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Hyperbolic uniformization of Fermat curves

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Abstract The ground-breaking research on the uniformization of curves was conducted at the beginning of the last century. Nevertheless, there are few examples in the literature of algebraic curves for which an explicit uniformization is known. In this article we obtain an explicit uniformization of the Fermat curves F_N, for each $N \geq 4$. The results presented here are based in part on an earlier study of the second author [6] in which each Riemann surface $F_N(\mathbb{C})$ was described as a quotient of the complex disk by a Fuchsian group Γ.

Keywords Hyperbolic uniformization · Fundamental domains · Fermat curves

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Secondary—11F30

Introduction

Fix a positive integer $N \geq 4$ and let F_N be the Fermat curve given in projective coordinates by the equation

$$X^N + Y^N = Z^N.$$

The curve is of degree N and of genus $g(N) = (N-1)(N-2)/2$.

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Let $\bar{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ contained in $\mathbb{C}$, and let $\mu_N \subset \overline{\mathbb{Q}}^*$ be the group of Nth-roots of unity. The group μ_N^3 acts on F_N by the formula

$$(\xi_1, \xi_2, \xi_3)(X, Y, Z) = (\xi_1 X, \xi_2 Y, \xi_3 Z).$$

This action factorizes through the quotient group $G_N = \mu_N^3/\iota(\mu_N)$, where $\iota\colon \mu_N \to \mu_N^3$ stands for the diagonal embedding. The group G_N is generated by the automorphisms

$$\alpha = (\zeta_N, 1, 1), \quad \beta = (1, \zeta_N, 1),$$

where $\zeta_N = e^{2\pi i/N}$. Moreover, $\mathrm{Aut}(F_N) = \langle \alpha, \beta, \sigma, \tau \rangle$, where σ, τ are defined by

$$\sigma(X, Y, Z) = (Z, X, Y), \quad \tau(X, Y, Z) = (Y, X, Z).$$

In accordance with Gross [9], a triple $(a, b, c) \in \mathbb{Z}^3$ is called admissible if $0 < a, b, c < N$ and $a + b + c \equiv 0 \pmod N$. For any admissible triple, we consider the differential form on F_N

$$w_{a,b} = x^{a-1} y^{b-1} \frac{dx}{y^{N-1}},$$

where $x = X/Z$, $y = Y/Z$. Gross proved that the set

$$\Omega = \{w_{a,b} : (a, b, c) \text{ admissible}\}$$

yields a basis for the algebraic de Rham cohomology group $H^1_{DR}(F_N/\mathbb{Q})$. The forms $w_{a,b}$ are eigenvectors for the natural action of α, β on differentials:

$$(\alpha^j \beta^k) w_{a,b} = \zeta^{aj+bk} w_{a,b}.$$

Rohrlich [13] showed that the singular homology group $H_1(F_N, \mathbb{Z})$ is a cyclic module over the group ring $\mathbb{Z}[\alpha, \beta]$ with a canonical generating cycle κ. If $I : [0, 1] \to F_N(\mathbb{R})$ denotes the one-dimensional simplex defined by

$$I(t) = (t^{1/N}, (1 - t)^{1/N}, 1), \quad t \in [0, 1],$$

then κ corresponds to the class of $(\alpha\beta)^{(N-1)/2}(1 - \alpha)(1 - \beta)I$ if N is odd, or to the class of $(1 - \alpha^{-1})(1 - \beta^{-1})I$ if N is even (cf. [7]).

Let $\mathcal{D}_r = \{z \in \mathbb{C} : z\bar{z} \leq r^2\}$ denote the complex disk of radius equal to r. Denote by Δ a Fuchsian triangular group of signature (N, N, N) acting on $\mathcal{D}_r$:

$$\Delta = \langle \alpha, \beta, \gamma : \alpha^N = \beta^N = \gamma^N = \mathrm{Id},\ \alpha\beta\gamma = \mathrm{Id} \rangle.$$

We know that a hyperbolic model of the Fermat curve is given through an isomorphism

$$\Gamma \backslash \mathcal{D}_r \simeq F_N(\mathbb{C}),$$

where $\Gamma = [\Delta, \Delta]$ denotes the commutator subgroup of Δ. A fundamental domain, $\mathcal{P}_N$, for the action of Γ on $\mathcal{D}_1$ was determined in [6].

Now we will go one step further and find a hyperbolic uniformization of the curves F_N. Our study focuses on the explicit determination of affine coordinate functions $\mathrm{sf}(z)$, $\mathrm{cf}(z)$, meromorphic on $\mathcal{D}_r$ and Γ-automorphic, realizing the isomorphism (0.1). Thus, we shall have

$$\mathrm{sf}^N(z) + \mathrm{cf}^N(z) = 1, \quad \text{for all } z \in \Gamma\backslash\mathcal{D}_r, \quad z \notin S,$$

for a certain finite subset S of $\Gamma\backslash\mathcal{D}_r$. When it is necessary to state the value of N explicitly, the *Fermat functions* sf, cf will be written $\mathrm{sf}(z; N)$, $\mathrm{cf}(z; N)$.

The multivalued inverse functions $(\mathrm{sf}^N)^{-1}$, $(\mathrm{cf}^N)^{-1}$ were particularly interesting. In fact, they are related to Legendre functions in a natural way.

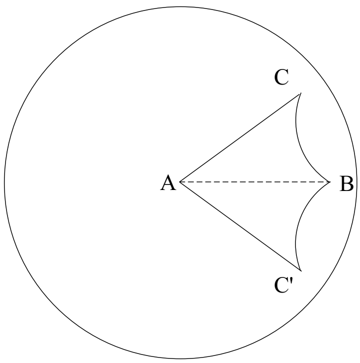
Our article is organized as follows. In Section 1, we provide a brief outline to the fundamental domains for the Fermat curves F_N and for some of their quotients. In Section 2, we obtain a Schwarzian differential equation whose integration, under convenient initial conditions, will provide uniformizing functions for genus zero quotients of F_N. In Section 3, we find explicit uniformizations of the curves F_N. In Section 4, we express the inverse functions $(\mathrm{sf}^N)^{-1}$, $(\mathrm{cf}^N)^{-1}$ in terms of Legendre functions. An accessory parameter is determined in Section 5. As an illustration of our method, some numerical data are collected in an appendix at the end of the paper. They involve the constants, the functions sf, cf, as well as the differentials attached to them.

