppose that we are given a subspace $E \subset F$ and linear form $\lambda \in F^*$ such that the conditions*

$$\begin{aligned} &\langle \lambda, E \rangle = 0 \\ &\langle J\lambda, E \rangle = 0 \\ &\Omega^{n-1} \wedge \lambda \wedge J\lambda \neq 0 \qquad (n = \dim F) \end{aligned} \tag{2.25}$$

are satisfied. Then $\dim E \leqq n - 1$.

Proof (due to Joe Harris). On the subspace $F' \subset F$ defined by the equations

$$\langle \lambda, f \rangle = \langle J\lambda, f \rangle = 0$$

the form Ω is nondegenerate by the third condition in (2.25). Since $E \subset F'$ and dim $F' = 2n-2$ the lemma follows from the previous linear algebra lemma. Q.E.D.

Now suppose $V = \mathbb{C}^2/\Lambda$ is an abelian surface where $\mathbb{C}^2$ has linear coordinates (z, w). The differentials

$$\omega = \sum_{\mu} dz_{\mu},$$

$$\omega' = \sum_{\nu} dw_{\nu},$$

$$\Omega = \sum_{\mu} dz_{\mu} \wedge dw_{\mu}$$

generate $\Omega^*(V^{(n)})$ — here $(z_1, w_1, \ldots, z_n, w_n)$ are coordinates on the cartesian product V^n. Then $\omega' = J\omega$ and

$$\Omega^{n-1} \wedge \omega \wedge \omega' = \left(\sum_{\alpha=1}^{n} dz_1 \wedge dw_1 \wedge \cdots \wedge \widehat{dz_{\alpha} \wedge dw_{\alpha}} \wedge \cdots \wedge dz_n \wedge dw_n \right) \wedge \left(\sum_{\beta, \gamma} dz_{\alpha} \wedge dw_{\gamma}\right)$$

$$\neq 0.$$

From the second linear algebra lemma we deduce:

If there is a $\mathbb{P}^m$ sitting rationally through a point $\Gamma = P_1 + \cdots + P_n \in V^{(n)}$ where $P_{\mu} \neq P_{\nu}$ for $\mu \neq \nu$, then $m \leqq n-1$.

We now prove that this estimate is sharp. Let $L \to V$ be a positive line bundle and $C, C' \in |L|$ curves meeting at $d = L \cdot L$ distinct points. According to (2.23) if we vary the pencil $|C + tC'|$ in $|L|$ we obtain a $\mathbb{P}^{d-2}$ passing through $\Gamma = C \cdot C'$. Now, to account for the remaining parameter we set $E_0 = L \oplus L$ and let $\sigma_0 \in H^0(E)$ correspond to Γ. Since

$$\dim H^1(Hom(L, L)) = \dim H^1(\mathcal{O}) = 2$$

we may obtain a $\mathbb{P}^1$ of inequivalent extensions

$$0 \to L \to E_{\eta} \to L \to 0.$$

The section σ_0 may be perturbed to $\sigma_{\eta} \in H^0(E_{\eta})$ due to $H^1(E_0) = H^1(L \oplus L) = 0$. In this way we find a $\mathbb{P}^{d-1}$ passing rationally through Γ. In summary we have proved:

(2.26) *If V is either a $K3$ or abelian surface, $L \to V$ a positive line bundle and $C, C' \in |L|$ curves meeting at $d = L \cdot L$ distinct points, then there is a $\mathbb{P}^r$ constituting an integral variety of $\Omega^*(V^{(d)})$ of maximal dimension passing through $\Gamma = C \cdot C'$.*

In these cases the Abel condition (2.17) is invertible.

We also note whereas in codimension 1 — say, points on a curve — the integral varieties of the differential system $\Omega^*(V^{(d)})$ are linear and therefore may be expected to be closed, the situation in codimension $\geqq 2$ is basically nonlinear and the general integral variety will not be closed.

In fact, suppose that we return to the $K3$ case and look at zero-cycles $\Gamma=\sum_{\nu=1}^{d} P_\nu \in V^{(d)}$ of degree d near a point Γ_0 where all P_ν are distinct. We may choose coordinates $(z_1, w_1, \ldots, z_d, w_d)$ around Γ_0 such that the 2-form Ω on $V^{(d)}$ is

$$\Omega=\sum_{\nu=1}^{d} dz_\nu \wedge dw_\nu .$$

The n-planes A on which Ω is zero have the following character: A general such A will have a basis

$$e_\nu=\frac{\partial}{\partial z_\nu}+\sum A_{\nu\mu}\frac{\partial}{\partial w_\mu}$$

and the condition $\langle \Omega, A\rangle=0$ is equivalent to

$$A_{\nu\mu}=A_{\mu\nu}.$$

Thus the local integral varieties of Ω having maximal dimension n and passing through Γ_0 form an $\frac{n(n+1)}{2}$-dimensional subvariety of the Grassmannian $G(n, 2n)$, whose dimension is n^2. What (2.20) implies is that if Γ_0 is a complete intersection, then exactly one of these local integral varieties will consist of complete intersections.

On the basis of this together with Bloch's results (footnote 1) and the discussion at the end of Section IIIa below, we might guess the following: If V is a regular surface with $p_g(V)\neq 0$ and Γ_0 is a 0-cycle of sufficiently large degree n, then the only way in which a nontrivial rational variation of Γ_0 can occur is by having $\Gamma_0=\Gamma'+\Gamma''$ where Γ' is a complete intersection and the variation is by moving Γ' as a complete intersection. Moreover, the maximal integral varieties of $\Omega^*(V^{(m)})$ passing through a complete intersection consist entirely of complete intersections[10].

(e) Abel's Theorem for Linear Spaces and Statement of a Converse

We consider a projective variety $V_n \subset \mathbb{P}^{n+r}$. V may be singular or reducible, but we do assume that every component of V has the same dimension n and multiplicity 1. As family of zero cycles we take the intersections

$$\Gamma_A = A\cdot V$$

of V with linear r-planes $A\in \mathbb{G}(r, n+r)$. A general A meets V in a finite number $d=$*degree* of V distinct points, so that the *incidence correspondence*

$$I\subset \mathbb{G}(r, n+r)\times V$$

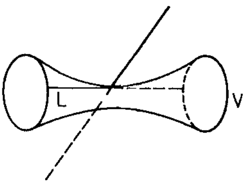
defined by

$$I=\{(A, P)\colon P\in A\}$$

10 See Notes Added in Proof

maps in a finite surjective manner onto the Grassmannian. The following example illustrates some of the possibilities for the special fibers of π: $I \to \mathbb{G}(r, n+r)$:

Suppose that $V \subset \mathbb{P}^3$ is a smooth quadric surface.



A general line $L \in \mathbb{G}(1,3)$ meets V in two distinct points; we write $L \cdot V = P + Q$ where $P \neq Q$. On a hypersurface $V^* \subset \mathbb{G}(1,3)$ the line L is tangent to V, so that $L \cdot V = 2P$. Finally, there is (reducible) curve C of lines L lying in the quadric surface; for these lines the fiber $\pi^{-1}(L) = L$ is positive-dimensional. Summarizing:

for $L \in \mathbb{G}(1,3) - V^*$, $\pi^{-1}(L) = P + Q$, $P \neq Q$
for $L \in V^* - C$, $\pi^{-1}(L) = 2P$
for $L \in C$, $\pi^{-1}(L) = L$.

We next discuss the conditions that a rational n-form ψ on V should be of the first kind with respect to this family $\{\Gamma_A\}_{A \in \mathbb{G}(r, n+r)}$ of zero cycles.

In case V is nonsingular, ψ is of the first kind $\Leftrightarrow \psi \in \Omega^n(V)$ is holomorphic on V.

Proof. Suppose that ψ is meromorphic with polar divisor D, and choose a linear space $\mathbf{A}_\cdot$ such that

$$\mathbf{A}_\cdot \cdot V = \mathbf{P} + \mathbf{Q}_1 + \cdots + \mathbf{Q}_{d-1}$$

where $\mathbf{P} \in D$ and the $\mathbf{Q}_i \in V - D$. (*Note:* If V were singular and $D \subset V_{\text{sing}}$, then this is not possible.) For A close to $\mathbf{A}_\cdot$, $A \cdot V = P + Q_1 + \cdots + Q_{d-1}$ where P is close to $\mathbf{P}$ and Q_i is close to $\mathbf{Q}_i$. Since the trace

$$\mathrm{Tr}_I(\psi)(A) = \psi(P) + \psi(Q_1) + \cdots + \psi(Q_{d-1}),$$

it is clear that $\mathrm{Tr}_I(\psi)$ has a singularity at the point $\mathbf{A} \in \mathbb{G}(r, n+r)$. Q.E.D.

Now suppose that $V_n \subset \mathbb{P}^{n+1}$ is a hypersurface having equation

$$f(x, y) = 0$$

in affine coordinates $(x, y) = (x_1, \dots, x_n, y)$. We write

$$\psi = \frac{p(x, y)\, dx_1 \wedge \cdots \wedge dx_n}{\dfrac{\partial f}{dy}(x, y)}$$

and will prove that:

(2.27) *ψ is of the first kind with respect to the family of lines in $\mathbb{P}^{n+1} \Leftrightarrow p(x, y)$ is a polynomial of degree at most $d - n - 2$.*

Proof. A Zariski open set in $\mathbb{G}(1, n+1)$ consists of lines $L(a, b)$ having equations

$$x_1 = a_1\, y + b_1 \qquad \vdots \qquad x_n = a_n\, y + b_n \tag{2.28}$$

where $(a, b) = (a_1, \dots, a_n; b_1, \dots, b_n)$ give local coordinates on the Grassmannian. We may assume that for $|a|, |b| < \varepsilon$ the intersection

$$L(a, b) \cdot V = P_1(a, b) + \cdots + P_d(a, b)$$

where the points $P_\nu(a, b) = (x_\nu(a, b), y_\nu(a, b))$ are finite, vary holomorphically with (a, b), and have distinct y-coordinates $y_\nu(a, b)$. The incidence correspondence is defined by

$$x_i = a_i\, y + b_i \quad (i = 1, \dots, n) \qquad f(x, y) = 0, \tag{2.29}$$

and for any function $q(x, y)$ we denote by

$$Q(y) = q(a_1\, y + b_1, \dots, a_n\, y + b_n, y)$$

the restriction of Q to I. We will now prove the beautiful formula

$$\boxed{(2.30) \quad \operatorname{Tr}_I(\psi) = \sum_A \pm \left\{ \sum_\nu \frac{y_\nu^{|A|}\, P(y_\nu)}{F'(y_\nu)} \right\} da_A \wedge db_{A^c}}$$

where A runs over index subsets $A = (i_1, \dots, i_k)$ of $(1, \dots, n)$, A^c is the complementary index set, and $|A| = k$ is the number of elements in A.

