GEODESICS, PERIODS, AND EQUATIONS OF REAL HYPERELLIPTIC CURVES

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Abstract
In this paper we start a new approach to the uniformization problem of Riemann surfaces and algebraic curves by means of computational procedures. The following question is studied: Given a compact Riemann surface S described as the quotient of the Poincaré upper half-plane divided by the action of a Fuchsian group, find explicitly the polynomial describing S as an algebraic curve (in some normal form). The explicit computation given in this paper is based on the numerical computation of conformal capacities of hyperbolic domains. These capacities yield the period matrices of S in terms of the Fenchel-Nielsen coordinates, and from there one gets to the polynomial via theta-characteristics. The paper also contains a list of worked-out examples and a list of examples—new in the literature—where the polynomial for the curve, as a function of the corresponding Fuchsian group, is given in closed form.

0. Introduction

The uniformization theorem of Koebe and Poincaré states that any Riemann surface has a universal covering conformally equivalent to either the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$, or the Poincaré upper half-plane $\mathbb{H}$. One of the consequences is that any smooth complex algebraic curve C of genus $g > 1$ is conformally equivalent to $\mathbb{H}/G$, where $G \subset \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian group. Conversely, any compact Riemann surface is isomorphic to an algebraic curve. Hence, any curve of genus $g > 1$ may be described in two ways, either by an equation or by a Fuchsian group. Going *explicitly* from one description to the other is, in either direction, a difficult problem. This is the classical uniformization problem. In this paper we study the direction from the Fuchsian groups to the curves. We also provide a large number of new examples, all in genus 2, where the correspondence is given in exact form.

Since $\mathbb{H}$ has a natural hyperbolic metric and this induces one on the corresponding curve C, one can reformulate the problem by asking how one relates explicitly

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equations for C with the hyperbolic metric on C. This is the approach we have taken, and we present here a method based on the computation of period matrices in terms of the hyperbolic metric.

If $\mathbb{H}/G$ has specific symmetries such that the associated algebraic curve is hyperelliptic and real, then the period matrix may be expressed in terms of conformal capacities of certain geodesic polygons. This reduces the computations to quite simple matters, and for this reason we have restricted this paper to real hyperelliptic curves, although the method is somewhat more general. For the conformal capacities we use approximation by harmonic polynomials, and combining this with Theta characteristics we obtain an equation of the curve in terms of the Fenchel-Nielsen coordinates of $\mathbb{H}/G$.

While experimenting with this method, we noticed a large number of examples that suggested that the correspondence between the Fenchel-Nielsen coordinates and the equation was expressible in exact form. For many of these we have proved that this is indeed the case by using a sort of uniformization "in families," that is, by exhibiting one-parameter families of curves which are algebraic with respect to the natural projective structure of the moduli space and which at the same time are defined by algebraic relations between the Fenchel-Nielsen coordinates. A list of exact correspondences and also some "possibly exact" correspondences, for which we have no proof, can be found in the final section.

1. Preliminaries

As a general reference we use [FK]; for more specific results on real curves we use [GH] or [SS], and for results on hyperbolic surfaces we use [Bu].

Let C be a smooth complex algebraic curve. Then C is naturally endowed with the structure of a Riemann surface. Conversely, any compact Riemann surface is conformally equivalent to an algebraic curve. By a classical theorem of Weil, C can be defined by real polynomial equations if and only if C admits an antiholomorphic involution σ. Moreover, the polynomials can be chosen such that σ is the involution induced by complex conjugation. Hence a real curve is a couple (C, σ), where C is a complex curve and σ is an antiholomorphic involution on C. We also say that σ is a real structure on C. If the antiholomorphic involution is clear, for example, if C is a curve already defined by polynomials with real coefficients and σ is complex conjugation, we simply say that C is a real curve. Note, however, that such a real curve may have additional real structures.

If C is of genus g, then the fixed-point set of σ, $C(\mathbb{R})$ can have at most $g+1$ connected components, and in this case we say that C is an M-curve. If C is an M-curve, then $C \smallsetminus C(\mathbb{R})$ has two connected components, each homeomorphic to a sphere with $g+1$ disks removed.

Let (C, σ) be a real curve of genus $g \geqslant 2$, and assume C is hyperelliptic with $C(\mathbb{R}) \neq \emptyset$. Then we can define (C, σ) by an equation of the form $y^2 = P(x)$, with P a real polynomial of degree either $2g+2$ or $2g+1$, with distinct roots and such that σ is induced by complex conjugation.

For such curves we note the following fairly trivial facts.

(i) (C, σ) is an M-curve if and only if all the roots of P are real.

(ii) C has two real structures. If the first is (C, σ), then the second is $(C, -\sigma)$, where $-\sigma$ is σ composed with the hyperelliptic involution. If an equation for the first is $y^2 = P(x)$, then an equation for the second is $y^2 = -P(x)$. We say that the real components of $(C, -\sigma)$ are the pure imaginary components of (C, σ). For M-curves the intersection points of the real and pure imaginary components are precisely the Weierstrass points of C.

By the uniformization theorem any algebraic curve of genus $g > 1$ carries a unique hyperbolic metric compatible with the underlying conformal structure. The terms geodesic and isometry should always be understood with respect to this metric.

The next lemma is the starting point of this paper.

LEMMA 1.1
Let (C, σ) be a real hyperelliptic M-curve of genus $g > 1$. Then the union of the real and pure imaginary components of (C, σ) separates C into four isometric geodesic right-angled $(2g+2)$-gons.

Conversely, let $\mathscr{P}$ be a hyperbolic geodesic right-angled $(2g+2)$-gon. Glue $\mathscr{P}$ and a mirror image of $\mathscr{P}$ so as to obtain a sphere $\mathscr{S}$ with $g+1$ disks removed. Let $\mathscr{S}'$ be a mirror image of $\mathscr{S}$. Glue $\mathscr{S}$ and $\mathscr{S}'$ in a way that preserves the mirror symmetries. Then the surface thus obtained is a real hyperelliptic M-curve.

Proof
Since σ is antiholomorphic, it is an orientation-reversing isometry. This implies that the real components of (C, σ) are simple closed geodesics. The same is true, of course, for the pure imaginary components. To see that they intersect at right angles, note that $-\sigma$ commutes with σ, so that σ fixes the pointwise invariant geodesics of $-\sigma$ and vice versa. As σ and $-\sigma$ are reflections along these sets, the sets intersect each other orthogonally. The rest of the first half of the lemma follows from the above considerations.

Call S the Riemann surface obtained in the second half. By construction S has 2 orientation-reversing symmetries, σ and σ'. Since by construction σ is an orientation-reversing isometry, it defines an antiholomorphic involution. Also by construction, the real curve (S, σ) has $g+1$ real components and hence is an M-curve. On the other hand, $\tau = \sigma \circ \sigma'$ defines a holomorphic involution. The fixed points of τ are the vertices of $\mathscr{P}$, and since there are $2g+2$ of these, τ is an hyperelliptic involution. $\square$

Remark 1.2

The preceding discussion does not extend immediately to genus-1 curves since an elliptic curve does not carry a natural hyperbolic metric. On the other hand, an elliptic curve with one point removed does, and we can do the following.

Let C be an elliptic curve defined by an equation of the form $y^2 = x(x-1)(x-a)$, $a \in \mathbb{R}$, and let p be the point at infinity. Then $\widetilde{C} = C \setminus \{p\}$ has a natural hyperbolic metric. The union of the real and pure imaginary components of C separate $\widetilde{C}$ into four isometric quadrangles with three right angles and one zero angle. (The corresponding vertex is a parabolic point.)

Conversely, starting with four copies of a hyperbolic quadrangle with three right angles and one zero angle, we can glue these so as to obtain an "ideal" pair of pants with two geodesic boundary components, β_1 and β_2, of equal length and a third of length zero. Gluing β_1 and β_2 with zero twist, we obtain an elliptic curve with one point p removed. For the same reasons as in the proof of Theorem 1.1, this curve has a real structure with two real components. Hence it has an equation of the form $y^2 = x(x-1)(x-a)$, $a \in \mathbb{R}$. Moreover, since by construction point p is at the intersection of a real and a pure imaginary component, we can choose a so that point p is the point at infinity.

2. The standard period matrices of real hyperelliptic M-curves

Let (C, σ) be a real hyperelliptic M-curve of genus g. Then by the considerations in Section 1, (C, σ) can be defined by an equation of the form

$$y^2 = P(x) = \prod_{i=1}^{2g+1} \big(x - x_i\big), \quad \text{with all } x_i \in \mathbb{R} \text{ and } x_1 < \cdots < x_{2g+1}. \tag{2.1}$$

We associate with it a standard form of the corresponding period matrices as follows.