1 A fundamental domain for Fermat curves

Since $N \geq 4$, we have that $\frac{1}{N} + \frac{1}{N} + \frac{1}{N} < 1$, and we may construct inside $\mathcal{D}_r$ an hyperbolic triangle $\mathcal{T}_N$ with vertices (A, B, C) and interior angles $(\pi/N, \pi/N, \pi/N)$. We take $A = 0$ and choose the vertex $B > 0$ on the real axis. We assume that $\mathcal{T}_N$ is oriented counterclockwise. We denote by $\mathcal{T}'_N$ the triangle with vertices (A, B, C') obtained from $\mathcal{T}_N$ by reflection on the side AB (cf. Fig. 1). We consider the Fuchsian triangle group $\Delta \subset \mathrm{Aut}(\mathcal{D}_r)$ attached to $\mathcal{T}_N$. It is generated by three hyperbolic rotations α, β, γ with centers at A, B, C, respectively, and rotation angles equal to $2\pi/N$. We make explicit the numerical data in Proposition 1.2.

For any given integer k, we let $\zeta_k = e^{2\pi i/k}$.

Fig. 1 Triangles $\mathcal{T}_N$, $\mathcal{T}'_N$



Proposition 1.1. *The group* $\mathrm{Aut}(\mathcal{D}_r)$ *consists of the following homographic transformations*

$$f(z) = r^2 e^{i\alpha} \frac{z + z_0}{\overline{z}_0 z + r^2}, \quad z \in \mathcal{D}_r,$$

for $\alpha \in \mathbb{R}$ *and* $|z_0| < r$.

The group $\mathrm{Aut}(\mathcal{D}_r)$ also admits the equivalent description

$$\mathrm{Aut}(\mathcal{D}_r) \simeq \left\{ \begin{bmatrix} a & br \\ \overline{b}/r & \overline{a} \end{bmatrix} : a, b \in \mathbb{C},\ |b| < |a| \right\} / \mathbb{R}^*.$$

Clearly, all the groups $\mathrm{Aut}(\mathcal{D}_r)$ are conjugate in $\mathbf{PGL}(2, \mathbb{C})$. By considering the homothety $h_r(z) = rz$, we have $\mathrm{Aut}(\mathcal{D}_r) = h_r \mathrm{Aut}(\mathcal{D}_1) h_r^{-1}$.

Proposition 1.2. *Let* $t = \sqrt{2\cos(\pi/N) - 1}$.

(i) The vertices of the triangles $\mathcal{T}_N, \mathcal{T}'_N$ *are*

$$A = 0, \quad B = r\,t, \quad C = \zeta_{2N} B, \quad C' = \bar{\zeta}_{2N} B.$$

(ii) The involution $\tau(z) = \dfrac{rz - rB}{(B/r)z - r}$ *defined by the matrix*

$$M(\tau) = \begin{bmatrix} ir & -iBr \\ iB/r & -ir \end{bmatrix}$$

is an element of $\mathrm{Aut}(\mathcal{D}_r)$ *which interchanges the points A and B, and the points C and C'.*

(iii) The hyperbolic rotations $\alpha, \beta, \gamma \in \mathrm{Aut}(\mathcal{D}_r)$ *generating* Δ *are represented by the matrices*

$$M(\alpha) = \begin{bmatrix} \zeta_{2N} & 0 \\ 0 & \overline{\zeta_{2N}}, \end{bmatrix}$$

$$M(\beta) = \begin{bmatrix} \cos\left(\frac{\pi}{N}\right)\left(1 + i\cot\left(\frac{\pi}{2N}\right)\right) & -irt\cot\left(\frac{\pi}{2N}\right) \\ i\cot\left(\frac{\pi}{2N}\right)\frac{t}{r} & \cos\left(\frac{\pi}{N}\right)\left(1 - i\cot\left(\frac{\pi}{2N}\right)\right) \end{bmatrix},$$

$$M(\gamma) = \begin{bmatrix} \cos\left(\frac{\pi}{N}\right)\left(1 + i\cot\left(\frac{\pi}{2N}\right)\right) & -irt\cot\left(\frac{\pi}{2N}\right)\zeta_{2N} \\ i\frac{t}{r}\cot\left(\frac{\pi}{2N}\right)\overline{\zeta}_{2N} & \cos\left(\frac{\pi}{N}\right)\left(1 - i\cot\left(\frac{\pi}{2N}\right)\right) \end{bmatrix}.$$

(iv) They satisfy

$$M(\alpha)M(\beta)M(\gamma) = -\mathrm{Id},$$
$$\mathrm{Inv}(M(\alpha)) = \mathrm{Inv}(M(\beta)) = \mathrm{Inv}(M(\gamma)) = 2\cos\left(\frac{\pi}{N}\right),$$

where, for any given matrix M, its invariant is defined by

$$\mathrm{Inv}(M) = \mathrm{Tr}(M)/(\mathrm{Det}(M))^{1/2}.$$

(v) The quadrilateral $\mathcal{Q} = AC'BC$ is a fundamental domain for the action of Δ in $\mathcal{D}_r$ (cf. [10]).