Assuming (2.30) for a moment, we may first deduce that the rational function $p(x, y)$ is necessarily a polynomial, and then by the *Lagrange interpolation formula* (1.10),

$$\sum_\nu \frac{y_\nu^{|A|}\, P(y_\nu)}{F'(y)} = \text{constant term in} \left\{ \frac{y^{|A|+1}\, P(y)}{F(y)} \right\} \tag{2.31}$$

which is zero $\Leftrightarrow \deg P + n + 2 \leqq \deg F$, thereby proving the assertion (2.27).

Now to the proof of (2.30). A convenient remark is that, for any index i,

$$\frac{dx_1 \wedge \cdots \wedge dx_n}{\dfrac{\partial f}{\partial y}(x, y)} = (-1)^{n-i+1} \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \wedge dy}{\dfrac{\partial f}{\partial x_i}(x, y)} \tag{2.33}$$

on the hypersurface V. This is because on V

$$0 = df = \sum_i \frac{\partial f}{\partial x_i}\, dx_i + \frac{\partial f}{\partial y}\, dy,$$

so that

$$\frac{dx_i}{\partial f / \partial y} \equiv -\frac{dy}{\partial f / \partial x_i} \text{ modulo } (dx_1, \cdots, \widehat{dx_i}, \dots, dx_n).$$

Differentiating the equations (2.29) of the incidence correspondence we obtain

$$(2.33)\quad \begin{aligned} &dx_i = a_i\,dy + y\,da_i + db_i \\ &\sum_j \frac{\partial f}{\partial x_j}\left(a_j \frac{\partial y}{\partial a_i} + \delta_{ij}\, y\right) + \frac{\partial f}{\partial y}\frac{\partial y}{\partial a_i} = 0 \\ &\sum_j \frac{\partial f}{\partial x_j}\left(a_j \frac{\partial y}{\partial b_i} + \delta_{ij}\right) + \frac{\partial f}{\partial y}\frac{\partial y}{\partial b_i} = 0. \end{aligned}$$

In these expressions we are considering just one of the points

$$P(a, b) = (x(a, b), y(a, b))$$

of intersection of the line $L(a, b)$ with V and differentiating x and y as functions of a and b. These equations may be rewritten as

$$(2.34)\quad \begin{aligned} &dx_i \equiv y\,da_i + db_i \qquad \textit{modulo } dy \\ &\frac{\partial y}{\partial a_i} = -\frac{y(\partial f/\partial x_i)}{F'(y)} \\ &\frac{\partial y}{\partial b_i} = -\frac{\partial f/\partial x_i}{F'(y)} \end{aligned}$$

where $F(y) = f(a_1 y + b_1, \ldots, a_n y + b_n, y)$ and $F'(y) = \sum_j a_j \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial y}$ is the y-derivative of $F(y)$. From the last two equations in (2.34)

$$dy = -\frac{1}{F'(y)}\left\{\sum_i \frac{\partial f}{\partial x_i}(y\,da_i + db_i)\right\}$$

so that

$$\begin{aligned} &dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dy \\ &= -\frac{1}{F'(y)}\left\{(y\,da_1 + db_1) \wedge \cdots \wedge (y\,da_{n-1} + db_{n-1}) \wedge \left(\sum_i \frac{\partial f}{\partial x_i}(y\,da_i + db_i)\right)\right\} \\ &= \frac{-\partial f/\partial x_n}{F'(y)}\{(y\,da_1 + db_1) \wedge \cdots \wedge (y\,da_n + db_n)\} \\ &= \frac{-\partial f/\partial x_n}{F'(y)}\{\sum_A \pm y^{|A|}\, da_A \wedge db_{A^c}\}. \end{aligned}$$

Combining this with (2.32) in the case $i = n$ yields the formula (2.30). Q.E.D.

It may be noted that the above computation is *local* around the points of intersection of the line $L(a, b)$ with V.

The principal new result of this paper is a converse to Abel's theorem for differentials of the first kind relative to the family of r-planes in $\mathbb{P}^{n+r}$.

Main Theorem. *Let A_0 be a fixed r-plane in $\mathbb{P}^{n+r}$. Suppose we are given distinct points $P_\nu \in A_0$ and in a neighborhood U_ν of P_ν a piece of n-dimensional analytic variety V_ν meeting A_0 transversely at P_ν. Suppose, moreover, that on each V_ν we are given a holomorphic n-form $\psi_\nu \not\equiv 0$. For r-planes A in a small neighborhood U of A_0 in $\mathbb{G}(r, n+r)$ we form the trace*

$$\mathrm{Tr}_I(\psi)(A) = \sum_\nu \psi_\nu(A \cdot V_\nu),$$

and assume that $\mathrm{Tr}_I(\psi)\equiv 0$. *Then there is an n-dimensional algebraic variety* $V\subset\mathbb{P}^{n+r}$ *and rational n-form* ψ *on* V *such that* ψ *is of the first kind with respect to the family of r-planes in* $\mathbb{P}^{n+r}$, *and such that each*

$$V_\nu\subset V,$$
$$\psi|V_\nu=\psi_\nu.$$

III. Residues and Abel's Theorem

(a) Local Properties of Residues; the Residue Theorem

We shall first discuss local properties of point residues. Complete proofs of the statements we shall make are scattered throughout the literature and are collected in Chapter V of the forthcoming book, *Analytic algebraic geometry*, by Joe Harris and the author to be published by Wiley.

Let U a connected open neighborhood of the origin in $\mathbb{C}^n$, e.g., the ball

$$\{z\in\mathbb{C}^n\colon \|z\|<\varepsilon\},\quad \text{and}\quad f_1,\dots,f_n\in\mathcal{O}(\bar U)$$

holomorphic functions defined in a neighborhood of the closure of U. We assume that the equations

$$f_1(z)=\cdots=f_n(z)=0$$

have the origin as isolated common zero. Equivalently, if $D_i=(f_i)$ is the divisor of f_i, then the set-theoretic intersection

$$D_1\cap\cdots\cap D_n=\{0\}.$$

We use the notations $U_i=U-D_i$, $D=D_1+\cdots+D_n$, and $U^*=U-\{0\}=\bigcup_{i=1}^{n}U_i$.

In $\bar U$ we consider meromorphic n-forms having polar divisor D. Any such ω may be written as

$$\omega=\frac{g(z)\,dz_1\wedge\cdots\wedge dz_n}{f_1(z)\dots f_n(z)},\qquad g\in\mathcal{O}(\bar U),$$

and although ω has polar set D it is pretty clear that the origin is, in some sense, the point of most intense singularity. We define the corresponding *residue* (or *point residue*)

$$\text{(3.1)}\quad \mathrm{Res}_{\{0\}}\,\omega=\left(\frac{1}{2\pi i}\right)^n\int_\Gamma\omega$$

where the path of integration is the real n-cycle

$$\Gamma=\{|f_1|=\cdots=|f_n|=\delta\}.^{11}$$

11 This residue is clearly a variant of the Cauchy integral in several variables. It may be defined purely algebraically – this essentially amounts to the *Grothendieck residue symbol* – and perhaps its most interesting property is the special case of Grothendieck's duality theorem stated below as property (ix) of the residue.

The earliest reference I can find is F. Severi, Funzoni analitiche e forme differenziali, Atti del Quarto Congresso dell'Unione Matematica Italiana, Vo I, ppg. 125–140, Casa Perrella Roma (1953)

As coordinate functions on an open dense set of Γ we may take the arguments $\arg f_i$, and then Γ is oriented by

$$d(\arg f_1) \wedge \cdots \wedge d(\arg f_n) > 0.$$

Here are some properties of the residue.

(i) $\mathrm{Res}_{\{0\}}\, \omega$ depends in an *alternating* fashion on the f_i and linearly on g. Moreover, the residue is zero in case g is in the ideal I_f generated by the f_i's.

Proof. If, say, $g = f_1$, we let $\Delta = \{|f_1| \leqq \delta, |f_2| = \cdots = |f_n| = \delta\}$. Then $\omega = \dfrac{dz_1 \wedge \cdots \wedge dz_n}{f_2(z) \ldots f_n(z)}$ is holomorphic in a neighborhood of the $(n+1)$-chain Δ, and by Stokes' theorem

$$0 = \int_\Delta d\omega = \int_{\partial\Delta} \omega = \int_\Gamma \omega. \quad \text{Q.E.D.}$$

(ii) Similarly, by Stokes' theorem, the residue depends only on the homology class of Γ in $H_n(U_1 \cap \cdots \cap U_n, \mathbb{Z})$ and de Rham cohomology class of ω in

$$H^n_{DR}(U_1 \cap \cdots \cap U_n).$$[12]

(iii) If we consider $\{U_i\}$ as a covering $\mathbf{U}$ of U^*, then $\omega \in \Omega^n(U_1 \cap \cdots \cap U_n)$ defines a Čech cochain in $C^{n-1}(\mathbf{U}, \Omega^n)$ and hence a cohomology class

$$[\omega] \in H^{n-1}(U^*, \Omega^n).$$

Under the *Dolbeault isomorphism*

$$H^{n-1}(U^*, \Omega^n) \simeq H^{n,n-1}_{\bar\partial}(U^*)$$

$[\omega]$ is represented by a C^∞ $\bar\partial$-closed form η_ω of type $(n, n-1)$ defined in U^*. Since $d = \bar\partial$ on forms of type (n, q), the linear functional

$$\eta \to \int_{\|z\| = \varepsilon} \eta$$

is well defined on $H^{n,n-1}_{\bar\partial}(U^*)$, and

$$\text{(3.2)} \quad \mathrm{Res}_{\{0\}}\, \omega = \int_{\|z\| = \varepsilon} \eta_\omega.$$

The explicit formula for η_ω is

$$\text{(3.3)} \quad \eta_\omega = \frac{C_n \sum_i (-1)^{i-1}\, g(z)\, \overline{df_1} \wedge \cdots \wedge \widehat{\overline{df_i}} \wedge \cdots \wedge \overline{df_n} \wedge dz_1 \wedge \cdots \wedge dz_n}{\|f(z)\|^{2n}}$$

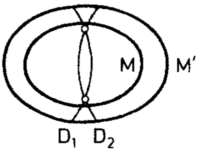
where $f(z) = (f_1(z), \ldots, f_n(z))$ and C_n is a constant depending only on n.

Passing to Dolbeault cohomology makes precise the sense in which ω has the origin as point singularity, and converts the original n-dimensional path of integration into one of real dimension $2n-1$. This, in turn, makes easy the proof of the residue theorem, which we discuss next.