Since P is nonzero in the upper half-plane $\mathbb{H}$ and the latter is simply connected, we can choose on $\mathbb{H}$ a determination of the square root $\sqrt{P(x)}$. Obviously we can extend this determination to $\mathbb{R}$ and even to strips below the $]x_i, x_{i+1}[$'s. We take the one that is positive on $[x_1, x_2]$. It is then negative on $[x_3, x_4]$, positive on $[x_5, x_6], \ldots,$ and so on; it is also pure imaginary with positive imaginary part on $]-\infty, x_1]$, pure imaginary with negative imaginary part on $[x_2, x_3], \ldots,$ and so on.

Let $\pi : C \to \mathbb{P}^1$ be the projection $(x, y) \mapsto x$. Let $\beta_i = \pi^{-1}([x_{2i-1}, x_{2i}])$ for $1 \leqslant i \leqslant g$, $\beta_{g+1} = \pi^{-1}([x_{2g+1}, \infty[)$, $\gamma_1 = \pi^{-1}(]-\infty, x_1])$, and $\gamma_{i+1} = \pi^{-1}([x_{2i}, x_{2i+1}])$ for $1 \leqslant i \leqslant g$. For $x \in \mathbb{H}$, the map $x \mapsto (x, \sqrt{P(x)})$ is a conformal inverse of π.

We choose on these cycles (which correspond to the real and pure imaginary components) the orientation defined by the map $x \mapsto (x, \sqrt{P(x)})$ and x increasing.

At the points $(x_i, 0)$, y is a local coordinate on C, and from this it is easy to compute the intersection numbers of the β_i's and γ_j's. We have $(\gamma_i \cdot \beta_i) = 1$ and $(\gamma_{i+1} \cdot \beta_i) = -1$, for $1 \leqslant i \leqslant g$, and $(\gamma_1 \cdot \beta_{g+1}) = -1$, all other intersection numbers being zero. Hence if $\alpha_i = -\sum_{k=1}^{i} \gamma_k$, then $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ defines a symplectic basis of $\mathrm{H}_1(C, \mathbb{Z})$, that is, one for which the intersection matrix is

$$\begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix},$$

where 1_g is the $g \times g$ identity matrix.

LEMMA 2.2

Let (C, σ) be a real hyperelliptic M-curve defined by an equation of the form in (2.1). Let $\sqrt{P(x)}$ be the above determination of the square root on $\mathbb{H}$, and set

$$A = \left(-\int_{-\infty}^{x_1} \frac{x^{j-1}\,dx}{\sqrt{P(x)}} - \sum_{k=1}^{i-1} \int_{x_{2k}}^{x_{2k+1}} \frac{x^{j-1}\,dx}{\sqrt{P(x)}} \right)_{i,j}, \qquad B = \left(\int_{x_{2i-1}}^{x_{2i}} \frac{x^{j-1}\,dx}{\sqrt{P(x)}} \right)_{i,j}.$$

Then $Z = A \cdot B^{-1}$ is a period matrix for C with $\Re e(Z) = 0$.

We say that a period matrix obtained in this way is a *standard period matrix* of (C, σ) or, more precisely, the standard period matrix associated to equation (2.1).

Proof

Let $\omega_j = x^{j-1}\,dx/y$. Then it is well known that the ω_j's form a basis of the space Ω^1_C of holomorphic 1-forms on C. Since $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$, α_i's and β_i's as above, is a symplectic basis of $\mathrm{H}_1(C, \mathbb{Z})$, this means that if we replace A and B by $(\int_{\alpha_i} \omega_j)_{i,j}$ and $(\int_{\beta_i} \omega_j)_{i,j}$, Z is a period matrix of C. But clearly

$$\int_{\alpha_i} \omega_j = -2 \int_{-\infty}^{x_1} \frac{x^{j-1}\,dx}{\sqrt{P(x)}} - 2 \sum_{k=1}^{i-1} \int_{x_{2k}}^{x_{2k+1}} \frac{x^{j-1}\,dx}{\sqrt{P(x)}} \qquad \text{and}$$

$$\int_{\beta_i} \omega_j = 2 \int_{x_{2i-1}}^{x_{2i}} \frac{x^{j-1}\,dx}{\sqrt{P(x)}}.$$

Finally, we note that B is real while A is pure imaginary, and this ends the proof. □

LEMMA 2.3

Let (C, σ) be as in Lemma 2.2, and let the β_i's be as above. Let

$$y^2 = \prod_{i=1}^{2g+1} \left(x - x_i'\right), \quad \textit{with } x_i' \in \mathbb{R} \textit{ and } x_i' < x_{i+1}',$$

be another equation defining (C, σ)*, and let* $\beta_1', \ldots, \beta_{g+1}'$ *be constructed in the same way as the* β_i*'s, but from the* x_i'*'s.*

Then the ordered set $\{\beta_1', \ldots, \beta_{g+1}'\}$ *is either a cyclic permutation of the ordered set* $\{\beta_1, \ldots, \beta_{g+1}\}$ *or a cyclic permutation of* $\{-\beta_{g+1}, \ldots, -\beta_1\}$.

Proof

To simplify notation, call (C', σ') the curve defined by the second equation. The hypothesis is now that (C, σ) and (C', σ') are isomorphic. Let $\psi : (C, \sigma) \to (C', \sigma')$ be an isomorphism. Then, since the hyperelliptic involution is in the centre of the automorphism group of an hyperelliptic curve, ψ also induces an isomorphism between $(C, -\sigma)$ and $(C', -\sigma')$. This implies that ψ sends real components to real components *and* pure imaginary components to pure imaginary components. Since an isomorphism respects the intersection form, this means that if ψ transforms β_i into β_j' (resp., $-\beta_j'$), then it must transform γ_i into γ_j' or $-\gamma_{j+1}'$ (resp., $-\gamma_j'$ or γ_{j+1}'), where the γ_i''s are again constructed in the same way as the γ_i's and $\gamma_{g+2} = \gamma_1$. Hence the cyclic order is either respected or reversed. To see that in the first case the orientations are preserved while they are reversed in the latter, recall how the cycles are oriented and recall the fact that ψ is induced by a projective transformation of $\mathbb{P}^1$, taking the x_i's to the x_i''s. (Hence the only possibilities are, in fact, β_i goes to β_j' and γ_j goes to γ_j' or β_i goes to $-\beta_j'$ and γ_j goes to $-\gamma_{j+1}'$.) □

COROLLARY 2.4

Let (C, σ) *be as in Lemma 2.2, and let* Z *and* Z' *be two standard period matrices of* (C, σ)*. Then* $Z = MZ'\,{}^tM$*, where* M *is in* $\mathfrak{G}_g$*, the subgroup of* $\mathrm{GL}_g(\mathbb{Z})$ *generated by the matrices*

$$N_1 = -\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & & & 1 & 0 \\ 0 & 0 & 1 & & & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \textit{and} \quad N_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & & 0 & -1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & -1 \end{pmatrix}.$$

Proof

Let ψ, the β_i''s, and the γ_i''s be as in the proof of Lemma 2.3.

Now recall that by construction we have $\beta_{g+1} = -\beta_1 - \beta_2 - \cdots - \beta_g$ and $\gamma_{g+1} = -\gamma_1 - \gamma_2 - \cdots - \gamma_g$. This, together with Lemma 2.3, implies that the matrix of ψ_* (the induced mapping), in the bases $\{\alpha_i, \beta_i\}$ and $\{\alpha'_i, \beta'_i\}$, is of the form

$$\begin{pmatrix} {}^tN^{-1} & 0 \\ 0 & N \end{pmatrix}$$

for a matrix N in $\mathfrak{G}_g$.

Applying this, with $M = N^{-1}$, to the construction of Z, we have Corollary 2.4. □

Remarks 2.5

(i) It is easily seen that the group $\mathfrak{G}_g$ is isomorphic to the dihedral group D_{g+1}.

(ii) Let $Z = (z_{ij})_{i,j}$. Then the diagonal elements of $N_2 Z\,{}^tN_2$ are

$$z_{gg}, \quad z_{11} - 2z_{1g} + z_{gg}, \quad z_{22} - 2z_{2g} + z_{gg}, \quad \ldots, \quad z_{g-1,g-1} - 2z_{g-1,g} + z_{gg}.$$

(Recall that Z is symmetric.) Those of $N_2^2 Z\,{}^tN_2^2$ are

$$z_{g-1,g-1} - 2z_{g-1,g} + z_{gg}, \quad z_{g-1,g-1}, \quad z_{11} - 2z_{1,g-1} + z_{g-1,g-1}, \quad \ldots,$$

and so forth.

This means that we can recover the coefficients of Z from the diagonal elements of the matrices $N_2^k Z\,{}^tN_2^k$.

3. Period matrices of real hyperelliptic M-curves in terms of capacities

Let (C, σ) again be a real hyperelliptic M-curve defined by an equation of the form in (2.1), and let notation be as in Section 2.