The proof of all these assertions follows from straightforward computations once the vertices of the triangle $\mathcal{T}_N$ are determined. We explain how to obtain the length of the side AB. Let $(A, \widetilde{B}, \widetilde{C})$ be the hyperbolic triangle obtained from the triangle (A, B, C) through a rotation of angle $-\pi/(2N)$ at the origin. We need only to compute $\widetilde{B}$. Observe that a circle of center $(c, 0)$ and radius R give rise to a hyperbolic line if and only if $c^2 - R^2 = r^2$. The side $\widetilde{B}\widetilde{C}$ corresponds to the value

$$c = \frac{r\cos(\frac{2\pi}{N})}{\cos(\frac{\pi}{2N})\sqrt{2\cos(\frac{\pi}{N}) - 1}},$$

and

$$\widetilde{B} = \left(r\cos\left(\frac{\pi}{2N}\right)\sqrt{2\cos\left(\frac{\pi}{N}\right) - 1}, -r\sin\left(\frac{\pi}{2N}\right)\sqrt{2\cos\left(\frac{\pi}{N}\right) - 1}\right).$$

As none of the vertices of $\mathcal{Q}$ is on the boundary of $\mathcal{D}_r$, the quotient $\mathcal{C} = \Delta\backslash\mathcal{D}_r$ is a compact and connected Riemann surface, and it is easy to check that it is of genus zero.

Consider the group homomorphism

$$\begin{aligned} \varphi_A\colon\ \Delta &\longrightarrow \mathbb{Z}/N\mathbb{Z} \\ \alpha &\mapsto 1 \\ \beta &\mapsto 0. \end{aligned}$$

The kernel of φ_A is the subgroup Δ_A of Δ generated by β and the commutator subgroup $[\Delta, \Delta]$. A fundamental domain for the action of Δ_A on $\mathcal{D}_r$ is a hyperbolic regular polygon, $\mathcal{P}_A$, centered at A with $2N$ sides and interior angles equal to π/N. It is composed by the quadrilaterals $\mathcal{Q}_i = \alpha^i(\mathcal{Q})$:

$$\mathcal{P}_A = \cup_{i=0}^{N-1}\mathcal{Q}_i.$$

Their vertices are the points $B_i = \alpha^i(B)$ and $C_i = \alpha^i(C')$. We enumerate the sides of the polygon from 0 to $2N - 1$ counterclockwise, starting from $\overline{C_0B_0}$. We denote by $\beta_i = \alpha^i\beta\alpha^{-i}$ the rotation of center B_i and angle $2\pi/N$. Similarly, we denote by

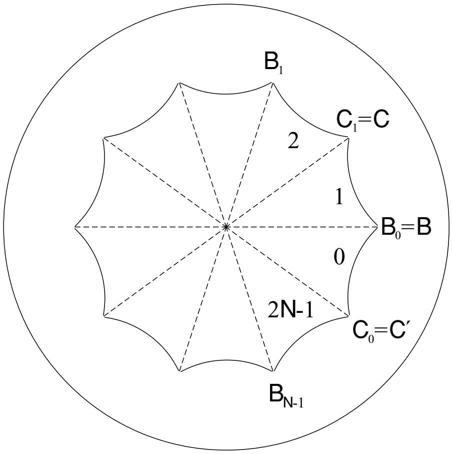


Fig. 2 Fundamental domain for $\mathcal{C}_A$

$\gamma_i = \alpha^i \gamma \alpha^{-i}$ the rotation of center C_i and angle $2\pi/N$. The case $N = 5$ is illustrated in Fig. 2.

The Riemann surface $\mathcal{C}_A := \Delta_A \backslash \mathcal{D}_r$ is a covering of degree N of $\mathcal{C}$, ramified at points A and C. The natural projection $\mathcal{C}_A \to \mathcal{C}$ maps every quadrilateral $\mathcal{Q}_i$ onto $\mathcal{Q}$. The group of automorphisms of $\mathcal{C}_A$ over $\mathcal{C}$ is $H_A = \Delta/\Delta_A = \langle \overline{\alpha} \rangle$, which is cyclic of order N.

We can mimic the construction of $\mathcal{C}_A$ by interchanging the roles of α and β. In this way, we obtain another Riemann surface, $\mathcal{C}_B$, of genus 0 which corresponds to the Fuchsian group Δ_B of Δ generated by α and $[\Delta, \Delta]$. A fundamental domain, $\mathcal{P}_B$, for Δ_B is composed by the quadrilaterals $\mathcal{Q}^j = \beta^j(\mathcal{Q})$:

$$\mathcal{P}_B = \cup_{j=0}^{N-1} \mathcal{Q}^j.$$

The group of automorphisms of $\mathcal{C}_B$ over $\mathcal{C}$ is $H_B = \Delta/\Delta_B = \langle \overline{\beta} \rangle$.

Since the genus of $\mathcal{C}_A$ is zero, there exists a Δ_A-automorphic function $\mathrm{sf}(z)$, defined on $\mathcal{D}_r$, establishing an analytic isomorphism between $\mathcal{C}_A$ and $\mathbb{P}^1(\mathbb{C})$. We assume the function sf normalized to satisfy

$$\mathrm{sf}(A) = 0, \quad \mathrm{sf}(B) = 1, \quad \mathrm{sf}(C) = \infty.$$

The function field $\mathbb{C}(\mathcal{C}_A)$ is isomorphic to $\mathbb{C}(\mathrm{sf})$.

Similarly, we can find a Δ_B-automorphic function $\mathrm{cf}(z)$ establishing an analytic isomorphism between $\mathcal{C}_B$ and $\mathbb{P}^1(\mathbb{C})$, satisfying

$$\mathrm{cf}(A) = 1, \quad \mathrm{cf}(B) = 0, \quad \mathrm{cf}(C) = \infty,$$

and such that the function field $\mathbb{C}(\mathcal{C}_B)$ is isomorphic to $\mathbb{C}(\mathrm{cf})$.

Proposition 1.3.

(i) $\mathrm{sf} \circ \tau = \mathrm{cf}$.