(iv) Let M be a connected complex manifold embedded as a relatively compact open set with smooth boundary in a complex manifold M'. The case where $M = M'$

12 $H^*_{DR}(M)$ denotes the *de Rham cohomology* {closed forms}/{exact forms} on a manifold M. Similarly, $H^*_{\bar\partial}(M)$ denotes the *Dolbeault cohomology* {$\bar\partial$-closed forms}/{$\bar\partial$-exact forms} on a complex manifold M

is compact is allowed. Suppose that $D_1, \ldots, D_n$ ($n = \dim M$) are divisors on M' meeting at a finite number of points interior to M.



Set $D = D_1 + \cdots + D_n$ and suppose that $\omega \in \Omega^n(M', D)$ is a meromorphic n-form on M' having polar divisor D.[13] This defines a class

$$[\omega] \in H^{n-1}(\mathbf{U}, \Omega^n)$$

where $\mathbf{U} = \{U_1, \ldots, U_n\}$ is the covering of $M^* = M - (D_1 \cap \cdots \cap D_n)$ by the open sets $U_i = M - M \cap D_i$. If η_ω is any $\bar\partial$-closed form of type $(n, n-1)$ on M^* representing $[\omega]$ under the Dolbeault isomorphism $H^{n-1}(M^*, \Omega^n) \simeq H_{\bar\partial}^{n,n-1}(M^*)$, then we have the

Residue Theorem. *If* $D_1 \cap \cdots \cap D_n = \sum_\nu P_\nu$, *then*

$$\boxed{(3.4)\quad \sum_{P_\nu} \operatorname{Res}_{P_\nu} \omega = \int_{\partial M} \eta_\omega.}$$

Proof. Let $B_\nu(\varepsilon)$ be a ball of small radius ε around P_ν. Then, by (iii),

$$\operatorname{Res}_{P_\nu} \omega = \int_{\partial B_\nu(\varepsilon)} \eta_\omega.$$

Applying Stokes' theorem to $M - \bigcup_\nu B_\nu(\varepsilon)$ gives

$$\begin{aligned} \int_{\partial M} \eta_\omega &= \sum_\nu \int_{\partial B_\nu(\varepsilon)} \eta_\omega \quad (\text{since } d\eta_\omega = 0) \\ &= \sum_\nu \operatorname{Res}_{P_\nu} \omega \end{aligned}$$

by what we just said. Q.E.D.

(v) In the *nondegenerate case* where the Jacobian $J_f(0) = \det\left(\frac{\partial f_i}{\partial z_j}(0)\right) \neq 0$,

$$(3.5)\quad \operatorname{Res}_{\{0\}} \omega = \frac{g(0)}{J_f(0)}.$$

This follows easily by applying the change of variables $w = f(z)$ to convert the residue integral into a standard Cauchy integral.

We observe that (3.2), (3.3), and (3.5) combine to yield the *Bochner-Martinelli formula.*

We also remark that $J_f(0) \neq 0$ is equivalent to saying that the D_i are nonsingular and meet transversely at the origin.

13 We are using the notation $\Omega^q(M, D) = H^0(M, \Omega^q([D]))$ for the meromorphic forms having polar divisor D

(vi) One of the most important applications of the residue theorem is to the *method of continuity:* If we are given functions $f_i(z, t)$, $g(z, t)$ which are holomorphic in $z \in \bar{U}$, continuous in t for $|t| < \varepsilon$, and if the $f_i(z, 0)$ have finitely many isolated zeros in the interior of $\bar{U}$ – which is equivalent to $\sum_i |f_i(z, 0)|^2 \geqq C > 0$ on the boundary of U – then the same is true of the $f_i(z, t)$ for small t. Setting $D_i(t) =$ divisor of $f_i(z, t)$, then the total number of points (counting multiplicities) in $D_1(t) \cap \cdots \cap D_n(t)$ is constant (c.f. (vii) below). If we write set-theoretically

$$D_1(t) \cap \cdots \cap D_n(t) = \sum_\nu P_\nu(t)$$

and set

$$\omega(t) = \frac{g(z, t)\, dz_1 \wedge \cdots \wedge dz_n}{f_1(z, t) \ldots f_n(z, t)},$$

then the method of continuity states:

The sum of the residues

$$\sum_\nu \operatorname{Res}_{P_\nu(t)} \omega(t)$$

is a continuous function of t.

Proof. Set $U^*(t) = U - (D_1(t) \cap \cdots \cap D_n(t))$ and consider $\omega(t)$ as a class in

$$H^{n-1}(U^*(t), \Omega^n) \simeq H_{\bar{\partial}}^{n, n-1}(U^*(t)).$$

If $\eta_\omega(t)$ is the Dolbeault representative (3.3) of $\omega(t)$, then $\eta_\omega(t)$ depends continuously on t, and by the residue theorem the same is true of

$$\sum_\nu \operatorname{Res}_{P_\nu(t)} \omega(t) = \int_{\partial U} \eta_\omega(t). \quad \text{Q.E.D.}$$

In particular, given $f_1, \ldots, f_n \in \mathcal{O}(\bar{U})$ with the origin as isolated common zero, we may find continuous perturbations $f_i(z, t)$ of $f_i(z)$ such that the divisors $D_i(t)$ are smooth and meet transversely for $t \neq 0$. Then the residue

$$\operatorname{Res}_{\{0\}} \omega = \lim_{t \to 0} \{\sum_\nu \operatorname{Res}_{P_\nu(t)} \omega(t)\}$$

is the limit of nondegenerate residues each of which may be "evaluated" by (3.5).

(vii) As an application of the method of continuity, we define the *intersection number*

$$(D_1, \ldots, D_n)_{\{0\}} = \operatorname{Res}_{\{0\}} k_f$$

where

$$k_f = \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n}$$

is the pull-back of the Cauchy kernel $\frac{dw_1}{w_1} \wedge \cdots \wedge \frac{dw_n}{w_n}$. It is easy to see that the intersection number is a positive integer. If we use the notation

$$(D_1', \ldots, D_n')_U = \sum_\nu (D_1', \ldots, D_n')_{P_\nu'}$$

for the total intersection number of divisors D_i' meeting at isolated points $P_\nu' \in U$, then since a continuous integer-valued function is constant, the intersection number

$$(D_1, \ldots, D_n)_{\{0\}} = (D_1(t), \ldots, D_n(t))_U$$

is simply the number of points in which generic perturbations of the D_i's will intersect.

(viii) Another important application of the continuity method is to the *transformation formula:* Suppose $\{f_i\}$ and $\{f_i'\}$ are two collections of n holomorphic functions each having the origin as isolated common zero. Suppose, moreover, that for some holomorphic matrix $(a_{ij}(z))$

$$f_i' = \sum_j a_{ij} f_j.$$

Then the transformation formula is

$$\text{(3.6)} \quad \operatorname{Res}_{\{0\}} \left(\frac{g\, dz_1 \wedge \cdots \wedge dz_n}{f_1, \ldots, f_n} \right) = \operatorname{Res}_{\{0\}} \left(\frac{g \det a_{ij}\, dz_1 \wedge \cdots \wedge dz_n}{f_1', \ldots, f_n'} \right).$$

This is proved first in the nondegenerate case by the explicit formula (3.5), and then in general by making generic perturbations and using (3.9).

(ix) The transformation formula, in turn, may be used to prove the *local duality theorem of Grothendieck:*[14]

Let $I_f = (f_1, \ldots, f_n)$ be the ideal generated by the f_i's and consider the pairing

$$\operatorname{Res}_f(\ ,\ ) : \mathcal{O}/I_f \times \mathcal{O}/I_f \to \mathbb{C}$$

given by

$$\operatorname{Res}_f(g, h) = \operatorname{Res}_{\{0\}} \left(\frac{g \cdot h\, dz_1 \wedge \cdots \wedge dz_n}{f_1, \ldots, f_n} \right)$$

The local duality theorem asserts that: *This pairing is nondegenerate.*

This may be verified directly when $f_i = z_i^{k_i}$, and then demonstrated in general using this case together with the transformation formula (3.6).

We now discuss some global applications of residues. A special case of (3.4) is the

Residue Theorem for Compact Manifolds. *Suppose that M_n is a compact, complex manifold and ω is a meromorphic n-form on M having polar divisor $D = D_1 + \cdots + D_n$ where the intersection $D_1 \cap \cdots \cap D_n = \sum_\nu P_\nu$. Then*

$$\text{(3.7)} \quad \sum_\nu \operatorname{Res}_{P_\nu} \omega = 0.$$

Here is an alternate proof of (3.7) without appealing to $\bar{\partial}$-cohomology. Let s_i be a global holomorphic section of the line bundle $[D_i] \to M$ whose divisor $(s_i) = D_i$. Choosing a metric in these line bundles, the real n-cycle

$$\Gamma_\varepsilon = \{|s_1| = \cdots = |s_n| = \varepsilon\}$$

14 C.f. R. Hartshorne, Residues and duality, Springer Lecture Notes No. 20 (1966)

will, for ε sufficiently small, decompose into a sum of n-cycles Γ_ν around the points P_ν with $\mathrm{Res}_{P_\nu}\,\omega = \int\limits_{\Gamma_\nu} \omega$. If now $\Delta = \{|s_1| \geqq \varepsilon, |s_2| = \cdots = |s_n| = \varepsilon\}$ then ω is holomorphic in a neighborhood of Δ and $d\omega = 0$ there. Consequently, by Stokes' theorem

$$\begin{aligned}\sum_\nu \mathrm{Res}_{P_\nu}\,\omega &= \int\limits_{\Gamma_\varepsilon} \omega \\ &= \int\limits_{\partial\Delta} \omega \\ &= 0,\end{aligned}$$

thereby re-proving (3.7).

Now we shall prove the following:

(3.8) **Converse to the Residue Theorem.** *Suppose that M and $D = D_1 + \cdots + D_n$ are as above, and assume that the D_i are positive in the sense of Kodaira. Given complex numbers c_ν, we may find a meromorphic n-form ω on M having polar divisor D and with* $\mathrm{Res}_{P_\nu}\,\omega = c_\nu$ *if, and only if,*

$$\sum_\nu c_\nu = 0.$$

Proof. We shall use the formalism of the *relative cohomology* $H^*(X, U; \mathscr{F})$ of a sheaf $\mathscr{F}$ on a space X modulo an open subset U of X. These groups may be defined, either by injective resolutions or by a Čech procedure, and for our purposes the two salient properties are the *exact cohomology sequence*

$$H^q(X, \mathscr{F}) \xrightarrow{\rho} H^q(U, \mathscr{F}) \to H^{q+1}(X, U; \mathscr{F}) \to H^{q+1}(X, \mathscr{F})$$

where ρ is the usual restriction mapping, and *excision*

$$H^*(X, U; \mathscr{F}) \simeq H^*(X - V, U - V, \mathscr{F})$$

where $V \subset U$ is a relatively compact open set.