Let B and Z be as in Lemma 2.2, and write ${}^tB^{-1} = (c_{ij})_{i,j}$ and $Z = (z_{ij})_{i,j}$.

Let

$$f_i : z \longmapsto \int_{x_1}^{z} \frac{\sum c_{ij} x^{j-1}\,dx}{\sqrt{P(x)}}.$$

By the choice of the determination of $\sqrt{P(x)}$, this defines holomorphic functions f_i on the simply connected domain formed by $\mathbb{H}$ and the vertical strips below the $]x_i, x_{i+1}[$'s.

By the choice of the c_{ij}'s we have

$$f_i(x_{2i}) - f_i(x_{2i-1}) = 1, \qquad f_i(x_{2j}) - f_i(x_{2j-1}) = 0, \quad \text{for } 1 \leqslant i, j \leqslant g \text{ and } i \neq j. \tag{3.1}$$

Recalling the definition of the α_i's and the fact that $\gamma_1 = -\gamma_2 - \cdots - \gamma_{g+1}$, we also have

$$\sum_{j=i}^{g} f_i(x_{2j+1}) - f_i(x_{2j}) = z_{ii}. \tag{3.2}$$

Define $u_i = \Re e(f_i)$ and $v_i = \Im m(f_i)$. These are harmonic functions on $\mathbb{H}$. To obtain the boundary behaviour of the u_i's, note that since $\sqrt{P(x)}$ is pure imaginary on the intervals $[x_{2i}, x_{2i+1}]$ and $]-\infty, x_1]$, we have by (3.1),

$$\begin{aligned} u_i(x) &= 0 \quad \text{on }]-\infty, x_1] \text{ and on } \left[x_{2j}, x_{2j+1}\right], \quad \text{for } j < i, \\ u_i(x) &= 1 \quad \text{on } \left[x_{2j}, x_{2j+1}\right], \quad \text{for } i \leqslant j. \end{aligned} \tag{3.3}$$

Writing $x = \xi + \sqrt{-1}\zeta$ with ξ and ζ real, we have from the Cauchy-Riemann equations

$$\frac{\partial u_i(x)}{\partial \zeta} = -\Im m\left(\frac{\partial f_i(x)}{\partial \xi}\right). \tag{3.4}$$

This holds, in particular, for $x \in]x_{2k-1,2k}[$, $k = 1, \dots, g$ and $x \in]x_{2g+1}, \infty[$. Since the imaginary part of f_i is constant on these intervals, we get

$$\frac{\partial u_i(x)}{\partial \zeta} = 0 \quad \text{for } x \in]x_{2k-1}, x_{2k}[\text{ and }]x_{2g+1}, \infty[. \tag{3.5}$$

Now recall that the capacity of a harmonic function h on a domain M is $c = \int_M \|\nabla h\|^2 dM$ and that by Green's theorem we can rewrite this as

$$c = \int_{\partial M} h \cdot \nu[h]\, d\mu,$$

where $\nu[h]$ is the derivative of h with respect to the outward pointing unit normal vector field. In our situation, where $M = \mathbb{H}$, the upper half-plane, and $h = u_i$, the corresponding capacity is

$$c_i = \int_{\mathbb{H}} \|\nabla u_i\|^2\, d\xi\, d\zeta = -\int_{\mathbb{R}} u_i(t) \cdot \frac{\partial u_i}{\partial \zeta}(t)\, dt. \tag{3.6}$$

Applying this and recalling (3.2), (3.3), (3.4), and (3.5), we find the following proposition.

PROPOSITION 3.7
Let (C, σ) be defined by $y^2 = \prod(x - x_i)$, $1 \leqslant i \leqslant 2g+1$, $x_i < x_{i+1}$. Let $Z = \sqrt{-1}(y_{ij})_{ij}$ be the standard period matrix associated to this equation, as in Lemma 2.2.

Let c_i be the capacity introduced in (3.6) for u_i, harmonic, satisfying conditions in (3.3) and (3.5).

Then, for $1 \leqslant i \leqslant g$, $c_i = y_{ii}$.

COROLLARY 3.8
Let (C, σ) be as in Proposition 3.7, and let $\mathscr{P}$ be the hyperbolic $(2g+2)$-gon

associated to (C,σ) as in Lemma 1.1. Then a standard period matrix of (C,σ) can be computed in terms of capacities of harmonic functions with mixed boundary conditions on $\mathscr{P}$.

Proof

There is a conformal map sending $\mathscr{P}$ to $\mathbb{H}\cup\{\infty\}$ in such a way that its vertices go to $x_1, x_2, \ldots, x_{2g+1}, \infty$. We label the sides of $\mathscr{P}$ successively $\widetilde{\beta}_1, \widetilde{\gamma}_1, \widetilde{\beta}_2, \ldots, \widetilde{\gamma}_{g+1}$, such that $\widetilde{\beta}_1$ goes to the interval $[x_1, x_2]$, $\widetilde{\gamma}_1$ to $[x_2, x_3]$, and so on.

We observe that y_{ii} is also the capacity c_i of the harmonic function h_i on $\mathscr{P}$ which is uniquely defined by the boundary conditions

$$\begin{aligned} \nu[h_i] &= 0 \quad \text{on } \widetilde{\beta}_1, \ldots, \widetilde{\beta}_{g+1}, \\ h_i &= 0 \quad \text{on } \widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_i, \\ h_i &= 1 \quad \text{on } \widetilde{\gamma}_{i+1}, \ldots, \widetilde{\gamma}_{g+1}. \end{aligned}$$

Now let $\kappa \in \{1, \ldots, g\}$, and set for $i = 1, \ldots, g+1$, $\widetilde{\beta}_i[\kappa] = \widetilde{\beta}_{i-\kappa}$, $\widetilde{\gamma}_i[\kappa] = \widetilde{\gamma}_{i-\kappa}$ (subscripts modulo $g+1$). This is just a cyclic renumbering of the sides. It induces a renumbering $\beta_i[\kappa] = \beta_{i-\kappa}$, $\gamma_i[\kappa] = \gamma_{i-\kappa}$, $i = 1, \ldots, g+1$, and provides the symplectic basis $\{\alpha_1[\kappa], \ldots, \alpha_g[\kappa], \beta_1[\kappa], \ldots, \beta_g[\kappa]\}$ with corresponding standard period matrix $Z[\kappa]$. A simple check using the relation $\beta_{g+1}[\kappa] = -\beta_1[\kappa] - \cdots - \beta_g[\kappa]$ shows that $\beta_j = \sum n_{ij}[\kappa]\beta_i[\kappa]$ with $(n_{ij})_{i,j} = N_2^\kappa$. Hence $Z[\kappa] = N_2^\kappa Z\, {}^t N_2^\kappa$.

The diagonal elements of $Z[\kappa] = \sqrt{-1}(y_{ij}[\kappa])_{i,j}$ are the capacities $c_i[\kappa]$ of the harmonic functions $h_i[\kappa]$ on $\mathscr{P}$ which are defined as the h_i's but with respect to the numbering $\widetilde{\beta}_1[\kappa], \widetilde{\gamma}_1[\kappa], \ldots$; that is, the $h_i[\kappa]$'s are the harmonic functions with respect to the boundary conditions

$$\begin{aligned} \nu[h_i[\kappa]] &= 0 \quad \text{on } \widetilde{\beta}_1, \ldots, \widetilde{\beta}_{g+1}, \\ h_i[\kappa] &= 0 \quad \text{on } \widetilde{\gamma}_{1-\kappa}, \ldots, \widetilde{\gamma}_{i-\kappa}, \\ h_i[\kappa] &= 1 \quad \text{on } \widetilde{\gamma}_{i+1-\kappa}, \ldots, \widetilde{\gamma}_{g+1-\kappa} \end{aligned} \tag{3.9}$$

(subscripts modulo $g+1$). Using Remark 2.5, we can easily compute the coefficients y_{ij}'s from these data. Explicitly, we have for $1 \leqslant i < j \leqslant g$,

$$2y_{ij} = y_{ii} + y_{jj} - y_{j-i,j-i}[g+1-i]. \qquad \square$$

Remark 3.10

All of the results of this section generalize to the case $g = 1$ in even simpler form since for $g = 1$ we only have to compute one capacity of the quadrangle introduced in Remark 1.2 to obtain the period of the corresponding real elliptic curve.

4. Computing the equation for the curve associated to a hyperbolic $(2g+2)$-gon
We detail in this section the method for computing an equation of the curve associated to a $(2g+2)$-gon as in Lemma 1.1 and Remark 1.2. The numerical part in our approach reduces to the computation of the conformal capacities of plane domains.

After testing various finite element methods based on Euclidean and hyperbolic triangulations, we found that the approximation of a harmonic function by harmonic polynomials—that is, functions of the form $\sum r^k(a_k\cos(kx)+b_k\sin(kx))$—is by far the fastest and the most accurate method. Moreover, it is simple to implement and uses only standard subroutines. For the achieved accuracy we refer to Section 8.