(ii) For some $r, s \in \mathbb{Z}$*, coprime with* N*, we have*

$$\mathrm{sf} \circ \alpha = \zeta_N^r \,\mathrm{sf}, \quad \mathrm{cf} \circ \beta = \zeta_N^s \,\mathrm{cf}.$$

(iii) $\mathbb{C}(\mathcal{C}) = \mathbb{C}(\mathrm{sf}^N) = \mathbb{C}(\mathrm{cf}^N)$.
(iv) For any $z \in \mathcal{C}$*,* $z \neq C$*, we have that*

$$\mathrm{sf}^N(z) + \mathrm{cf}^N(z) = 1.$$

Finally we define the group homomorphism

$$\begin{aligned} \varphi\colon \ \Delta &\longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \\ \alpha &\mapsto \ (1, 0) \\ \beta &\mapsto \ (0, 1), \end{aligned}$$

whose kernel is $\Gamma_N := [\Delta, \Delta]$. Let us denote by $\mathcal{C}_N = [\Delta, \Delta]\backslash\mathcal{D}_r$ the corresponding Riemann surface, and by $H_N = \Delta/[\Delta, \Delta]$ the automorphism group of $\mathcal{C}_N$ over $\mathcal{C}$. As representatives of the classes in H_N we can take the elements $\{\beta_i^j \alpha^i\}_{i,j=0}^{N-1}$. Put $\mathcal{Q}_{i,j} = \beta_i^j \alpha^i(\mathcal{Q}) = \beta_i^j(\mathcal{Q}_i) = \alpha^i(\mathcal{Q}^j)$. Then, it is clear that the polygon

$$\mathcal{P}_N = \cup_{i,j=0}^{N-1} \mathcal{Q}_{i,j}$$

is a fundamental domain for $\mathcal{C}_N$.

From now on, we shall consider all indices as integers modulo N. For $0 \le i \le N-1$, the quadrilaterals $\mathcal{Q}_{i,0}, \mathcal{Q}_{i,1}, \ldots, \mathcal{Q}_{i,N-1}$ form a $2N$-sided regular polygon $\mathcal{T}_i$, centered at point B_i. We label its vertices $C_{i,j}$, starting from point A_i and moving counterclockwise, so that $C_{i,2j} = \beta_i^j(A)$, $C_{i,2j+1} = \beta_i^j(C_i)$. Finally, we denote by $b_{i,j}$ the side of $\mathcal{T}_i$ which goes from point $C_{i,j}$ to point $C_{i,j+1}$. With this notation, the boundary of the polygon $\mathcal{P}_N$ is described by the sides

$$b_{0,1}, b_{0,2}, \ldots, b_{0,2N-2}; b_{1,1}, \ldots b_{1,2N-2}; \ldots; b_{N-1,1} \ldots b_{N-1,2N-2}.$$

The case $N = 5$ is illustrated in Figure 3.

Proposition 1.4. *We have that* $\mathcal{C}_N = F_N(\mathbb{C})$*. Thus,*

$$\mathbb{C}(F_N) = \mathbb{C}(\mathrm{sf}, \mathrm{cf}).$$

The functions sf, cf which uniformize the curve F_N are Γ-automorphic. Both of them define mappings $F_N(\mathbb{C}) \to \mathbf{P}^1(\mathbb{C})$. We have the following diagram:

$$\begin{tikzcd}
 & & \mathcal{C}_N \arrow[ddll] \arrow[ddrr] & & \\
 & & (x,y) \arrow[dl] \arrow[dr] & & \\
\mathcal{C}_A \arrow[dd] & x \arrow[d] & & y \arrow[d] & \mathcal{C}_B \arrow[dd] \\
 & x^N \arrow[rr] & & y^N & \\
\mathcal{C} \arrow[rrrr] & & & & \mathcal{C}
\end{tikzcd}$$

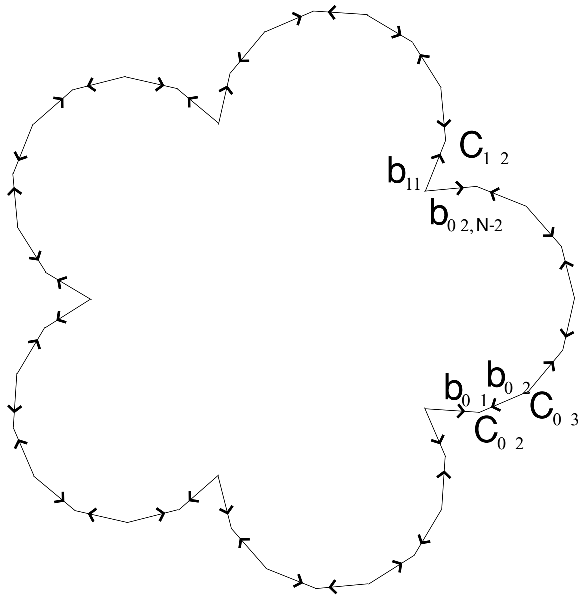


Fig. 3 Polygon $\mathcal{P}_N$

We may interpret the function sf as a Δ_A-automorphic function, and, similarly, the function cf as a Δ_B-automorphic function. This point of view allows us to interpret them as uniformizing functions

$$\mathrm{sf} : \mathcal{C}_A \to \mathbf{P}^1(\mathbb{C}), \quad \mathrm{cf} : \mathcal{C}_B \to \mathbf{P}^1(\mathbb{C}).$$

2 A Schwarzian differential equation

In this section, we shall determine a Schwarzian differential equation whose integration will provide the Δ-automorphic function sf^N.