We first apply the relative cohomology where $X = M$, $U = M^* = M - \bigcap\limits_\nu D_\nu$, and $\mathscr{F} = \Omega^n$. Using $H^n(M^*, \Omega^n) = 0$ and the Kodaira-Serre duality $H^n(M, \Omega^n) \simeq \mathbb{C}$, we obtain

$$(3.9) \quad H^{n-1}(M^*, \Omega^n) \to H^n(M, M^*; \Omega^n) \to \mathbb{C} \to 0$$

from the general exact cohomology sequence.

Next, we let U_ν be a small ball around P_ν and apply the exact cohomology sequence when $X = U_\nu$, $U = U_\nu^* = U_\nu - \{P_\nu\}$, and $\mathscr{F} = \Omega^n$. Using $H^q(U_\nu, \Omega^n) = 0$ for $q > 0$ we obtain

$$(3.10) \quad H^{n-1}(U_\nu^*, \Omega^n) \to H^n(U_\nu, U_\nu^*; \Omega^n) \to 0;$$

this is an isomorphism for $n \geqq 2$.

Finally, we combine (3.9) and (3.10) by applying excision to the case $X = M$, $U = M^*$, $V = \bigcup\limits_\nu U_\nu$; we obtain the exact sequence

$$(3.11) \quad H^{n-1}(M^*, \Omega^n) \to \bigoplus_\nu H^{n-1}(U_\nu^*, \Omega^n) \xrightarrow{R} \mathbb{C} \to 0.$$

Actually, we have only proved (3.11) under the assumption $n \geqq 2$ so that (3.10) is an isomorphism; the modifications necessary to treat the $n=1$ case are easy to fill in.

The mapping R may be identified as

$$(3.12)\quad R\big(\bigoplus_{\nu} \omega_\nu\big)=\sum_{\nu} \operatorname{Res}_{P_\nu} \omega_\nu$$

where the local residue map is defined by

$$H^{n-1}(U_\nu^*, \Omega^n) \simeq H_{\bar\partial}^{n, n-1}(U_\nu^*) \xrightarrow{R_\nu} \mathbb{C}$$

with the first map being the Dolbeault isomorphism and the second the integration map,

$$R_\nu(\eta)=\int_{\partial U_\nu} \eta.$$

A meromorphic form ω on M with polar divisor D defines a class $[\omega] \in H^{n-1}(M^*, \Omega^n)$, and from (3.11) and (3.12) we have still another proof of the residue theorem (3.7).

Note that the same is true if we only assume ω is holomorphic in $M-D$, having perhaps even an essential singularity on D.

Conversely, suppose we are given complex numbers c_ν. By the local duality theorem (ix) discussed above, we may find a meromorphic form ω_ν in U_ν having polar variety $D \cap U_\nu$ and with $\operatorname{Res}_{P_\nu} \omega_\nu = c_\nu$. We consider ω_ν as defining a class $[\omega_\nu] \in H^{n-1}(U_\nu^*; \Omega^n)$. Referring to (3.11), there is a global class $\kappa \in H^{n-1}(M^*, \Omega^n)$ with $\kappa | U_\nu = [\omega_\nu]$ if, and only if, the residue relation $\sum_\nu c_\nu = 0$ is satisfied.

Assume this to be the case, and consider the open covering $\mathbf{U}=\{U_i = M - D_i\}$ of M^*. If the *acyclicity condition*

$$(3.13)\quad H^q(U_{i_0} \cap \cdots \cap U_{i_p}, \Omega^n)=0, \qquad q>0,$$

is verified, then by Leray's theorem the cohomology

$$H^*(M^*, \Omega^n) \simeq H^*(\mathbf{U}, \Omega^n)$$

is computible from the Čech complex for this covering. In this case we may find a holomorphic form ω on $M-D$, perhaps having essential singularities along D, but whose class $[\omega]=\kappa$ in $H^{n-1}(M^*, \Omega^n)$.

Now by assumption the D_i are positive, and this gives us something much stronger than (3.13). Namely, by the Kodaira vanishing theorem

$$(3.14)\quad H^q(M, \Omega^n(D_{i_0}+\cdots+D_{i_p}))=0, \qquad q>0,$$

where $\Omega^n(*)$ is the sheaf on M of meromorphic forms having poles on a divisor $*$. We may think of (3.14) as giving (3.13) but with only first-order poles allowed on $D_{i_0}+\cdots+D_{i_p}$. Then, if one thinks through how (3.13) is used in the proof of Leray's theorem, it follows by the very same argument that $H^*(M^*, \Omega^n)$ may be calculated from the Čech complex $\bigoplus_{(i_0, \ldots, i_p)} H^0(M, \Omega^n(D_{i_0}+\cdots+D_{i_p}))$. In particular, there is a surjective mapping

$$H^0(M, \Omega^n(D)) \to H^{n-1}(M^*, \Omega^n) \to 0,$$

and we may find ω having polar divisor D and cohomology class $[\omega]=\kappa$. Q.E.D.

We now discuss the residue theorem in some special cases – the case of $\mathbb{P}^n$ will be dealt with extensively in the next section.

In case M is a compact Riemann surface, what we have found are the so-called *elementary differentials of the third kind.*

Suppose next that M is an algebraic surface on which we have line bundles L and L'. Assume that we are given curves $C\in|L|$ and $C'\in|L'|$ meeting transversely, and denote by $|K+L+L'|$ the complete linear system of zero divisors of meromorphic 2-forms ω having polar curve $C+C'$. We will prove the

(3.15) **Proposition.** *If a curve $E\in|K+L+L'|$ passes through all but one point of $C\cdot C'$, then it contains this intersection entirely.*[15]

Proof. We suppose that C and C' are given by holomorphic sections $\sigma\in H^0(\mathcal{O}(L))$ and $\sigma'\in H^0(\mathcal{O}(L'))$, and E by $\tau\in H^0(\Omega^2(L\otimes L'))$. Then

$$\omega=\frac{\tau}{\sigma\cdot\sigma'}$$

is a meromorphic 2-form on M having polar divisor $C+C'$ and vanishing at all but one point of $C\cap C'$. From (3.5) we see that for $P\in C\cap C'$, $\mathrm{Res}_P\,\omega=c_P\,\tau(P)$ $(c_P\neq 0)$, and the proposition follows from $\sum_{P\in C\cap C'}\mathrm{Res}_P\,\omega=0$. Q.E.D.

In the second paper of this series we shall systematically discuss results of this sort and show how they may be used to characterize complete intersections. Here, a propros our discussion of the $K3$ surface in Section II d, we give the following illustration in this special case of how (3.15) may be utilized:

Suppose M is a $K3$ surface and $L\to M$ is a positive line bundle. Then the complete intersections

$$\Gamma=C\cdot C'\qquad (C,\,C'\in|L|)$$

depend on

$$\delta=L\cdot L$$

parameters – this was proved when we previously discussed $K3$ surfaces (c.f. (2.23)). The zero cycle Γ also has degree equal to δ, and hence depends on 2δ parameters. For a general $\Gamma\in M^{(\delta)}$ – not necessarily a complete intersection – we consider the condition:

($*$) *Every curve $E\in|2L|$ which passes through all but one point of Γ necessarily contains Γ.*

According to (3.15) a complete intersection has the property ($*$), and we shall now prove a converse. Precisely, we shall show:

($**$) *If a zero cycle Γ has the property ($*$), then Γ lies on a pencil $|C+t\,C'|$ of curves in $|L|$. If, e.g., one of these curves is irreducible, then $\Gamma=C\cdot C'$ is a complete intersection.*

15 This proposition may be viewed as the geometric form of the residue theorem (3.7)

Proof. Although not essential it will make the argument run more smoothly if we assume that the complete linear system $|L|$ induces a projective embedding

$$M \to \mathbb{P}^{\frac{\delta}{2}+1}$$

$\left(\dim|L| = \frac{L\cdot L}{2}+1 \text{ by (2.21)}\right)$. Thus the curves $C\in|L|$ are hyperplane sections for this embedding.

Now we make two simple observations from linear algebra. Suppose that $P_1, \ldots, P_N$ is a set of N distinct points in $\mathbb{P}^n$ where $N \geqq n$. Then the P_ν lie on a $\mathbb{P}^{n-2}$ if, and only if, any $n-1$ of them are linearly dependent. This is clear. Now suppose that $P_{\nu_1}, \ldots, P_{\nu_{n-1}}$ is any subset of $n-1$ points selected from all the $\{P_\nu\}$. Then these are linearly dependent if, and only if, any hyperplane H passing through all but one of them necessarily contains this last point as well. Putting these two statements together we deduce:

(***) *In order that $\Gamma = P_1 + \cdots + P_\delta$ $(\delta = L\cdot L)$ should lie on a pencil in $|L|$, it is sufficient that for any subset containing $\frac{\delta}{2}$ of these points, say, $P_1, \ldots, P_{\delta/2}$ for simplicity of notation, that any curve $C\in|L|$ which passes through all but one P_ν necessarily contains P_ν also—here $\nu = 1, \ldots, \frac{\delta}{2}$.*

We must prove that (*)⇒(***).

Since, by the Riemann-Roch (2.21), $\dim|L| = \frac{\delta}{2}+1$, we may find a curve $C'\in|L|$ passing through $P_{\delta/2+1}, \ldots, P_\delta$. Suppose first (as will generally be the case) that C' does not contain any of $P_1, \ldots, P_{\delta/2}$. Then for any $C\in|L|$ passing through $P_1, \ldots, \hat{P}_\nu, \ldots, P_{\delta/2}$, $C + C' \in |2L|$ passes through $\Gamma - \{P_\nu\}$, and hence contains Γ entirely according to (*). Since P_ν does not lie on C', it must lie on C and (***) is verified.

In the degenerate case we cannot directly use (***) and must argue as follows: Suppose, after renumbering, that C' contains $P_{\lambda+1}, \ldots, P_{\delta/2}$ but none of the points $P_1, \ldots, P_\lambda$. For $1 \leqq \nu \leqq \lambda$ choose $C\in|L|$ passing through $P_1, \ldots, \hat{P}_\nu, \ldots, P_\lambda$. Then, as before, $C + C'$ contains Γ, and so P_ν lies on C. But then $P_1, \ldots, P_\lambda$, and hence $P_1, \ldots, P_{\delta/2}$ are linearly dependent.

In summary we have shown that *any* $\frac{\delta}{2}$ points selected from $P_1, \ldots, P_\delta$ are linearly dependent, and hence these points must lie on a $\mathbb{P}^{\delta/2-1}$ in $\mathbb{P}^{\delta/2+1}$. Q.E.D.