The practical computation of an approximating harmonic polynomial on a polygon domain with respect to given boundary conditions is as follows. First we realize the polygon as a domain in the unit disk and distribute a finite number of points along the boundary, more or less equidistantly (with respect to the Euclidean metric). Imposing that the polynomial satisfy the boundary conditions in these points gives us a family of linear equations in the a_k's and b_k's. We overdetermine the system and solve it in the sense of least squares using a standard SVD (singular values decomposition) subroutine. Using the Green-Riemann formula, we get the capacity of the function directly out of the a_k's and b_k's.

From the capacities we can, using the method described in Section 3, recover a standard period matrix Z of the curve. Since the curve is hyperelliptic, we can use Theta characteristics to obtain an equation of the curve (see, e.g., [FK, pp. 348–350]).

For genus 2 the domain is a hyperbolic hexagon $\mathscr{H}$ in the unit disk with sides labeled $l_1, \widehat{l_3}, l_2, \widehat{l_1}, l_3, \widehat{l_2}$, in that order (see Figure 1).

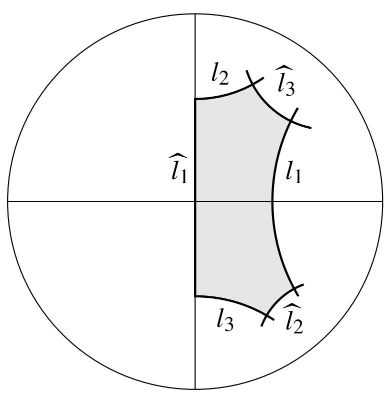

Figure 1

We assume that the sides $\{l_i\}$ (resp., $\{\widehat{l_i}\}$) are mapped by the inclusion $\mathscr{H} \hookrightarrow C$ into the real and pure imaginary components, respectively. The three capacities c_1, c_2, and c_3 are computed corresponding, respectively, to u_1, which is zero on $\widehat{l_1}$, 1 on $\widehat{l_2}$ and $\widehat{l_3}$; u_2, which is 1 on $\widehat{l_2}$, zero on $\widehat{l_1}$ and on $\widehat{l_3}$; and u_3, which is 1 on $\widehat{l_3}$, zero on $\widehat{l_1}$ and on $\widehat{l_2}$. From Section 3 we conclude that the matrix

$$Z = i \begin{pmatrix} c_1 & \frac{1}{2}(c_1 + c_2 - c_3) \\ \frac{1}{2}(c_1 + c_2 - c_3) & c_2 \end{pmatrix}$$

is a period matrix of C.

To compute an equation for C, let

$$\vartheta \begin{bmatrix} 2\alpha \\ 2\beta \end{bmatrix} (Z) = \sum_{n \in \mathbb{Z}^g} \exp\left(\pi i\, {}^t(n+\alpha) Z (n+\alpha) + 2\pi i\, {}^t(n+\alpha)\beta\right)$$

for 2α and 2β in $\mathbb{Z}^g$, and set

$$\begin{aligned} x_1 &= \left(\frac{\vartheta\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right](Z) \cdot \vartheta\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right](Z)}{\vartheta\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right](Z) \cdot \vartheta\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right](Z)} \right)^2, \\ x_2 &= \left(\frac{\vartheta\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right](Z) \cdot \vartheta\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right](Z)}{\vartheta\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right](Z) \cdot \vartheta\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right](Z)} \right)^2, \\ x_3 &= \left(\frac{\vartheta\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right](Z) \cdot \vartheta\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right](Z)}{\vartheta\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right](Z) \cdot \vartheta\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right](Z)} \right)^2. \end{aligned} \tag{4.1}$$

Then $y^2 = x(x-1)(x-x_1)(x-x_2)(x-x_3)$ is an equation for C (see, e.g., [FK, pp. 348–350]).

The generalization of this method for $g > 2$ is straightforward.

Actually, this method also works in the hyperbolic genus-1 case as considered in Remark 1.2 and is in fact even simpler.

Let $\mathcal{Q}$ be a hyperbolic quadrangle with three right angles and with one zero angle, that is, with one vertex at infinity. We denote by l and $\widehat{l}$ the two sides of $\mathcal{Q}$ with finite length.

Let C be the real curve associated with $\mathcal{Q}$ such that the inclusion map $\mathcal{Q} \hookrightarrow C$ sends l and $\widehat{l}$ upon a real and a pure imaginary component, respectively.

We let τ be the capacity of the harmonic function u on $\mathcal{Q}$ which is zero on $\widehat{l}$, 1 on the opposite side of $\widehat{l}$, and which has zero normal derivatives on the remaining two sides.

By the considerations of Section 3, there is a conformal map ψ_1 sending $\mathcal{Q}$ onto the rectangle $(0, 1, 1 + i\tau, i\tau)$ and sending vertices to vertices. Let

$$x_1 = \frac{\exp(\pi\tau)}{16} \prod_{n=1}^{\infty} \frac{\left(1 + \exp\left(-(2n-1)\pi\tau\right)\right)^8}{\left(1 + \exp(-2n\pi\tau)\right)^8}. \tag{4.2}$$

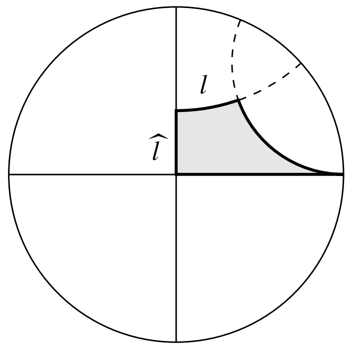

Figure 2

Then it is well known (see, e.g., Z. Nehari [Ne]) that there exists a conformal map ψ_2 sending the rectangle $(0, 1, 1 + i\tau, i\tau)$ onto the upper half-plane and such that $\psi_2(0) = 0$, $\psi_2(1) = 1$, $\psi_2(1 + i\tau) = x_1$, and $\psi_2(i\tau) = \infty$.

But this means that $y^2 = x(x-1)(x-x_1)$ is an equation for C, and that if h is the projection $(x, y) \mapsto x$, then the composition

$$\mathcal{Q} \hookrightarrow C \xrightarrow{h} \mathbb{P}^1$$

is equal to $\psi_2 \circ \psi_1$ and maps $\mathcal{Q}$ conformally onto the upper half-plane $\mathbb{H}$.

For the numerical computation of τ, we place $\mathcal{Q}$ as in Figure 2. Let, as before, a_k and b_k be the coefficients of the harmonic polynomial. Then all b_k's and also all a_{2k}'s are zero and the capacity becomes

$$\tau = \sum (-1)^k a_{2k+1} |p|^{2k+1},$$

where p is the nonzero end point of side $\widehat{l}$. In this case the computation is particularly fast and efficient.

With a variant of this method we can also handle "half-twists." We describe this here in the case $g = 1$; for $g = 2$ we refer to the end of Section 7.

We first need some remarks on real elliptic curves with one real component.

Let C_1 be the elliptic curve defined by the lattice Λ_1 generated by 1 and $1/2 + i\mu$, $\mu \in \mathbb{R}$, $\mu > 0$. Then C_1 is a real curve with one real component, the image in C_1 of the horizontal lines $\mathbb{R} + ni\mu$, $n \in \mathbb{Z}$. C_1 has also one pure imaginary component, the image of the vertical lines $n/2 + i\mathbb{R}$, $n \in \mathbb{Z}$. We note also that we can normalize equations of such curves in the form $y^2 = (x^2 + 1)(x^2 - \alpha)$, $\alpha \in \mathbb{R}$, $\alpha > 0$.

Now let C_2 be the elliptic curve defined by the lattice Λ_2 generated by $1/2$ and $i\mu$. The inclusion $\Lambda_1 \subset \Lambda_2$ yields a double covering of C_2 by C_1. Such a double covering is easy to describe in terms of equations. If the equation of C_1 is as above, then the equation of C_2 is $y^2 = x(x+1)(x-\alpha)$, the covering map being $(x, y) \mapsto (x^2, xy)$.

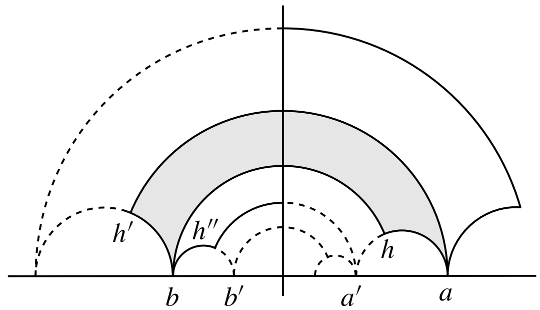

Figure 3

Of course, the procedure can be reversed, and this is what we are going to do.