Let $f(z)$ be a meromorphic function. We recall that the Schwarzian derivative $D_s(f(z), z)$ and the automorphic derivative $D_a(f(z), z)$ are defined by

$$D_s(f(z), z) := \frac{2f'(z)f'''(z) - 3f''(z)^2}{f'(z)^2},$$

$$D_a(f(z), z) := \frac{D_s(f(z), z)}{f'(z)^2}.$$

Let us write $w = f(z)$. Then the Schwarzian and automorphic derivatives are related by the formula

$$D_a(f(z), z) = -D_s(f^{-1}(w), w).$$

In order to determine the Schwarzian differential equation satisfied by the function sf^N, we take into account the isotropy groups of the vertices of the quadrilateral $\mathcal{Q}$ under the action of Δ.

Proposition 2.1. *The isotropy groups of the points A, B, C under the action of Δ are $\langle\alpha\rangle$, $\langle\beta\rangle$, $\langle\gamma\rangle$, respectively. All of them are cyclic groups of order N.*

This information is enough to allow us to write down the Schwarzian differential equation to be satisfied by sf^N (cf. [14]).

Proposition 2.2. *The function $\mathrm{sf}^N(z; N)$ is a solution of the differential equation:*

$$D_a(f(z), z) = -R(f(z)), \tag{2.1}$$

where

$$R(w) = \frac{1 - 1/N^2}{(w - w_A)^2} + \frac{1 - 1/N^2}{(w - w_B)^2} + \frac{m_A}{w - w_A} + \frac{m_B}{w - w_B}.$$

Here $w_A = 0$ and $w_B = 1$ are the values of sf^N at points A, B; m_A, m_B are two constants determined by the local conditions at point C, where sf^N takes the value ∞:

$$\left.\begin{array}{l} m_A + m_B = 0 \\ m_A w_A + m_B w_B + 1 - 1/N^2 = 0 \end{array}\right\}.$$

We have written the equation in terms of the values of sf^N at A, B to show how to write the equation corresponding to a different uniformizing parameter $\frac{a\,\mathrm{sf}^N + b}{c\,\mathrm{sf}^N + d}$, $ad - bc \neq 0$, for the curve $\mathcal{C}$.

It is straightforward to obtain that, with our choices, $m_A = -m_B = 1 - N^{-2}$. Thus,

$$R(w) = \frac{(N-1)(N+1)(w^2 - w + 1)}{N^2(w-1)^2 w^2}. \tag{2.2}$$

3 Taylor expansion of sf and cf

Now we formally integrate the Eq. (2.1). We shall write its solutions as series expansions around point A. Since the isotropy group $\langle\alpha\rangle$ at this point is cyclic of order N, a good local parameter at a neighbourhood of A is z^N. By imposing the initial condition $f(A) = 0$, the solutions we are looking for will be of the shape

$$f(z) = a_N z^N + a_{2N} z^{2N} + a_{3N} z^{3N} + a_{4N} z^{4N} + \dots$$

The direct substitution of $f(z)$ in Eq. (2.1) yields a computationally hard linear system of equations in the coefficients of $f(z)$. To overcome this difficulty, we perform a suitable change of variables. It is easy to see that the automorphic derivative satisfies the following chain rule:

$$D_a(f(q(z)), z) = D_a(f(q), q) + \frac{D_a(q, z)}{D(f(q), q)^2},$$

where the operator D denotes the usual derivative. When we apply this formula to the function $q(z) = z^N$, we obtain

$$D_a(q, z) = \frac{1 - N^2}{N^2} q^{-2}.$$

Hence,

$$\begin{aligned}
D_a(f(q(z)), z) = {} & \frac{1 - N^2}{N^2 a_N^2} q^{-2} + \frac{4(-1 + N^2) a_{2N}}{N^2 a_N^3} q^{-1} \\
& + \frac{6((2 - 4N^2) a_{2N}^2 + (-1 + 3N^2) a_N a_{3N})}{N^2 a_N^4} \\
& + \frac{4(8(-1 + 4N^2) a_{2N}^3 + 9(1 - 5N^2) a_N a_{2N} a_{3N} + 2(-1 + 7N^2) a_N^2 a_{4N})}{N^2 a_N^5} q \\
& + \dots
\end{aligned} \tag{3.1}$$

We equate the series in (3.1) to $R(f(q))$ and produce a homogeneous linear system for the coefficients a_{kN} which is computationally simpler. Its solutions are:

$$\begin{aligned}
a_{1N} &= a_N, \\
a_{2N} &= -\frac{a_N^2}{2}, \\
a_{3N} &= \frac{(1 + 11N^2) a_N^3}{2^4(-1 + 2N)(1 + 2N)}, \\
a_{4N} &= -\frac{3(1 + N^2) a_N^4}{2^4(-1 + 2N)(1 + 2N)}, \\
a_{5N} &= \frac{(-13 - 138N^2 + 1593N^4 + 718N^6) a_N^5}{2^8(-1 + 2N)^2(1 + 2N)^2(-1 + 4N)(1 + 4N)}, \\
a_{6N} &= -\frac{15(-5 + 42N^2 + 87N^4 + 20N^6) a_N^6}{2^9(-1 + 2N)^2(1 + 2N)^2(-1 + 4N)(1 + 4N)}, \\
&\ \ \vdots
\end{aligned}$$

We obtain in this way a parametric family of solutions $f(z) = f(z; a_N)$. By taking into account the initial condition $\mathrm{sf}^N(B) = 1$, we shall obtain a particular value λ_N of the parameter a_N for which

$$\mathrm{sf}^N(z; N) = \lambda_N z^N - \frac{\lambda_N^2}{2} z^{2N} + \frac{(1 + 11N^2)\lambda_N^3}{2^4(-1 + 2N)(1 + 2N)} z^{3N} + \cdots$$

Now we extract the Nth-root of $f(z; a_N)$ to deduce a series expansion $g(z; b_1)$ around point A such that

$$g(z; b_1)^N = f(z; a_N) = \sum_{j\geq 1} a_{jN} z^{jN}. \tag{3.2}$$

We write

$$g(z; b_1) = \sum_{k\geq 0} b_{kN+1} z^{kN+1},$$

and deduce by substitution in equality (3.2) a linear system of equations for the coefficients b-s. When we solve it, we obtain

$$g(z; b_1) = b_1 z - \frac{b_1^{N+1}}{2N} z^{N+1} + \frac{(-1+3N)(2+N)(1+N)b_1^{2N+1}}{2^4 N^2(1+2N)(-1+2N)} z^{2N+1} + \dots, \tag{3.3}$$

where $b_1^N = a_N$.