It is perhaps interesting to compare (**) with the previous condition (2.26) in this example. Suppose that $M \subset \mathbb{P}^3$ is a nonsingular quartic surface and that L_0 is a general line meeting M in four distinct points P_1, P_2, P_3, P_4. We want to describe these points Q_i close to P_i such that Q_1, Q_2, Q_3, Q_4 lie on a line L.

Let ω be the holomorphic 2-form on M and choose holomorphic coordinates (z_i, w_i) near P_i such that $\omega = h_i(z_i, w_i)\, dz_i \wedge dw_i$. According to (2.26) the set of zero-cycles $\Gamma = Q_1 + Q_2 + Q_3 + Q_4$ of the form $L\cdot M$ constitute an integral variety of the differential system

$$(\#)\quad \sum_{i=1}^{4} h_i(Q_i)\, dz_i(Q_i) \wedge dw_i(Q_i) = 0.$$

There are many 4-dimensional integral varieties of this system passing through $\Gamma_0 = P_1 + P_2 + P_3 + P_4$, but only one of these describes the geometric property "four points lie on a line."

On the other hand, let $X = [X_0, X_1, X_2, X_3]$ be homogeneous coordinates for $\mathbb{P}^3$ and $F_1(X), \ldots, F_{10}(X)$ a basis for the homogeneous quadric polynomials. According to $(**)$ the conditions that points P_i close to Q_i lie on a line is

$$(\#\,\#)\quad rank \begin{pmatrix} F_1(Q_1), \ldots, F_{10}(Q_1) \\ F_1(Q_2), \ldots, F_{10}(Q_2) \\ F_1(Q_3), \ldots, F_{10}(Q_3) \\ F_1(Q_4), \ldots, F_{10}(Q_4) \end{pmatrix} \leqq 3.$$

It does not seem obvious how to directly relate $(\#)$ and $(\#\,\#)$; in particular, note that $(\#\,\#) \Rightarrow (\#)$?

In the next section we will show that the residue theorem (3.7) and classical Abel theorem from Section Ia are closely related. On the other hand, we may view (3.15) as a geometric form of the residue theorem. Consequently, proving the converse $(**)$ may again be interpreted as inverting the Abel-type conditions $(*)$.

(b) Residues and the Classical Abel Theorem

On $\mathbb{P}^2$ with affine coordinates (x, y) we consider two algebraic curves C and D having respective equations $f(x, y) = 0$ and $g(x, y) = 0$ of degrees m and n. We assume further that C and D have no common component, but in contrast to our previous discussions multiple components of either one individually are allowed. The most general meromorphic 2-form ω on $\mathbb{P}^2$ having polar curve $C + D$ together possibly with the line at infinity has an expression

$$(3.16)\quad \omega = \frac{p(x, y)\, dx \wedge dy}{f(x, y)\, g(x, y)}.$$

If $p(x, y)$ has degree d and we set $x = \frac{1}{x'}$, $y = \frac{y'}{x'}$, then

$$\omega = \frac{1}{(x')^{m+n-d-3}} \frac{\tilde{p}(x', y')\, dx' \wedge dy'}{\tilde{f}(x', y')\, \tilde{g}(x', y')}$$

where $\tilde{f}(x', y') = x'^m f\left(\frac{1}{x'}, \frac{y'}{x'}\right)$, etc. From this we deduce:

The rational 2-forms on $\mathbb{P}^2$ having polar curve $C + D$ are given by expressions (3.16) *where* $\deg p \leqq m + n - 3$.

If $C \cdot D = \sum P_\nu$, then by the residue theorem (3.7)

$$\sum_\nu \operatorname{Res}_{P_\nu} \omega = 0.$$

In case the $P_\nu = (x_\nu, y_\nu)$ are mn distinct finite points, this reduces to the *Jacobi relation*

$$(3.17)\quad \sum \frac{p(x_\nu, y_\nu)}{\dfrac{\partial(f, g)}{\partial(x, y)}(x_\nu, y_\nu)} = 0,$$

already utilized in Section I c and concerning which we shall have several remarks in this series of papers.

For the moment, we observe that the n-variable version of (3.17) is

$$(3.18)\quad \sum_\nu \frac{p(x_\nu)}{\dfrac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}(x_\nu)} = 0$$

where $f_1(x), \ldots f_n(x)$ are polynomials of degrees $d_1, \ldots, d_n$ whose divisors meet transversely at $d_1 \ldots d_n$ distinct finite points, and where $p(x)$ is a polynomial of degree at most $d_1 + \cdots + d_n - n - 1$. For $n = 1$ we again obtain the ubiquitous Lagrange interpolation formula (1.10)

$$\sum_\nu \frac{p(x_\nu)}{f'(x_\nu)} = 0, \qquad \deg p \leqq \deg f - 2,$$

and (3.18) may be regarded as a generalization of this formula. Indeed, this is the context in which it was originally taken up by Jacobi.[16]

Turning now to the converse (3.8) of the residue theorem (3.7) in the case at hand, we assume that C and D meet at mn distinct finite points (x_ν, y_ν) and set $j_\nu = \dfrac{1}{\dfrac{\partial(f, g)}{\partial(x, y)}(x_\nu, y_\nu)}$. Given complex numbers c_ν we seek a polynomial $p(x, y)$ of degree $m + n - 3$ with $p(x_\nu, y_\nu) = c_\nu$.

(3.19) *The relation*

$$\sum_\nu c_\nu j_\nu = 0$$

is the necessary and sufficient condition for the existence of the desired polynomial.

Indeed, by the converse to the residue theorem we may find a rational 2-form ω having polar curve $C + D$ and residues $\operatorname{Res}_{P_\nu} \omega = c_\nu j_\nu$ provided that the relation (3.18) is satisfied. Since ω has an expression (3.16) our result follows.

From a sheaf theoretic viewpoint, if we set $\Gamma = C \cdot D$ and let $I_\Gamma(k) \subset \mathcal{O}_{\mathbb{P}^2}(k)$ be the sections of the k-th power of the hyperplane line bundle which vanish on Γ,

16 C.f. C. G. J. Jacobi, Theoremata nova algebraica circa systema duarum aequationum inter duas variabiles propositarum, Gesammelte Werke, Band III, pp. 285–294, and De relationibus quae locum habere debent inter punta intersectionis duarum curvarum vel truim superficierum algebraicarum dati ordinis, simul cum enodational paradoxi algebraici, G. Werke III, pp. 329–354

then from the exact cohomology sequence of

$$0 \to I_\Gamma(m+n-3) \to \mathcal{O}_{\mathbb{P}^2}(m+n-3) \to \mathcal{O}_\Gamma(m+n-3) \to 0$$

and $H^1(\mathcal{O}_{\mathbb{P}^2}(k))=0$ for all k, what we just proved is equivalent to

(3.20) $\dim H^1(\mathbb{P}^2, I_\Gamma(m+n-3))=1.$

On the other hand, if not both $m \leqq 2$ and $n \leqq 2$, then a general zero cycle $\Gamma \in (\mathbb{P}^2)^{(mn)}$ has

$$H^1(\mathbb{P}^2, I_\Gamma(m+n-3))=0.$$

In Part II we shall discuss the manner in which (3.20) characterizes complete intersections, not only on $\mathbb{P}^2$ but on general surfaces as well.

We now relate (3.18) to the trace form of the Abel theorem for linear spaces given in Section IIe. For this we suppose that $f(x,y)=f(x_1,\dots,x_n,y)=0$ is a hypersurface V of degree d in $\mathbb{P}^{n+1}$ having no multiple components. We may then assume that the line $x_1=\dots=x_n=0$ meets V in d finite points having distinct y coordinates. Consider now the intersection of V with a variable line $L(a,b)$ having equations

(3.21)
$$\begin{aligned} l_1(x,y)&=x_1-a_1y-b_1\\ &\vdots\\ l_n(x,y)&=x_n-a_ny-b_n. \end{aligned}$$

We want to apply the residue theorem to the differential

$$\omega=\frac{q(x,y)\,dx_1\wedge\dots\wedge dx_n\wedge dy}{l_1(x,y)\dots l_n(x,y)\,f(x,y)}$$

where $\deg q(x,y) \leqq d-2$. For this we need to compute the Jacobian of the functions in the denominator, and this is accomplished by the

(3.22) **Lemma.** *For the $l_i(x,y)$ given by* (3.21),

$$dl_1\wedge\dots\wedge dl_n\wedge df=F'(y)\,dx_1\wedge\dots\wedge dx_n\wedge dy$$

at a point (x,y) satisfying $l_i(x,y)=f(x,y)=0$ and where

$$F(y)=f(a_1y+b_1,\dots,a_ny+b_n,y).$$

Proof.

$$\begin{aligned} &\bigwedge_{i=1}^{n}(dx_i-a_i\,dy)\wedge\left(\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}dx_j+\frac{\partial f}{\partial y}dy\right)\\ &\qquad=\left(\frac{\partial f}{\partial y}+\sum a_i\frac{\partial f}{\partial x_i}\right)dx_1\wedge\dots\wedge dx_n\wedge dy\\ &\qquad=F'(y)\,dx_1\wedge\dots\wedge dx_n\wedge dy. \quad \text{Q.E.D.} \end{aligned}$$

Writing the intersection $L(a,b)\cdot V=\sum_\nu P_\nu(a,b)$ where $P_\nu=(x_\nu,y_\nu)$, the residue theorem (3.18) gives

(3.23) $\displaystyle\sum_\nu \frac{Q(y_\nu)}{F'(y_\nu)}=0$

where $Q(y)=q(a_1y+b_1,\dots,a_ny+b_n\,y)$.

At this point we have come full circle. Referring to (2.30) we have again proved that the trace $\mathrm{Tr}_I \psi \equiv 0$ where

$$\psi = \frac{p(x,y)\, dx_1 \wedge \cdots \wedge dx_n}{\dfrac{\partial f}{\partial y}(x,y)}, \qquad \deg p \leqq d-n-2$$

and I is the incidence correspondence given by points in $\mathbb{P}^{n+1}$ lying both on V and on a line. The direct connection between this residue proof and the one given in Section II d will be brought out in the next section when we discuss Poincaré residues.