Thus, consider again the quadrangles and the "ideal" pair of pants described in Remark 1.2, but assume this time that we glue the two boundary components β_1 and β_2 with a half-twist. We again obtain a torus with one point removed. It also has an obvious orientation-reversing symmetry with a connected fixed-point set and hence corresponds to a real genus-1 curve with one real component. Call C this curve.

Looking at two copies of the quadrangle, the situation is as in Figure 3. Note that in the corresponding Riemann surface the points $a, a', b, b', \ldots$ are identified and so are the points $h, h', h'', \ldots$. For the orientation-reversing symmetry we can take the one induced by reflection along the (h, b) geodesic, which is the same as the one induced by taking reflection along the (a, h') geodesic. In this case the real part consists of the images of these two geodesic arcs. The pure imaginary components consist of the images of the geodesic arcs (a, h) and (b, h'). But now we are exactly in the situation described above. (In terms of fundamental parallelograms the situation is as in Figure 4.)

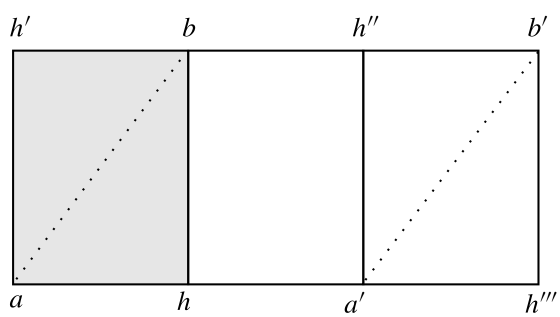

Figure 4

In particular, if τ is the capacity of a harmonic function on the shaded area a, h, b, h', with boundary conditions zero on (a, h') and 1 on (b, h) and zero normal derivative on the two remaining sides, then the period of the curve C obtained with a half-twist is $1/2+i(\tau/2)$. Using the Legendre map we can easily find from this period an equation for C. We can also use the above remarks to simplify the computations. Namely, let x_1 be associated to τ as in (4.1), and let $\alpha = x_1 - 1$. Then an equation of C is $y^2 = (x^2+1)(x^2-\alpha)$.

5. A D_5 action on a subspace of the real genus-2 moduli space

In Section 6 we construct various examples of real genus-2 curves with three real components for which the uniformization is given exactly. The construction uses group actions and curve families which have an explicit algebraic description in terms of the coefficients of equations of algebraic curves. At the same time these actions and curve families have an explicit description that is algebraic in terms of the Fenchel-Nielsen coordinates. In this section we describe the group actions.

We use the following notation. The group of all orientation-preserving automorphisms of a real curve $C = (C, \sigma)$ preserving the real structure σ is denoted by $\mathrm{Aut}^+_\sigma(C)$. The real moduli space of the real genus-2 curves with three components is denoted by $\mathscr{M}^{(2,3,0)}_{\mathbb{R}}$. As usual we write $C \in \mathscr{M}^{(2,3,0)}_{\mathbb{R}}$ to say that the isomorphism class of C belongs to $\mathscr{M}^{(2,3,0)}_{\mathbb{R}}$. For any finite group G we let $\mathscr{M}^{(2,3,0)}_{\mathbb{R}}(G)$ denote the subspace of all $C \in \mathscr{M}^{(2,3,0)}_{\mathbb{R}}$ with $G \subset \mathrm{Aut}^+_\sigma(C)$.

Let $\mathscr{P}$ be the moduli space of pairs of pants. Taking for any $P \in \mathscr{P}$ the Schottky double, we get a Riemann surface S_P with a real structure.

This sets up a natural isomorphism between $\mathscr{P}$ and $\mathscr{M}^{(2,3,0)}_{\mathbb{R}}$, and we identify $\mathscr{P}$ with $\mathscr{M}^{(2,3,0)}_{\mathbb{R}}$.

An element P in $\mathscr{P}$ consists of two copies of a hyperbolic geodesic hexagon. The lengths of the sides of the hexagon listed in cyclic order are denoted by $l_1, \widehat{l_3}, l_2, \widehat{l_1}, l_3, \widehat{l_2}$. We also denote by h_i the common orthogonal between l_i and $\widehat{l_i}$.

The hexagon is determined up to isometry by l_1, l_2, l_3. The lengths of the remaining quantities are given by the following formulas, where we abbreviate $u_i = \cosh(l_i)$, $i = 1, 2, 3$:

$$\cosh^2\left(\widehat{l_i}\right) = \frac{\left(u_i + u_{i-1}u_{i+1}\right)^2}{\left(u_{i-1}^2 - 1\right)\left(u_{i+1}^2 - 1\right)} \tag{5.1}$$

(subscripts modulo 3),

$$\cosh^2(h_i) = \frac{u_1^2 + u_2^2 + u_3^2 + 2u_1u_2u_3 - 1}{u_i^2 - 1}. \tag{5.2}$$

P is obtained by pasting the hexagons together along the sides $\widehat{l_i}$ and has boundary geodesics of lengths $2l_1$, $2l_2$,$2l_3$. We use the unordered triple $\{l_1, l_2, l_3\}$ as a set of coordinates for $P \in \mathscr{P}$. Since the pasting that gives S_P has zero twist, $\{l_1, l_2, l_3\}$ also serves as a set of coordinates for S_P.

Equations for the elements in $\mathscr{M}_{\mathbb{R}}^{(2,3,0)}$ are written either in the form

$$y^2 = x(x-1)(x-x_1)(x-x_2)(x-x_3) \tag{5.3}$$

as earlier, with x_1, x_2, x_3 real, or in the form

$$y^2 = (x-d)(x+1)(x-c)(x-a)(x-1)(x-b), \; d < -1 < c < a < 1 < b. \tag{5.4}$$

Now let $\mathscr{P}_2$ be the subspace of $\mathscr{P}$ formed by the pairs of pants with two boundary components of equal length. From the point of view of hyperbolic geometry, $\mathscr{P}_2$ may be characterized as the set of elements in $\mathscr{P}$ with coordinates $\{l_1, l_2, l_2\}$. To describe $\mathscr{P}_2$ in terms of coefficients of equations, we note that any $P \in \mathscr{P}_2$ has an orientation-preserving involution with one fixed point. It induces an involution of S_P with exactly two fixed points. Since there is also the hyperelliptic involution, we have $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset \mathrm{Aut}_\sigma^+(S_P)$. Conversely, any $C \in \mathscr{M}_{\mathbb{R}}^{(2,3,0)}$ with $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset \mathrm{Aut}_\sigma^+(C)$ is obtained in this way. Hence, under the above identification of $\mathscr{P}$ with $\mathscr{M}_{\mathbb{R}}^{(2,3,0)}$, the subspace $\mathscr{P}_2 \subset \mathscr{P}$ is identified with $\mathscr{M}_{\mathbb{R}}^{(2,3,0)}(\mathbb{Z}/2 \times \mathbb{Z}/2)$ and we identify $\mathscr{P}_2$ with $\mathscr{M}_{\mathbb{R}}^{(2,3,0)}(\mathbb{Z}/2 \times \mathbb{Z}/2)$.

In (5.4) we may choose the constants such that the fixed points of the nonhyperelliptic involution are $(0, \pm y)$. This allows us to normalize equations of elements in $\mathscr{P}_2$ in the form

$$y^2 = \left(x^2 - a\right)\left(x^2 - 1\right)\left(x^2 - b\right), \quad \text{with } 0 < a < 1 < b. \tag{5.5}$$

Using Lemma 2.3, it is straightforward to show that two curves C, C' with equations in this form are real isomorphic if and only if $a = a'$ and $b = b'$. Each element in $\mathscr{P}_2$ is therefore represented by a unique equation, so that we have here a one-to-one correspondence from $\mathscr{P}_2$ to the set of all $(a, b) \in \mathbb{R}^2$, with $0 < a < 1 < b$.

Our first action is $\varphi : \mathscr{P} \to \mathscr{P}$ defined by $\varphi : \{l_1, l_2, l_3\} \mapsto \{\widehat{l_1}, \widehat{l_2}, \widehat{l_3}\}$. From the point of view of equations, this corresponds to replacing the curve defined by $y^2 = P(x)$ by the one defined by $y^2 = -P(x)$. If the equation of the curve is written in the form of (5.4), then this action can also be described by $(a, b, c, d) \mapsto (1/b, 1/a, 1/d, 1/c)$ or by $(a, b, c, d) \mapsto (-1/c, -1/d, -1/a, -1/b)$ (which in certain cases is more useful). This defines an algebraic action on $\mathscr{M}_{\mathbb{R}}^{(2,3,0)}$. Of course, φ restricts to an action on $\mathscr{P}_2 = \mathscr{M}_{\mathbb{R}}^{(2,3,0)}(\mathbb{Z}/2 \times \mathbb{Z}/2)$. In terms of the coordinates (a, b) based on (5.5), $\varphi : \mathscr{P}_2 \to \mathscr{P}_2$ is described by

$$\varphi : (a, b) \longmapsto \left(\frac{1}{b}, \frac{1}{a}\right), \quad 0 < a < 1 < b. \tag{5.6}$$

To describe the action of φ in terms of the Fenchel-Nielsen coordinates we recall that for $P \in \mathscr{P}_2$, $l_2 = l_3$ and all twist parameters are zero. Hence the action can be described by

$$(l_1, l_2) \longmapsto \left(\widehat{l_1}, \widehat{l_2}\right), \quad l_1, l_2 > 0, \tag{5.7}$$

where $\widehat{l_1}$ and $\widehat{l_2}$ are given by (5.1).