For a particular value μ_N of the parameter b_1, we shall have $\mathrm{sf}(z; N) = g(z; \mu_N)$ and $\lambda_N = \mu_N^N$. By performing analogous computations, we find the Taylor series around the point A of the function $\mathrm{cf}(z)$. Thus,

$$\mathrm{sf}(z; N) = \mu_N z - \frac{\mu_N^{N+1}}{2N} z^{N+1} + \frac{(-1+3N)(2+N)(1+N)\mu_N^{2N+1}}{2^4 N^2(1+2N)(-1+2N)} z^{2N+1} + \dots \tag{3.4}$$

$$\mathrm{cf}(z; N) = \sqrt[N]{1 - \mathrm{sf}^N(z)} = 1 - \mu_N^N z^N + \frac{\mu_N^{2N}}{2N^2} z^{2N} + \dots \tag{3.5}$$

We have determined both coordinate functions sf, cf up to the parameter μ_N. The appearance of this accessory parameter is not surprising because, up to now, we have not imposed the initial condition $\mathrm{sf}(B) = 1$. Since the function $\mathrm{sf}(z)$ has a pole at the point C, point B lies on the boundary of the convergence disk of the series (3.4). Hence, in order to obtain a good numerical approximation of μ_N, it is not advisable to compute many terms of the series $\mathrm{sf}(z)$ and then impose $\mathrm{sf}(B) = 1$. The determination of μ_N will be taken up in Section 5.

In fact, the indeterminacy of the coefficient b_1 reflects the random choice of the radius r of the disk $\mathcal{D}_r$ that we have taken to build up the fundamental domains for our curves. Or, perhaps better, it reflects the random choice of the conjugacy class of the fuchsian group Γ used to uniformize the curves.

The series (3.4), (3.5) provide a finer version of Proposition 1.3:

$$\begin{aligned} &\mathrm{sf}(\alpha(z)) = \zeta_N \mathrm{sf}(z), \qquad \mathrm{cf}(\alpha(z)) = \mathrm{cf}(z), \\ &\mathrm{sf}(\beta(z)) = \mathrm{sf}(z), \qquad \mathrm{cf}(\beta(z)) = \zeta_N \mathrm{cf}(z). \end{aligned} \tag{3.6}$$

In this way we recover the automorphisms $\alpha, \beta \in \mathrm{Aut}(F_N)$ quoted in the introduction. We also observe that the equality $\mathrm{sf}(\tau(z)) = \mathrm{cf}(z)$ allows the computation of the Taylor series of both functions around point B.

4 Inverse functions

The study of the multivalued inverse functions arc sf and arc cf deserves special attention. As is well known, Schwarzian differential equations are closely related to a certain class of ordinary linear differential equations of the second order. More precisely, the solutions of the differential equation

$$D_s(g(w), w) = R(w) \tag{4.1}$$

are the inverse functions of the solutions of the differential equation

$$D_a(f(z), z) = -R(f(z)) \tag{4.2}$$

and a closer examination reveals that the solutions of (4.1) are quotients of two linearly independent solutions of

$$u''(w) + \frac{1}{4}R(w)u(w) = 0. \tag{4.3}$$

The substitution $v(w) = s(w)u(w)$ transforms Eq. (4.3) into an equation of the type

$$v''(w) + P(w)v'(w) + Q(w)v(w) = 0, \tag{4.4}$$

where

$$P(w) = -2\frac{d}{dw}\log s(w), \qquad Q(z) = \frac{1}{4}R(w) + \frac{2s'(w)^2 - s(w)s''(w)}{s(w)^2}.$$

By a suitable election of $s(w)$, Eq. (4.4) turns out to be a hypergeometric equation. In our case, by taking $s(w) = (w(w-1))^{\frac{1-N}{2N}}$ we arrive at the equation

$$w(w-1)v''(w) + \frac{N-1}{N}(2w-1)v'(w) + \frac{(N-3)(N-1)}{4N^2}v(w) = 0. \tag{4.5}$$

The general solution, $v(w)$, of equation (4.5) is

$$c_1 F\left(\frac{N-1}{2N}, \frac{N-3}{2N}, \frac{N-1}{N}; w\right) + c_2 w^{\frac{1}{N}} F\left(\frac{N+1}{2N}, \frac{N-1}{2N}, \frac{N+1}{N}; w\right),$$

where

$$F(a,b,c;w)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tw)^{-a}dt$$
$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{w^n}{n!} \tag{4.6}$$

is the well known hypergeometric function.

Now we know that the inverse function arc $f(z;1)$ and a certain quotient of solutions of (4.5) must coincide, but it should be noted that we do not know which. When we study the values at three points of the fundamental solutions of the hypergeometric equation, we deduce that the solutions involved are those corresponding to the constants $c_1=0,\ c_2=1$ and $c_1=1,\ c_2=0$. The precise result is stated in the next proposition.

Proposition 4.1. *The inverse function* arc $f(z;1)$ *of* $f(z;1)$ *is the quotient of two hypergeometric functions:*

$$\operatorname{arc} f(z;1)(w)=w^{\frac{1}{N}}\frac{F\left(\frac{N+1}{2N},\frac{N-1}{2N},\frac{N+1}{N};w\right)}{F\left(\frac{N-1}{2N},\frac{N-3}{2N},\frac{N-1}{N};w\right)}$$
$$=w^{\frac{1}{N}}\left(1+\frac{1}{2N}w+\frac{(N+1)(13N^2-5N-2)}{16N^2(2N+1)(2N-1)}w^2\right.$$
$$\left.+\frac{(N+1)(23N^2-15N-2)}{96N^3(2N-1)}w^3+\dots\right). \tag{4.7}$$

The advantage of the expression in proposition (4.1) is that it provides an easy way of computing the series arc $f(z;1)$. This offers an alternative approach to the computation of $f(z;1)$.