At the moment, we wish to note that for a nonsingular hypersurface $V \subset \mathbb{P}^{n+1}$ of degree d and holomorphic form $\psi = p(x,y)\, dx_1 \wedge \cdots \wedge dx_n / \partial f / \partial y(x,y)$ $(\deg p \leqq d-n-2)$ on V, we have proved the Abel theorem

$$\mathrm{Tr}_I \psi \equiv 0$$

in *three* ways: First, by the local properties of the trace and global property $\Omega^n(\mathbb{G}(1, n+1)) = 0$ of the Grassmannian; secondly, by the explicit computation centered around the formula (2.30); and thirdly, as a special case of the residue theorem. Of these the last seems most penetrating – e.g., since there is no restriction that the f_i in the denominator of the rational form ω should not have repeated factors, we obtain Abel-type realtions for more general intersections than straight lines, etc. These matters will be further pursued in our second paper on Abel's theorem.

(c) Poincaré Residues and the Proof of the Main Theorem

Let M be a complex n-manifold and ω a meromorphic n-form on M whose polar divisor is $V + W$ where each component of V has multiplicity 1 and $\operatorname{codim}(V \cap W) \geqq 2$. Locally,

$$\omega = \frac{h(z)\, dz_1 \wedge \cdots \wedge dz_n}{f(z)\, g(z)}$$

where f, g, h are holomorphic functions and $V = (f)$, $W = (g)$ with f, g being relatively prime. The *Poincaré residue* $R_V(\omega)$ will be a meromorphic $(n-1)$ form on V whose singularities are contained in $D = V_{\mathrm{sing}} + (V \cap W)$. To define it we note that at each point $z^* \in V - D$ some derivative $\dfrac{\partial f}{\partial z_i}(z_i^*) \neq 0$, and since $0 \equiv df|_V = \sum_{i=1}^{n} \dfrac{\partial f}{\partial z_i} dz_i$ implies

$$\frac{dz_i}{\dfrac{\partial f}{\partial z_j}(z^*)} \equiv \frac{-dz_j}{\dfrac{\partial f}{\partial z_i}(z^*)} \text{ modulo } \{dz_1, \ldots, \widehat{dz_i}, \ldots, \widehat{dz_j}, \ldots, dz_n\},$$

the locally defined expression

$$R_V(\omega) = (-1)^{n-i} \frac{h(z^*)\, dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n}{g(z^*) \dfrac{\partial f}{\partial z_i}(z^*)} \qquad \text{(restricted to } f=0) \tag{3.24}$$

is independent of the index i and is holomorphic on $V-D$. It is also straightforward to verify that it is independent of the local coordinates and defining functions of V and W, and serves to define the Poincaré residue.

In case V is smooth we may take $f(z)=z_n$ and then

$$R_V\left(\frac{h(z)\,dz_1\wedge\cdots\wedge dz_n}{z_n}\right)=h(z)\,dz_1\wedge\cdots\wedge dz_{n-1}$$

tying it in with the usual notion of residue. For such smooth V the Poincaré residue operator induces an exact sheaf sequence

$$0\to\Omega_M^n\to\Omega_M^n(V)\xrightarrow{R_V}\Omega_V^{n-1}\to 0,$$

so that, e.g., every holomorphic $(n-1)$-form on V is a Poincaré residue in case

$$H^1(M,\Omega_M^n)=0.$$

In ω is a meromorphic n-form with polar divisor $V+W+Z$ where all components of V and W have multiplicity 1 and the obvious general position requirements are met, we may iterate the Poincaré residue, and when this is done

(3.25) $\mathrm{Res}_W(\mathrm{Res}_V(\omega))=-\mathrm{Res}_V(\mathrm{Res}_W(\omega))$.

Continuing along this line, if ω has polar divisor $D_1+\cdots+D_n$ where the D_i are smooth and meet transversely at a point P, we may iterate the Poincaré residue operator n times and we claim that:

(3.26) $\mathrm{Res}_{D_1}(\mathrm{Res}_{D_2}(\ldots\mathrm{Res}_{D_{n-1}}(\mathrm{Res}_{D_n}(\omega))\ldots))=\mathrm{Res}_P(\omega)$

where the right-hand side is the previously defined point residue (3.1) (note that both sides are alternating in the D_i). Since both sides of (3.26) are independent of choices, we may take

$$\omega=\frac{h(z)\,dz_1\wedge\cdots\wedge dz_n}{z_1\ldots z_n}$$

where $(z_i)=D_i$, and then both sides are equal to $h(0)$ by (3.5). Obviously, (3.26) may be proved under less stringent hypotheses, but this will suffice for our purposes.

As a first use of Poincaré residues we suppose as in Section IIe that $V\subset\mathbb{P}^{n+1}$ is an algebraic hypersurface having no multiple components and affine equation $f(x,y)=0$. The most general meromorphic $(n+1)$-form ω on $\mathbb{P}^{n+1}$ having polar divisor $V+W$ where $\mathrm{codim}(V\cap W)\geqq 2$ has an expression

$$\omega=\frac{r(x,y)\,dx_1\wedge\cdots\wedge dx_n\wedge dy}{f(x,y)}$$

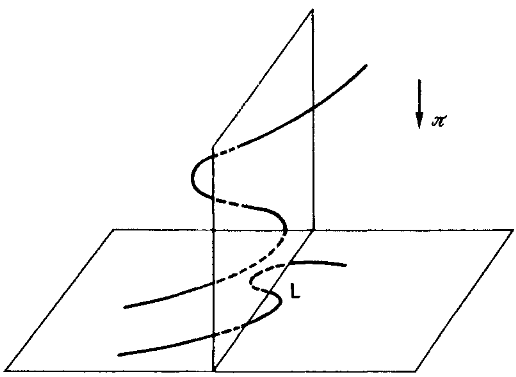
where $r(x,y)$ is a rational function not identically equal to ∞ on V. The Poincaré residue is given by

(3.27) $\mathrm{Res}_V(\omega)=\dfrac{r(x,y)\,dx_1\wedge\cdots\wedge dx_n}{\partial f/\partial y(x,y)}$;

this, at last, explains why the form (1.8) was so convenient for the study of abelian integrals.

We now come to the proof of the main theorem.

The first step is to reduce to the case of a hypersurface in $\mathbb{P}^{n+1}$. If $V_n \subset \mathbb{P}^{n+r}(r \geqq 2)$ is either a global algebraic variety or germ of analytic variety, then we may project V from a generic linear space $\mathbb{P}^{r-2} \subset \mathbb{P}^{n+r} - V$ to obtain a hypersurface in $\mathbb{P}^{n+1}$. This is done by choosing a generic $\mathbb{P}^{n+1}$ not meeting $\mathbb{P}^{r-2}$. Then each point $A \in \mathbb{P}^{n+r} - \mathbb{P}^{r-2}$ together with $\mathbb{P}^{r-2}$ spans a $\mathbb{P}^{r-1}(A)$ meeting $\mathbb{P}^{n-1}$ in a point $\pi(A)$. The resulting projection $V \to \pi(V) \subset \mathbb{P}^{n+1}$ is birational in case V is a global algebraic variety and biholomorphic if V is a germ of analytic variety. Here is a picture for $n=1, r=2$.



The inverse image of a line $L \subset \mathbb{P}^{n+1}$ is an r-plane $A(L)$ in $\mathbb{P}^{n+r}$, and generically

$$L \cdot \pi(V) \simeq A(L) \cdot V$$

under the projection mapping. If ψ is a meromorphic n-form on V having trace zero on $\mathbb{G}(r, n+r)$, then $\pi(\psi)$ will be a meromorphic n-form on $\pi(V)$ having trace zero on $\mathbb{G}(1, n+1)$. Based on this we may reduce to proving our result in the co-dimension 1 case.

We now explain the idea behind the proof.[17] Given an algebraic hypersurface V of degree d and rational n-form ψ on V, by (3.27) we may write $\psi = \operatorname{Res}_V(\Psi)$ as the Poincaré residue of a rational $(n+1)$-form on $\mathbb{P}^{n+1}$. If ψ is of the first kind relative to the lines in $\mathbb{P}^{n+1}$, then we uniquely have

$$\Psi = \frac{p(x, y)\, dx_1 \wedge \cdots \wedge dx_n \wedge dy}{f(x, y)}$$

where $p(x, y)$ is a polynomial of degree $\leqq d-n-2$ and $f(x, y)=0$ is the equation of V. We consider lines L having equations

$$(3.28) \qquad \begin{aligned} l_1 &= x_1 - a_1 y - b_1 = 0 \\ &\vdots \\ l_n &= x_n - a_n y - b_n = 0 \end{aligned}$$

17 Further explanation – only with the advantage of hindsight – is given at the conclusion of the proof, where two formal identities (*) and (**) which underlie the argument are discussed – c.f. also (3.29) below

Any such line L has y as linear coordinate and meets V in d points $y_\nu(L)$. On L there will be an equation

$$(3.29)\quad \frac{p(x,y)\,dy}{f(x,y)} = \sum_\nu \rho_\nu(L) \frac{dy}{y - y_\nu(L)}$$

where the $\rho_\nu(L)$ are residues of some sort (to be made precise later). *The main observation is that because of this the right-hand side of* (3.29) *depends only on* (V,ψ) *near the points of intersection* $L \cap V$, *while the left-hand side contains the equation of* V. Consequently, given the local data (V_ν, ψ_ν) of our main theorem, we may attempt to reverse our steps and *define*

$$(3.30)\quad \Psi = \left\{ \sum_\nu \rho_\nu(L) \frac{dy}{y - y_\nu(L)} \right\} \wedge dx_1 \wedge \cdots \wedge dx_n.$$

If we can show that this makes sense in some neighborhood of a line L_0 in $\mathbb{P}^{n+1}$, then by a Levi-Hartogs type of theorem, Ψ will extend to a meromorphic $(n+1)$-form on $\mathbb{P}^{n+1}$ and this will prove our result. The trace $\equiv 0$ condition is what will allow us to carry this idea out.

We now make this all precise, still assuming for a while that we have the global data (V,ψ) with $\psi = \mathrm{Res}_V(\Psi)$ as above. Consider the *incidence correspondence*

$$I \subset \mathbb{P}^{n+1} \times \mathbb{G}(1, n+1)$$

consisting of all pairs

$$\{(P, L) \colon P \in L\}.$$

The equations of I are just (3.28) relative to our standard affine coordinate system $(x, y; a, b)$ on $\mathbb{P}^{n+1} \times \mathbb{G}(1, n+1)$. The projection map

$$\pi \colon I \to \mathbb{P}^{n+1}$$

is a fibration with fiber $\pi^{-1}(A)$ the projective space $\mathbb{P}^n(A)$ of lines through a point $A \in \mathbb{P}^{n+1}$. We wish to show that:

(3.31) *The pull-back* $\pi^* \Psi$ *may be expressed in terms of the local behavior of* V *and* ψ *near the points of intersection of* V *with a line.*

Once this has been accomplished it is possible to begin reversing the reasoning. Writing $L \cdot V = \sum_\nu P_\nu$ where $P_\nu = (x_\nu, y_\nu)$ the formula which will establish (3.31) is

$$(3.32)\quad \pi^* \Psi = \sum_A \pm \left\{ \sum_\nu \mathrm{Res}_{P_\nu} \left(\frac{\psi}{l_1 \ldots l_n} \right) \frac{dy}{y - y_\nu} \right\} y^{|A|}\, da_A \wedge db_{A^c}$$

where $A = (i_1 < \cdots < i_k)$ runs over index sets selected from $(1, \ldots, n)$, $|A| = k$, and A^c is the complementary index set.