Remarks 5.8

(i) From the point of view of complex isomorphism classes, the action of φ is of course trivial. The action on real isomorphism classes, however, is nontrivial, and we see that *even* from the complex point of view it is quite important (see Remark 5.18).

(ii) The action of φ can also be described in terms of period matrices. If we take the standard period matrix as in Section 3, then the action is $Z = iY \mapsto (i/\det(Y))Y$. The fixed space is simply characterized by the condition $\det(Y) = 1$.

To describe the next action we begin with the genus-3 curves given by an equation

$$y^2 = \left(x^2 - x_1^2\right)\left(x^2 - x_2^2\right)\left(x^2 - \frac{1}{x_1^2}\right)\left(x^2 - \frac{1}{x_2^2}\right), \tag{5.9}$$

where $0 < x_1 < x_2 < 1$. We denote this family by $\mathscr{C}_2$. Any curve $C \in \mathscr{C}_2$ has the two fixed-point free involutions

$$j_1 : (x, y) \longmapsto (-x, -y), \qquad j_2 : (x, y) \longmapsto \left(\frac{1}{x}, \frac{-y}{x^4}\right).$$

Each j_i fixes two components of the real part of C and interchanges the other two.

The mapping $\psi_1 : (x, y) \mapsto (x^2, xy)$ is the natural projection onto the quotient of C by j_1 and thus is a twofold covering from C to the genus-2 curve Γ_1 given by the equation

$$y^2 = x\left(x - x_1^2\right)\left(x - x_2^2\right)\left(x - \frac{1}{x_1^2}\right)\left(x - \frac{1}{x_2^2}\right).$$

We write $\Gamma_1 = \gamma_1(C)$. To obtain an equation for Γ_1 in the normal form in (5.5), we use the transformation $z \mapsto ((1 - x_1^2)/(1 + x_1^2))((1 + z)/(1 - z))$, $z \in \mathbb{P}^1$. The equation becomes

$$\Gamma_1 : y^2 = \left(x^2 - a\right)\left(x^2 - 1\right)\left(x^2 - b\right),$$

where

$$a = \left(\frac{1 - x_1^2}{1 + x_1^2}\right)^2, \qquad b = \left(\frac{1 - x_1^2}{1 + x_1^2}\frac{1 + x_2^2}{1 - x_2^2}\right)^2. \tag{5.10}$$

Note that the mapping $(x_1, x_2) \mapsto (a, b)$ is one-to-one onto $]0, 1[\times]1, \infty[$. This mapping is just the coordinate description of γ_1, and thus $\gamma_1 : \mathscr{C}_2 \to \mathscr{P}_2$ is a bijection.

The mapping $\psi_2 : (x, y) \mapsto (x + 1/x, (x^2 - 1)y/x^3)$ is the natural projection onto the quotient of C by j_2 and is a twofold covering from C to the curve $\Gamma_2 = \gamma_2(C)$ given by the equation

$$y^2 = \left(x^2 - 4\right)\left(x^2 - \left(x_1 + \frac{1}{x_1}\right)^2\right)\left(x^2 - \left(x_2 + \frac{1}{x_2}\right)^2\right).$$

The normal form for Γ_2 is obtained via the transformation $z \mapsto (x_2/(x_2^2 + 1))z$ and becomes

$$\Gamma_2 : y^2 = \left(x^2 - a'\right)\left(x^2 - 1\right)\left(x^2 - b'\right)$$

with

$$a' = \left(\frac{2x_2}{x_2^2 + 1}\right)^2, \qquad b' = \left(\frac{x_2\left(x_1^2 + 1\right)}{x_1\left(x_2^2 + 1\right)}\right)^2. \tag{5.11}$$

Here too the mapping $(x_1, x_2) \mapsto (a', b')$ is one-to-one onto $]0, 1[\times]1, \infty[$ and is the coordinate description of γ_2.

Altogether γ_1 and γ_2 are one-to-one correspondences from $\mathscr{C}_2$ onto $\mathscr{P}_2$, and we obtain an action $\eta = \gamma_2 \circ \gamma_1^{-1} : \mathscr{P}_2 \to \mathscr{P}_2$ which in terms of coefficients of equations is given by

$$\eta : (a, b) \longmapsto \left(\frac{b - a}{b}, \frac{b - a}{b(1 - a)}\right). \tag{5.12}$$

Note that $\eta^2 = Id$.

To describe η in terms of the Fenchel-Nielsen coordinates, we need to introduce some notation. We let $\pi : C \to \mathbb{P}^1$ be the projection $(x, y) \mapsto x$ and consider the following cycles on C:

$$\begin{aligned}
\lambda_1 &= \pi^{-1}\left(\left[-\frac{1}{x_2}, -x_2\right]\right), & \lambda_1' &= \pi^{-1}\left(\left[x_2, \frac{1}{x_2}\right]\right),\\
\lambda_2 &= \pi^{-1}\left(\left[-x_1, x_1\right]\right), & \lambda_2' &= \pi^{-1}\left(\left]-\infty, -\frac{1}{x_1}\right] \cup \left[\frac{1}{x_1}, \infty\right]\right),\\
\mu_1 &= \pi^{-1}\left(\left[-\frac{1}{x_1}, -\frac{1}{x_2}\right]\right), & \mu_1' &= \pi^{-1}\left(\left[\frac{1}{x_2}, \frac{1}{x_1}\right]\right),\\
\mu_2 &= \pi^{-1}\left(\left[-x_2, -x_1\right]\right), & \mu_2' &= \pi^{-1}\left(\left[x_1, x_2\right]\right).
\end{aligned} \tag{5.13}$$

Furthermore, we let μ_3, μ_3' be the two cycles that project upon the imaginary axis, and let ν, ν' be the cycles that project upon the unit circle.

All these cycles are geodesics with respect to the hyperbolic metric. This follows from the fact that each of them is part of the fixed-point set of some orientation-reversing automorphism of C.

Consider the quotient maps $\psi_i : C \to \Gamma_i$, $i = 1, 2$, where the equations of Γ_1, Γ_2 are in the normal form in (5.5). The images under ψ_i of the cycles are as follows, where $\pi_i : \Gamma_i \to \mathbb{P}^1$ is again the natural projection:

$$\begin{aligned} &\pi_1^{-1}\big([-\sqrt{b}, -1]\big) = \psi_1(\mu_1) = \psi_1\big(\mu_1'\big),\\ &\pi_1^{-1}\big([-1, -\sqrt{a}\,]\big) = \psi_1\big(\lambda_2'\big),\\ &\pi_1^{-1}\big([-\sqrt{a}, \sqrt{a}\,]\big) = \psi_1(\mu_3) = \psi_1\big(\mu_3'\big),\\ &\pi_1^{-1}\big([\sqrt{a}, 1]\big) = \psi_1\big(\lambda_2\big),\\ &\pi_1^{-1}\big([1, \sqrt{b}]\big) = \psi_1\big(\mu_2\big) = \psi_1\big(\mu_2'\big),\\ &\pi_1^{-1}\big([\sqrt{b}, \infty] \cup\,]-\infty, -\sqrt{b}]\big) = \psi_1\big(\lambda_1\big) = \psi_1\big(\lambda_1'\big). \end{aligned} \tag{5.14}$$

Furthermore, $\psi_1(\nu)$ and $\psi_1(\nu')$ are the cycles on Γ_1 which project upon the imaginary axis. Here too all cycles are geodesics.

On Γ_2 we have the geodesics

$$\begin{aligned} &\pi_2^{-1}\big([-\sqrt{b'}, -1]\big) = \psi_2\big(\mu_1\big) = \psi_2\big(\mu_2\big),\\ &\pi_2^{-1}\big([-1, -\sqrt{a'}\,]\big) = \psi_2\big(\lambda_1\big),\\ &\pi_2^{-1}\big([-\sqrt{a'}, \sqrt{a'}\,]\big) = \psi_2(\nu) = \psi_2\big(\nu'\big),\\ &\pi_2^{-1}\big([\sqrt{a'}, 1]\big) = \psi_2\big(\lambda_1'\big),\\ &\pi_2^{-1}\big([1, \sqrt{b'}\,]\big) = \psi_2\big(\mu_2'\big) = \psi_2\big(\mu_1'\big),\\ &\pi_2^{-1}\big([\sqrt{b'}, \infty] \cup\,]-\infty, -\sqrt{b'}\,]\big) = \psi_2\big(\lambda_2\big) = \psi_2\big(\lambda_2'\big), \end{aligned} \tag{5.15}$$

plus the geodesics $\psi_2(\mu_3)$ and $\psi_2(\mu_3')$ which project upon the imaginary axis.