An interesting feature of the particular hypergeometric Eq. (4.5) is that it can be reduced to a Legendre differential equation:

$$(1-w^2)y''(w)-2wy'(w)+\left(n(n+1)-\frac{m^2}{1-w^2}\right)y(w)=0. \tag{4.8}$$

For this purpose, we first make the change of variable $x=2w-1$, so that equation 4.5 becomes

$$(1-x^2)y''(x)-2\frac{N-1}{N}xy'(x)-\frac{(N-1)(N-3)}{4N^2}y(x)=0.$$

Substituting in this equation the quadrature $y(x)=t(x)y_1(x)$ in order to transform it into a Legendre equation, we see that a good choice is $t(x)=(x^2-1)^{1/(2N)}$. The reduced equation is a Legendre equation with parameters $n=\frac{1-N}{2N}$, $m=1/N$:

$$(1-x^2)y''(x)-2xy'(x)+\left(\frac{1-N^2}{4N^2}+\frac{1}{N^2(1-x^2)}\right)y(x)=0. \tag{4.9}$$

The general solution of Eq. (4.9) is

$$y(x) = c_1 P^{1/N}_{\frac{1-N}{2N}}(x) + c_2 Q^{1/N}_{\frac{1-N}{2N}}(x),$$

where $P^{1/N}_{\frac{1-N}{2N}}$, $Q^{1/N}_{\frac{1-N}{2N}}$ denote what are known as associated Legendre functions.

Proposition 4.2. *The inverse function* arc $f(z;1)$ *of* $f(z;1)$ *is the quotient of two associated Legendre functions:*

$$\operatorname{arc} f(z;1)(w) = \frac{P^{1/N}_{\frac{1-N}{2N}}(2w-1)}{Q^{1/N}_{\frac{1-N}{2N}}(2w-1)}.$$

5 Determination of μ_N

In order to determine the coefficient b_1 of series (3.4), (3.5), we shall use the condition $\mathrm{sf}(B) = 1$ and the results just stated. Thus

$$1 = \mathrm{sf}^N(B;N) = f(B;\lambda_N) = f(\mu_N B;1),$$

$$\mu_N B = \operatorname{arc} f(z;1)(1) = \frac{F\left(\frac{N+1}{2N}, \frac{N-1}{2N}, \frac{N+1}{N};1\right)}{F\left(\frac{N-1}{2N}, \frac{N-3}{2N}, \frac{N-1}{N};1\right)} = \frac{P^{1/N}_{\frac{1-N}{2N}}(1)}{Q^{1/N}_{\frac{1-N}{2N}}(1)}.$$

By taking into account the value of the hypergeometric function at 1 in terms of the Γ-function

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

we obtain

$$\mu_N B = \frac{\Gamma\left(\frac{N+1}{N}\right)\Gamma\left(\frac{N-1}{2N}\right)}{\Gamma\left(\frac{N-1}{N}\right)\Gamma\left(\frac{N+3}{2N}\right)}.$$

Since the number on the right hand side is real and B is equal to $r\sqrt{2\cos(\pi/N)-1}$, we deduce that the constant μ_N is real and that the radius $r = r_N$ and this constant are inversely proportional.

From the simple observation of the case $N = 2$ (cf. [3]), it may be inferred that for each value of N there exists a special value of B which is the most natural one to parametrize F_N. We take as B the length π_N of the one-dimensional simplex contained in $F_N(\mathbb{R})$ joining the points $(1, 0)$ and $(0, 1)$:

$$\pi_N := \int_0^1 \sqrt{1 + D(\mathrm{cf}(\mathrm{sf}), \mathrm{sf})^2}\, d(\mathrm{sf}) = \int_0^1 \sqrt{1 + \left(\frac{\mathrm{sf}}{\mathrm{cf}}\right)^{2N-2}}\, d(\mathrm{sf}).$$

Combining all these results, we obtain:

$$\mathrm{sf}(0) = 0, \ \mathrm{cf}(0) = 1; \quad \mathrm{sf}(\pi_N) = 1, \quad \mathrm{cf}(\pi_N) = 0,$$
$$r_N = \frac{\pi_N}{\sqrt{2\cos(\pi/N) - 1}}, \quad \mu_N = \frac{1}{\pi_N} \frac{\Gamma\left(\frac{N+1}{N}\right)\Gamma\left(\frac{N-1}{2N}\right)}{\Gamma\left(\frac{N-1}{N}\right)\Gamma\left(\frac{N+3}{2N}\right)}.$$

Appendix

In this appendix, we tabulate some constants, functions and differential forms attached to Fermat curves throughout the paper.