Proof of (3.32). For any function $q(x,y)$ on $\mathbb{P}^{n+1}$, we let

$$Q(y, a, b) = Q(y) = q(a_1 y + b_1, \ldots, a_n y + b_n, y)$$

be the pull-back of q to I. Since, by (3.28),

$$dx_i \equiv y\, da_i + db_i \text{ modulo } dy$$

we find that

(3.33) $\pi^*(dx_1 \wedge \cdots \wedge dx_n) \equiv \sum_A y^{|A|}\, da_A \wedge db_{A^c}$ *modulo* dy.

Consequently,

(3.34) $$\pi^* \Psi = \frac{P(y)\,dy}{F(y)} \wedge \{\sum_A y^{|A|}\, da_A \wedge db_{A^c}\}.$$

On the incidence variety I, for fixed (a, b) the differential form

$$\varphi(a, b) = \frac{P(y)\,dy}{F(y)}$$

is rational on the line $L(a, b)$ given by (3.28) and has first-order poles at the $y_\nu = y_\nu(a, b)$. The residue at $y = y_\nu$ is

$$\rho_\nu(a, b) = \frac{P(y_\nu)}{F'(y_\nu)},$$

and, as remarked on previous occasions, the relation $\sum_\nu \rho_\nu = 0$ follows from the Lagrange interpolation formula. These residues uniquely determine φ; in fact,

$$\varphi = \sum_\nu \rho_\nu \frac{dy}{y - y_\nu}.$$

Combining this with (3.34) gives

(3.35) $$\pi^* \Psi = \sum_A \pm \left\{\sum_\nu \rho_\nu \frac{dy}{y - y_\nu}\right\} y^{|A|}\, da_A \wedge db_{A^c},$$

which is the first step in the proof of (3.32).

Next, we define

$$\omega = \frac{\psi}{l_1 \ldots l_n} = \operatorname{Res}_V(\Omega)$$

where

(3.36) $$\Omega = \frac{p(x, y)\, dx_1 \wedge \cdots \wedge dx_n \wedge dy}{(x_1 - a_1 y - b_1) \cdots (x_n - a_n y - b_n) f(x, y)}.$$

The forms ω and Ω have point residues at $P_\nu = (x_\nu, y_\nu)$. Letting H_i be the hyperplane $l_i = 0$, these point residues may be calculated by twice applying the formulae (3.25) and (3.26) for iteration of Poincaré residues. Thus

$$\begin{aligned}
\operatorname{Res}_{P_\nu}(\omega) &= \pm \operatorname{Res}_{H_1}(\operatorname{Res}_{H_2}(\ldots \operatorname{Res}_{H_n}(\omega))\ldots) \\
&= \pm \operatorname{Res}_{H_1}(\operatorname{Res}_{H_2}(\ldots \operatorname{Res}_{H_n}(\operatorname{Res}_V(\Omega)\ldots)) \\
&= \pm \operatorname{Res}_V(\operatorname{Res}_{H_1}(\operatorname{Res}_{H_2}(\ldots \operatorname{Res}_{H_n}(\Omega))\ldots) \\
&= \pm \operatorname{Res}_V \left\{\frac{P(y)\,dy}{F(y)}\right\} \\
&= \pm \rho_\nu.
\end{aligned}$$

We may combine this with (3.35) to prove (3.32).

Now are almost ready to reverse our steps. In doing this one comes across the need for two lemmas whose statement and proof we shall give first:

(3.37) **Lemma.** *Let η be a meromorphic function defined in neighborhood U of a line L_0 in $\mathbb{P}^{n+1}$. Then η extends to a rational function on $\mathbb{P}^{n+1}$.*

Proof. This is a variant of the Levi-Hartogs' theorem and has to do with—but is not implied by—the *pseudo-concavity* of $\mathbb{P}^{n+1}-U$. Use of the incidence variety gives a convenient method for proving the lemma, and we shall carry this out for $\mathbb{P}^2$, which is the essential case. We may then think of $\mathbb{P}^2-U$ as being the complement $\{(x,y)\colon \|x\|^2+\|y\|^2 \geqq R^2\}$ of a ball in $\mathbb{C}^2$. If $\xi(x,y)$ is a meromorphic function in $U\cap\mathbb{C}^2$ whose polar locus is contained in the coordinate axis $\{x=0\}$, then for some large k, $x^k\,\xi(x,y)$ will be holomorphic in the exterior of the ball, and hence by Hartog's theorem extends to a holomorphic function on $\mathbb{C}^2$. This is the extension result we shall use in a little while.

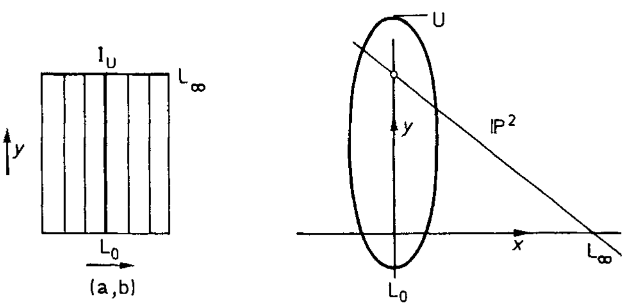
Now choose affine coordinates (x,y) on $\mathbb{P}^2$ such that $L_0=\{x=0\}$. Lines $L(a,b)$ near L_0 have equations $\{x=a\,y+b\colon |a|,|b|<\varepsilon\}$, and we let

$$I_U\subset\mathbb{P}^2\times\mathbb{P}^{2*}$$

$=\{(P,L(a,b))\colon P\in L(a,b) \text{ and } |a|,|b|<\varepsilon\}$ be the restriction of the incidence correspondence to U. Then I_U is the *disjoint* union of the lines in U, and the projection

$$I_U\xrightarrow{\pi}U$$

has fiber the $\mathbb{P}^1$ of lines through a point in U.



We assume that the polar curve C of $\eta(x,y)$ does not meet the line L_∞ given by $y=\infty$ in U. Then C meets each line $L(a,b)$ in finite points $y_1(a,b),\ldots,y_d(a,b)$, and we consider on I_U the function

$$\Phi(y,a,b)=e^{\mu(a,b)}\prod_{\nu=1}^{d}(y-y_\nu(a,b)),$$

where $\mu(a,b)$ is holomorphic for $|a|,|b|<\varepsilon$. Φ is meromorphic in I_U and has polar locus $d\cdot L_\infty$. If we can choose μ such that $\Phi=\pi^*\varphi$ comes from U, then

$$\xi=\varphi\cdot\eta$$

will be a function to which Hartogs' theorem, as discussed above, will apply. Since the zero curve for ξ is the same as that for η, it follows that the zero curve $\{\eta=0\}$

has a global equation $f(x, y)=0$ where f is a polynomial of degree d. Similarly, the polar curve $\{\eta=\infty\}$ is given by $g(x, y)=0$. Then $\eta=C(f/g)$ where C is a nonzero constant.

It remains to show that Φ comes from U; i.e., we must choose μ so that $\Phi(y, a, b)$ depends only on $x=a\,y+b$ and y. A general version of this will be discussed in the next lemma, but in our case it is pretty clear that the necessary and sufficient condition that $\Phi=\varphi(x, y)$ is

$$d\Phi\equiv 0 \text{ modulo } \{dx, dy\},$$

or, equivalently,

$$d\Phi\wedge dx\wedge dy\equiv 0.$$

When computed out this latter equation is

$$\left(\frac{\partial}{\partial a}-y\frac{\partial}{\partial b}\right)\Phi=0.$$

Denote by X the vector field on the left-hand side. Then

$$X\cdot\log\Phi=X\cdot\mu+\sum_{a,b}X\cdot\log(y-y_v(a,b)).$$

The right-hand side of this equation is holomorphic for finite y since $X\cdot\log\eta\equiv 0$, and at $y=\infty$ it remains bounded. Consequently, by the maximum principle

$$X\cdot\log\Phi$$

depends only on (a, b), and so we may choose $\mu(a, b)$ to make $X\cdot\log\Phi\equiv 0$. Q.E.D.

It will be seen that the proof of this lemma contains the germ of the argument for the main theorem.

(3.38) **Lemma.** *Given connected manifolds M and N, a surjective smooth mapping*

$$f\colon M\to N$$

having maximal rank everywhere, and a smooth form α on M, then $\alpha=f^*\beta$ for a (unique!) form β if, and only if,

$$\langle\alpha,\xi\rangle=0=\langle d\alpha,\xi\rangle$$

for all tangent vectors $\xi\in T(M)$ with $f_*(\xi)=0$.

We shall say that a form γ satisfying $\langle\gamma,\xi\rangle=0$ for all ξ with $f_*(\xi)=0$ is *horizontal* (these tangent vectors ξ are *vertical*). In case $\deg\alpha=\dim N$ the conditions of the lemma are

$$(3.39)\qquad \begin{aligned}&\alpha \text{ is horizontal}\\ &d\alpha=0.\end{aligned}$$

Proof of (3.38). The conditions are obviously necessary. Also, β is unique in any open set where it exists. Consequently, it will suffice to prove the sufficiency around a point on M where we have product coordinates $(u, v)=(w_1, \dots, u_n; v_1, \dots, v_m)$ with $f(u, v)=u$. If α is horizontal, then

$$\alpha=\sum_I \alpha_I(u, v)\,du_I$$

where the $\alpha_I(w, v)$ are C^∞ functions. Consequently,

$$d\alpha \equiv \sum_I \left(\sum_\mu \frac{\partial \alpha_I(u, v)}{\partial v_\nu} dv_\mu\right) \wedge du_I$$

modulo terms containing only du's. If $d\alpha$ is horizontal, then

$$\frac{\partial \alpha_I(u, v)}{\partial v_\nu} = 0$$

for all α_I, ν and $\alpha = \pi^*(\sum_I \alpha_I(u)\, du_I)$. Q.E.D.