In Γ_1 the geodesics $\psi_1(\lambda_2)$ and $\psi_1(\lambda_2')$ have the same length, so that this length is $2l_2$ and the length of $\psi_1(\lambda_1)$ is $2l_1$. (Recall that all l_k are half-lengths.) As $\psi_1 \mid \lambda_1$ is one-to-one and $\psi_1 \mid \lambda_2$ is two-to-one, this yields $2l_1 = \text{length}(\lambda_1)$, $2l_2 = (1/2)\text{length}(\lambda_2)$.

In Γ_2 the geodesics $\psi_2(\lambda_1)$ and $\psi_2(\lambda_1')$ have the same length, so that this length is $2l_2'$ and the length of $\psi_2(\lambda_1)$ is $2l_1'$. Here $\psi_2 \mid \lambda_2$ is one-to-one and $\psi_2 \mid \lambda_1$ is two-to-one, so that $2l_1' = \text{length}(\lambda_2)$, $2l_2' = (1/2)\text{length}(\lambda_1)$.

Figure 5 shows one of the four octagons defining C drawn as a hyperbolic polygon. The quotients Γ_1 and Γ_2 are obtained from the two distinguished hexagons.

Altogether we get the description $\eta : (l_1, l_2) \mapsto (2l_2, l_1/2)$. We display this more transparently writing $(l_1, l_2, h_1, \widehat{l_1}, \widehat{l_2})$ instead of (l_1, l_2). The description of η

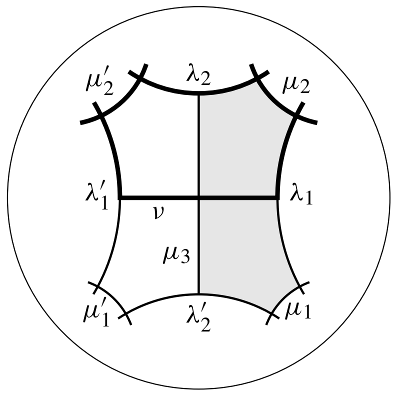

Figure 5

then becomes $\eta : (l_1, l_2, h_1, \widehat{l_1}, \widehat{l_2}) \mapsto (2l_2, l_1/2, \widehat{l_1}/2, 2h_1, \widehat{l_2})$, and that of φ is $\varphi : (l_1, l_2, h_1, \widehat{l_1}, \widehat{l_2}) \mapsto (\widehat{l_1}, \widehat{l_2}, h_1, l_1, l_2)$.

Setting $\psi = \varphi \circ \eta$, we get the following result.

PROPOSITION 5.16
The transformations $\varphi : (l_1, l_2, h_1, \widehat{l_1}, \widehat{l_2}) \mapsto (\widehat{l_1}, \widehat{l_2}, h_1, l_1, l_2)$ and $\psi : (l_1, l_2, h_1, \widehat{l_1}, \widehat{l_2}) \mapsto (2h_1, \widehat{l_2}, \widehat{l_1}/2, 2l_2, l_1/2)$ generate an action of the dihedral group D_5 on $\mathscr{P}_2$. This induces an algebraic action of D_5 on $\mathscr{M}_{\mathbb{R}}^{(2,3,0)}(\mathbb{Z}/2 \times \mathbb{Z}/2)$, defined by $\varphi : (a, b) \mapsto (1/b, 1/a)$ and $\psi : (a, b) \mapsto (b(1-a)/(b-a), b/(b-a))$ when the equations of the curves are written in the form $y^2 = (x^2 - a)(x^2 - 1)(x^2 - b)$, $0 < a < 1 < b$.

We summarize in Table 5.17 the action of D_5.

Remarks 5.18
(i) From the point of view of real isomorphy classes this action of D_5 is effective in the sense that if the curve is not in one of the fixed spaces of the involutions $\varphi\psi^k$ (see Section 6), then no two curves in the orbit are real isomorphic (for the obvious reason that a real isomorphism is an isometry and sends real components to real components). From the point of view of complex isomorphy classes, we note that although φ acts trivially and η is of order 2, ψ is of order 5. (This follows from the fact that a complex genus-2 curve has at most two distinct real structures with three real components (see, e.g., [Na]).) Hence in each orbit not intersecting the fixed subspaces we find five nonisometric surfaces.

Table 5.17

Id	$\left(l_1, l_2, h_1, \widehat{l_1}, \widehat{l_2}\right)$	(a, b)
ψ	$\left(2h_1, \widehat{l_2}, \frac{\widehat{l_1}}{2}, 2l_2, \frac{l_1}{2}\right)$	$\left(\frac{b(1-a)}{b-a}, \frac{b}{b-a}\right)$
ψ^2	$\left(\widehat{l_1}, \frac{l_1}{2}, l_2, 2\widehat{l_2}, h_1\right)$	$\left(\frac{b-1}{b-a}, \frac{1}{a}\right)$
ψ^3	$\left(2l_2, h_1, \widehat{l_2}, l_1, \frac{\widehat{l_1}}{2}\right)$	$\left(\frac{1}{b}, \frac{b-a}{b(1-a)}\right)$
ψ^4	$\left(2\widehat{l_2}, \frac{\widehat{l_1}}{2}, \frac{l_1}{2}, 2h_1, l_2\right)$	$\left(\frac{b-a}{b}, \frac{b-a}{b-1}\right)$
φ	$\left(\widehat{l_1}, \widehat{l_2}, h_1, l_1, l_2\right)$	$\left(\frac{1}{b}, \frac{1}{a}\right)$
$\varphi\psi$	$\left(2l_2, \frac{l_1}{2}, \frac{\widehat{l_1}}{2}, 2h_1, \widehat{l_2}\right)$	$\left(\frac{b-a}{b}, \frac{b-a}{b(1-a)}\right)$
$\varphi\psi^2$	$\left(2\widehat{l_2}, h_1, l_2, \widehat{l_1}, \frac{l_1}{2}\right)$	$\left(a, \frac{b-a}{b-1}\right)$
$\varphi\psi^3$	$\left(l_1, \frac{\widehat{l_1}}{2}, \widehat{l_2}, 2l_2, h_1\right)$	$\left(\frac{b(1-a)}{b-a}, b\right)$
$\varphi\psi^4$	$\left(2h_1, l_2, \frac{l_1}{2}, 2\widehat{l_2}, \frac{\widehat{l_1}}{2}\right)$	$\left(\frac{b-1}{b-a}, \frac{b}{b-a}\right)$

(ii) The group D_5 is the mapping class group of the pentagon. And we do have a pentagon in the picture (see Figure 5): the one with sides $\{l_1/2, \widehat{l_2}, l_2, \widehat{l_1}/2, h_1\}$. In this setting the action of D_5 corresponds exactly to the different ways one can build a symmetric hexagon starting with the pentagon. This suggests the idea that we can do this for other hyperbolic polygons, and this is indeed the case. Starting with a hexagon we can glue eight copies of the same, without twist. We obtain in this way a real hyperelliptic genus-3 curve with a real genus-2 quotient. The action of D_6 can be seen as generated by cyclic permutation of the sides and replacing lengths by dual lengths—that is, replacing $y^2 = P(x)$ by $y^2 = -P(x)$. The first action leaves the genus-2 quotient fixed (up to complex isomorphism, from the real point of view we have, in general, two quotients). On the level of equations we can describe the action in the following way.

Let $y^2 = P(x) = (x^2 - a)(x^2 - b)(x^2 - c)(x^2 - 1)$, $(0 < a < b < c < 1)$ be the equation of the genus-3 curve. An equation of the genus-2 quotient is then

$y^2 = x(x-a)(x-b)(x-c)(x-1)$. The cyclic permutation of the sides of the hexagon can be written $(a,b,c) \mapsto (1-c, (1-c)/(1-a), (1-c)/(1-b))$.

(iii) The D_5 action is more difficult to describe in terms of period matrices, but we can indicate some relations between the coefficients of the period matrices of the transforms.