Table 1 Values of π_N, r_N, μ_N

N	π_N	r_N	μ_N
4	1.75442	2.72598	0.917155
5	1.79861	2.28787	0.835412
6	1.82943	2.13819	0.779984
7	1.85211	2.06822	0.740087
8	1.86949	2.03043	0.710054
9	1.88323	2.00823	0.686653
10	1.89435	1.99448	0.667917
11	1.90354	1.98568	0.652585
12	1.91127	1.97992	0.639808
13	1.91785	1.97613	0.629000
14	1.92352	1.97364	0.619738
15	1.92846	1.97203	0.611715
16	1.9328	1.97104	0.604697
17	1.93665	1.97049	0.598507
18	1.94007	1.97024	0.593007
19	1.94315	1.97021	0.588088
20	1.94593	1.97034	0.583662

Table 2 Taylor coefficients of $\mathrm{sf}(z; 4)$, $\mathrm{cf}(z; 4)$ at 0

n	$\mathrm{sf}(z; 4)$	n	$\mathrm{cf}(z; 4)$
1	1	0	1
5	$-\frac{15}{5!}$	4	$-\frac{6}{4!}$
9	$\frac{7425}{9!}$	8	$\frac{1260}{8!}$
13	$-\frac{18822375}{13!}$	12	$-\frac{2316600}{12!}$
17	$\frac{159120014625}{17!}$	16	$\frac{15081066000}{16!}$
21	$-\frac{3416758559589375}{21!}$	20	$-\frac{261570317580000}{20!}$
25	$\frac{154667733894382190625}{25!}$	24	$\frac{9957261810295800000}{24!}$
29	$-\frac{13152597869424682778484375}{29!}$	28	$-\frac{729754600219383538800000}{28!}$

Table 3 Reduced Taylor coefficients of sf(z; 37), cf(z; 37) at 0

n	sf(z; 37)	n	cf(z; 37)
1	1	0	1
38	$-\frac{1}{74}$	37	$-\frac{1}{37}$
75	$\frac{2717}{1998740}$	74	$\frac{1}{2738}$
112	$-\frac{13091}{147906760}$	111	$-\frac{10739}{73953380}$
149	$\frac{744697343}{166669797434672}$	148	$\frac{21113}{5472550120}$
186	$-\frac{61959482923}{154169562627071600}$	185	$-\frac{43945156471}{77084781313535800}$
223	$\frac{563138467716575}{1485757975523610357228 8}$	222	$\frac{279990543017}{11408547634403298400}$
260	$-\frac{5081316514596887}{24541537988559635364940 00}$	259	$-\frac{2158310701054223}{8589538295995872377729 00}$
297	$\frac{21158496912821252478247}{18396943935707987680699411155840 0}$	296	$\frac{682866283188190333}{5085006671229556447615568000}$
334	$-\frac{1118759058186208413686224 37}{1014223519175581360836958537021459 2000}$	333	$-\frac{698788932021482486947390 21}{5071117595877906804184792685107296000}$
371	$\frac{16215411705290449514944498325753}{19842515723540739801545393811531836343872000}$	370	$\frac{7093186512624366848393199 47}{93815675523741275877418664674484976 0000}$

Table 4 Reduced Taylor coefficients at 0 of a basis of $H^0(F_4(\mathbb{C}), \Omega^1)$

n	$w_{1,1}$(z; 4)	$w_{1,2}$(z; 4)	n	$w_{2,1}$(z; 4)
0	1	1	1	1
4	$\frac{1}{8}$	$-\frac{1}{8}$	5	0
8	$-\frac{3}{896}$	$-\frac{3}{896}$	9	$\frac{1}{672}$
12	$\frac{1}{21504}$	$-\frac{1}{21504}$	13	0
16	$\frac{3}{1605632}$	$\frac{3}{1605632}$	17	$-\frac{5}{1705984}$
20	$\frac{11}{93585408}$	$-\frac{11}{93585408}$	21	0
24	$\frac{4589}{1687532077056}$	$\frac{4589}{1687532077056}$	25	$-\frac{41}{131838443520}$
28	$-\frac{107}{86787363962880}$	$\frac{107}{86787363962880}$	29	0
32	$-\frac{1160753}{2812373458338447360}$	$-\frac{1160753}{2812373458338447360}$	33	$-\frac{39203}{32957501464903680}$
36	$-\frac{12444583}{491763587572322795520}$	$\frac{12444583}{491763587572322795520}$	37	0
40	$-\frac{116637329}{42838072517411230187520}$	$-\frac{116637329}{42838072517411230187520}$	41	$-\frac{56303}{1470168225346423357 44}$

References

1. Alsina. M., Bayer, P.: Quaternion orders, quadratic forms and shimura curves. CRM Monograph Series **22**, AMS (2004)
2. Bayer, P.: Uniformization of certain Shimura curves. In: T. Crespo and Z. Hajto (eds.), Differential Galois Theory, Banach Center Publications **58**, 13–26 (2002)
3. Bayer, P., Guàrdia, J.: A la recerca de pi, Butlletí de la Societat Catalana de Matemàtiques **17**, 7–19 (2002)
4. Farkas, H., Kra, I.: Riemann surfaces. Graduate Texts in Mathematics **71**, Springer, New York, NY (1992)
5. Ford, L.: Automorphic functions, (Chelsea, 1951)

6. Guàrdia, J.: A fundamental domain for the Fermat curves and their quotients. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales **94**, 391–396 (2000)
7. Hai Lim, C.: The Jacobian of a cyclic quotient of the Fermat curve. Nagoya Math. J. **125**, 73–92 (1992)
8. Lang, S.: Introduction to algebraic and abelian functions. Graduate Text in Mathematics **89**, (Springer, 1982)
9. Gross, B.: On the periods of abelian integrals and a formula of Chowla and Selberg. Inventiones math. **45**, 193–211 (1978)
10. Lehner, J.: Discontinuous groups and automorphic functions. AMS Mathematical Surveys **8**, Amer. Math. Society (1964)
11. Magnus, W.: Noneuclidean tesselations and their groups, Academic Press, 1974
12. Nehari, Z.: Conformal mapping. Dover Publications (1952)
13. Rohrlich, D.: The periods of the Fermat curve, appendix to Gross, B., On the periods of abelian integrals and a formula of Chowla and Selberg. *Inventiones Math.* **45**, 193–211 (1978)
14. Sansone, G., Gerretsen, J.: Lectures on the theory of functions of a complex variable. Wolters-Noordhoff Publishing (1969)
15. Schwarz, H. A.: Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine *algebraische* Function ihres vierten Elementes darstellt. J. reine u. angew. Mathematik **75**, 292–335 (1873)