Now we have all the ingredients to complete the proof of the main theorem. Given the local pieces of analytic variety V_ν and meromorphic n-forms ψ_ν as in the statement, we let U be a neighborhood of our fixed line $L_0 = \{x_1 = \cdots = x_n = 0\}$ such that for $L \in U$ the trace is defined and

$$\text{(3.40)} \quad \sum_\nu \psi_\nu(L \cdot V_\nu) \equiv 0$$

Set $I_U = \{(P, L): P \in L \subset U\} = \pi^{-1}(U)$ and invert (3.32) by defining the meromorphic $(n+1)$-form Φ in I_U by

$$\text{(3.41)} \quad \Phi = \sum_A \pm \left\{\sum_\nu \operatorname{Res}_{P_\nu} \left(\frac{\psi_\nu}{l_1 \ldots l_n}\right) \frac{dy}{y - y_\nu}\right\} y^{|A|}\, da_A \wedge db_{A^c}.$$

Since, by (3.33)

$$\pi^*(dx_1 \wedge \cdots \wedge dx_n \wedge dy) = \sum_A \pm y^{|A|}\, da_A \wedge db_{A^c} \wedge dy,$$

we see that Φ is horizontal for the fibration

$$I_U \xrightarrow{\pi} U.$$

If we can prove that

$$\text{(3.42)} \quad d\Phi = 0,$$

then, according to Lemma (3.38) in the form (3.39) we will have

$$\Phi = \pi^* \Psi$$

where Ψ is a meromorphic $(n+1)$-form in U. Writing $\Psi = \eta\, dx_1 \wedge \cdots \wedge dx_n \wedge dy$ we may then apply Lemma (3.37) to η to conclude that Ψ extends to a *rational* form on $\mathbb{P}^{n+1}$. The polar set of Ψ is our desired V and $\psi = \operatorname{Res}_V(\Psi)$.

Thus it all comes down to proving (3.42). To do this we put the assumption (3.40) in convenient form. Choose small open sets $U_\nu \subset \mathbb{P}^{n+1}$ and holomorphic functions $f_\nu(x, y) \in \mathcal{O}(U_\nu)$ such that V_ν is the divisor of f_ν. Then we may find $g_\nu \in \mathcal{O}(U_\nu)$ with

$$\text{(3.43)} \quad \psi_\nu = \operatorname{Res}_{V_\nu} \left\{\frac{g_\nu(x, y)\, dx_1 \wedge \cdots \wedge dx_n \wedge dy}{f_\nu(x, y)}\right\}$$

$$= \frac{g_\nu(x, y)\, dx_1 \wedge \cdots \wedge dx_n}{\dfrac{\partial f_\nu}{\partial y}(x, y)}.$$

The exact same computation as in the proof of the basic formula (2.30) gives

$$\sum_{\nu} \psi_\nu(L \cdot V_\nu) = \sum_A \pm \left\{ \sum_\nu y_\nu^{|A|} \frac{G_\nu(y_\nu)}{F'_\nu(y_\nu)} \right\} da_A \wedge db_{A^c}$$

where, as before, $F(y) = f(a_1 y + b_1, \ldots, a_n y + b_n, y)$, etc. The hypothesis (3.41) is equivalent to

$$\sum_\nu \frac{y_\nu^k G_\nu(y_\nu)}{F'_\nu(y_\nu)} = 0 \quad (k = 0, \ldots, n).$$

On the other hand, referring to the proof of (3.32)

$$\begin{aligned} \rho_\nu &= \mathrm{Res}_{P_\nu} \left(\frac{\psi_\nu}{l_1 \cdots l_n} \right) \quad \text{(definition)} \\ &= \mathrm{Res}_{P_\nu} \left(\frac{\Psi_\nu}{l_1 \cdots l_n} \right) \end{aligned}$$

where Ψ_ν is the form on the right side of (3.43)

$$= \frac{G_\nu(y_\nu)}{F'_\nu(y_\nu)}$$

by the same argument as used to prove (3.26).

Consequently, the assumption (3.41) is the same as

$$(3.44) \quad \sum_\nu y_\nu^k \rho_\nu = 0 \quad (k = 0, \ldots, n).$$

We now finally compute $d\Phi$. Writing

$$(3.45) \quad d\Phi = \sum_{I, J} \Phi_{IJ}(y, a, b)\, dy \wedge da_I \wedge db_J,$$

it is clear that Φ_{IJ} has at most *second-order poles* at $y = y_\nu(a, b)$. In fact, $d\Phi$ is holomorphic on the divisor $y = y_\nu$ in I_U: Near $y = y_\nu$ we may use a Taylor's expansion to write

$$\begin{aligned} \Phi &= \left(\frac{G_\nu(y_\nu)}{F'_\nu(y_\nu)} \frac{dy}{y - y_\nu} \right) \wedge dx_1 \wedge \cdots \wedge dx_n + (\text{holomorphic form}) \\ &= \left(\frac{G_\nu(y)\, dy}{F_\nu(y)} \right) \wedge dx_1 \wedge \cdots \wedge dx_n + (\text{holomorphic form}), \end{aligned}$$

since $F(y) = F'(y_\nu)(y - y_\nu) + \dfrac{F''(y_\nu)}{2}(y - y_\nu)^2 + \cdots$, and so

$$\begin{aligned} d\Phi &= d \left(\frac{G_\nu(y)\, dy \wedge dx_1 \wedge \cdots \wedge dx_n}{F(y)} \right) + (\text{holomorphic form}) \\ &= 0 + (\text{holomorphic form}). \end{aligned}$$

The point in this calculation is to eliminate the dependence of y_ν on (a, b), at least modulo holomorphic terms.

Now look at $d\Phi$ near the divisor $y=\infty$. Set $u=\frac{1}{y}$ and use

$$\frac{1}{1-v}=\sum_{k=0}^{\infty} v^k$$

to write near $u=0$

$$\begin{aligned}\sum_v \frac{\rho_v\, dy}{y-y_v} &= -\sum_v \left(\frac{\rho_v}{1-y_v u}\right)\frac{du}{u} \\ &= -\left\{\sum_{k=0}^{\infty}\left(\sum_v \rho_v y_v^k\right) u^k\right\}\frac{du}{u} \\ &= u^n \times (\text{holomorphic term})\end{aligned}$$

by (3.44). Referring to the definition (3.41), we set that Φ is holomorphic near the divisor $y=\infty$, hence the same is true of $d\Phi$ and thus of the $\Phi_{IJ}(a, b, y)$, and then finally, $d\Phi\equiv 0$ since the Φ_{IJ} are holomorphic for finite y also. Q.E.D.

Having proved the result, we may extract the essence of the argument as being the *converse of the Lagrange interpolation formula* in the following sense:

Given distinct points $\{y_v\}$ and complex numbers $\{\rho_v\}$, the conditions that we can find a rational 1-form φ on $\mathbb{P}^1(y)$ having first order poles at $y=y_v$ with residue ρ_v and a zero of order n at $y=\infty$ are

$$\sum y_v^k \rho_v = 0 \qquad (k=0, \dots, n+1)$$

The form φ may be written in the two ways

$$(*) \quad \sum_v \rho_v \frac{dy}{y-y_v} = \frac{p(y)\, dy}{f(y)}$$

where $f(y)=\prod_{v=1}^{d}(y-y_v)$ is the defining equation of the $\{y_v\}$ and $p(y)$ is a polynomial of degree $\leqq d-n-2$. The connection between the two sides of $(*)$ is

$$(**) \quad \rho_v = \operatorname{Res}_{y_v}(\varphi(y)) = \frac{p(y_v)}{f'(y_v)}.$$

Our proof of the main theorem is just an extension of this observation allowing dependence on parameters.

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Notes Added in Proof

3 Along a different line, it seems pretty clear that results at a deeper level than our main theorem can be expected in the presence of the "maximum number" of addition theorems, rather than just one such relation. This study was initiated by Blaschke-Bol for the situations corresponding to curves in the plane and 3-space, and the general version for curves is the object of work in progress by S. S. Chern and the present author. Roughly speaking, the presence of the maximum number of addition theorems allows us to, in general, dispense with the assumption that the A's are global and linear, and thus allows one to invert the maximum number of addition relations when these are given in a purely local form.

10 Here, we wish to mention a conjecture of Spencer Bloch to the effect that, on a surface S with geometric genus $p_g(S)=0$, the Chow group $C(V)\simeq\mathbb{Z}$. He has verified this in numerous special cases including the Enriques and Godeaux surfaces – c.f. Hartshorne's article referred to in footnote 7. Our preceding remarks about the maximal closed integral varieties possibly providing rational equivalence are very closely related to his conjecture. It is, however, premature to call our suggestion a "conjecture," as the following example and observations will point out.

Thus far, our discussion about inverting the zero-cycle conditions in Abel's theorem has been centered around complete intersections, and we now give an example of inversion in a noncomplete intersection case. Let $S\subset\mathbb{P}^3$ be a smooth quartic surface with holomorphic 2-form ω inducing by (2.10) the differential form $\Omega\in\Omega^2(S^{(n)})$ on the symmetric products. We consider the family $\{C\}$ of rational normal curves $C\subset\mathbb{P}^3$. These are cubic space curves, certainly not complete intersections, and any two are projectively equivalent. Thus the family of all such has dimension equal to

$$\dim(\mathrm{Aut}(\mathbb{P}^3))-\dim(\mathrm{Aut}(\mathbb{P}^1))=15-3=12.$$

By the remark in the paragraph preceding Equation (2.8), the trace

$$\sum\omega(C\cdot S)\equiv 0,$$

where the sum is over the $12=3\cdot 4$ points of intersection of C and S. Letting $\Gamma=C\cdot S\in S^{(12)}$, the maximal integral varieties of Ω passing through Γ have dimension equal to $12=\frac{1}{2}(\dim S^{(12)})$, and this is exactly equal to the dimension of the variable zero cycles $\Gamma'=C'\cdot S$ for C' a rational normal curve. Consequently,

If $S\subset\mathbb{P}^3$ is a smooth quartic surface and $C\subset\mathbb{P}^3$ a rational normal curve, then a maximal integral variety of $\Omega^*(S^{(12)})$ passing through $\Gamma=C\cdot S$ will consist of intersections $\Gamma'=C'\cdot S$ where $C'\subset\mathbb{P}^2$ is a variable rational normal curve.

So, once again, we have inversion of the conditions in Abel's theorem.

It may be noted that our examples thus far have concerned surfaces with $p_g=1$. In case $p_g(S)>1$, say S has very ample canonical bundle, we should expect that the maximal closed integral varieties of $\Omega^*(S^{(n)})$ should have dimension $<n$, especially in view of (2.21). This is the case for purely local reasons, at least provided none of the points in $\Gamma=\sum_{\nu=1}^{n}P_\nu$ is constrained to vary on a curve, in some examples such as as a quintic surface in $\mathbb{P}^3$. Moreover, in this case, a local integral variety of maximal dimension is automatically globally closed and gives a rational equivalence.