We have described the action of φ in (5.8). To describe the action of $\eta = \varphi\psi$, let C again be the curve defined by (5.9), and let Z be the standard period matrix associated to equation (5.9) as in Lemma 2.2. Writing out the action of the automorphisms j_1 and j_2 on $\mathrm{H}_1(C,\mathbb{Z})$ and $\Omega^1(C)$, it is straightforward to check that Z is of the form

$$i\begin{pmatrix} y_1 & \frac{y_2}{2} & y_{13} \\ \frac{y_2}{2} & y_2 & \frac{y_2}{2} \\ y_{13} & \frac{y_2}{2} & y_1 \end{pmatrix}.$$

From the explicit description of how the quotients are obtained, it is again straightforward to show that a period matrix for Γ_1 is

$$i\begin{pmatrix} 2y_2 & y_2 \\ y_2 & y_1 + y_{13} \end{pmatrix},$$

which is equivalent to

$$i\begin{pmatrix} y_1 + y_{13} & y_1 + y_{13} - y_2 \\ y_1 + y_{13} - y_2 & y_1 + y_{13} \end{pmatrix},$$

and that a period matrix for Γ_2 is

$$i\begin{pmatrix} 2y_1 - \frac{y_2}{2} & 2y_{13} - \frac{y_2}{2} \\ 2y_{13} - \frac{y_2}{2} & 2y_1 - \frac{y_2}{2} \end{pmatrix}.$$

Note that this description is in terms of a covering and that it does not yield a direct method to pass from a period matrix of Γ_1 to one of Γ_2.

6. Special families and special curves

In this section we describe families of surfaces defined by simple relations between their Fenchel-Nielsen coordinates and algebraic relations between the coefficients of the equations defining the associated algebraic curves. We use the notation of Section 5.

Our first family is defined by the condition $l_1 = \widehat{l_1}$ (which implies $l_2 = \widehat{l_2}$ and $l_3 = \widehat{l_3}$). For such surfaces we can find an equation of the form (5.4) with $c = -1/b$ and $d = -1/a$.

Let us now work again in $\mathscr{P}_2$, so that we have $l_2 = l_3$. Then the normalized equation (5.5) becomes

$$y^2 = (x^2 - a)(x^2 - 1)\left(x^2 - \frac{1}{a}\right).$$

Hence the correspondence of the following conditions defining the same family in $\mathscr{P}_2$ (notation as in Table 5.17):

$$\widehat{l_1} = l_1, \qquad b = \frac{1}{a}, \quad 0 < a < 1 < b. \tag{6.1}$$

(Note that this family can also be defined by the conditions $\cosh(h_1) = (\cosh(l_1) + 1)/(\cosh(l_1) - 1)$ or $\cosh^2(l_2) = 2\cosh(l_1)/(\cosh(l_1) - 1)$.) We also note that, conversely, any curve with $b = 1/a$ in a normalized equation (5.5) has $\widehat{l_1} = l_1$.

The one parameter family described in (6.1) is the fixed subspace under the action of φ of Table 5.17. Its images under the D_5 action are the fixed subspaces of the 5 involutions $\varphi\psi^k$ and can be described by

$$l_1 = 2l_2, \qquad a = \frac{b}{b+1}, \quad 0 < a < 1 < b. \tag{6.2}$$

This family can also be defined by the conditions $\widehat{l_1} = 2h_1$ or $\cosh(\widehat{l_1}) = 2\cosh(\widehat{l_2}) + 1$ or $\cosh(\widehat{l_2}) = (\cosh(l_1) + 1)/(\cosh(l_1) - 1)$,

$$l_2 = h_1, \qquad a = b(2 - b), \quad 0 < a < 1 < b, \tag{6.3}$$

which can also be defined by the condition $l_1 = 2\widehat{l_2}$,

$$l_1 = 2h_1, \qquad b = a + 1, \quad 0 < a < 1 < b, \tag{6.4}$$

$$\widehat{l_2} = h_1, \qquad b = \frac{a^2}{2a - 1}, \quad 0 < a < 1 < b. \tag{6.5}$$

Note that the last two are images under φ of (6.2) and (6.3).

We next consider the family defined by the condition $l_1 = l_2 = l_3$.

LEMMA 6.6

The curves associated to the surfaces in $\mathscr{P}$ satisfying the conditions $l_1 = l_2 = l_3$ have equations of the form

$$y^2 = (x^2 - a)(x^2 - 1)(x^2 - b),$$

with

$$a = \frac{(3 - t)^2}{(t^2 + t)^2}, \qquad b = \frac{(3 + t)^2}{(t^2 - t)^2} \quad \text{and } 1 < t < 3.$$

Conversely, any curve with such an equation is associated to a surface in $\mathscr{P}$ with $l_1 = l_2 = l_3$.

Proof
The condition $l_1 = l_2 = l_3$ implies that $\mathbb{Z}/3 \subset \mathrm{Aut}^+_{\mathbb{R}}(C)$. Conversely, if the curve has a nontrivial automorphism of order 3, then it must permute the real components (since Weierstrass points must be sent to Weierstrass points) and hence the surface satisfies $l_1 = l_2 = l_3$.

Fix t, $1 < t < 3$. The curve defined by the equation in Lemma 6.6 is isomorphic to the one defined by

$$y^2 = \bigl(x^2 - t^2 a\bigr)\bigl(x^2 - t^2\bigr)\bigl(x^2 - t^2 b\bigr),$$

and on this curve $(x, y) \mapsto ((x+3)/(1-x), 8y/(x-1)^3)$ defines an automorphism of order 3. To see that this parameterizes the complete family, we only need to note that it is a one parameter family and that $t = 1$ corresponds to $l_1 = 0$ while $t = 3$ corresponds to $l_1 = \infty$. □

Taking images by the D_5-action, we find four more families,

$$l_1 = 4l_2 \quad \bigl(\text{or } \widehat{l_2} = 2h_1\bigr), \qquad a = \frac{16t\bigl(3+t^2\bigr)}{(3+t)^2(t+1)^2}, \tag{6.7}$$
$$b = \frac{16t}{(t+1)^3(3-t)}, \quad 1 < t < 3,$$

$$h_1 = 2l_2 \quad \bigl(\text{or } \widehat{l_1} = 2l_1\bigr), \qquad a = \frac{(t+1)^3(3-t)}{16t}, \tag{6.8}$$
$$b = \frac{\bigl(t^2+t\bigr)^2}{(3-t)^2}, \quad 1 < t < 3,$$

and the images of these two by φ,

$$l_2 = 2h_1, \qquad a = \frac{(3+t)^3(t-1)}{16t^3}, \qquad b = \frac{(3+t)^2(t+1)^2}{16t\bigl(t^2+3\bigr)}, \quad 1 < t < 3, \tag{6.9}$$

$$l_1 = 2\widehat{l_1}, \qquad a = \frac{(t^2-t)^2}{(3+t)^2}, \qquad b = \frac{16t^3}{(3+t)^3(t-1)}, \quad 1 < t < 3. \tag{6.10}$$

The intersection points of these families give us a first set of special curves that we can uniformize explicitly. Our first example is at the common point of intersection of the families (6.1)–(6.10) and is in fact the fixed point of the D_5 action:

$$\begin{gathered} \cosh(l_1) = 2 + \sqrt{5}, \qquad \cosh(l_2) = \frac{1+\sqrt{5}}{2}, \\ a = \frac{\sqrt{5}-1}{2}, \qquad b = \frac{1+\sqrt{5}}{2}. \end{gathered} \tag{6.11}$$

The hexagon defining the surface is a double of the right-angled regular pentagon, and we refer to this curve as obtained from the regular pentagon.

Remark 6.12
Note that the curve defined by (6.11) *is not* isometric to the one with a $\mathbb{Z}/5$ action—that is, the curve defined by $y^2 = (x^5 - 1)$. The reason for this is that, up to isomorphism, there are only two distinct real structures on this last curve and each has only one real component.

Our second example is the curve at the intersections of (6.1) and (6.6),

$$\cosh(l_1) = 2, \qquad \cosh(l_2) = 2, \qquad a = 7 - 4\sqrt{3}, \qquad b = 7 + 4\sqrt{3}. \tag{6.13}$$

The hexagon defining this curve is the regular hexagon, with all sides equal to $\operatorname{arccosh}(2)$. This curve is isometric to the one defined by $y^2 = (x^6 - 1)$, but it is not real isomorphic. In this second form the curve has only one real component, and for this reason we prefer to present it as obtained from the hexagon $\cosh(l_1) = 5$, $\cosh(l_2) = \cosh(l_3) = 4$, pasting with zero twist along the side with length $2l_1$ and a half-twist along the two other sides. (One can deduce this description from [KN].)

Our third example is obtained by taking a twofold quotient of the genus-3 curve defined by the regular octagon. To obtain an equation for this curve, we start with $y^2 = \prod(x - \exp(ki\pi/4))$ and transform by $x \mapsto i((1+x)/(1-x))$ to obtain

$$y^2 = x\Big(x^2 - \big(\sqrt{2} - 1\big)^2\Big)\big(x^2 - 1\big)\Big(x^2 - \big(1 + \sqrt{2}\big)^2\Big)$$

as an equation for the corresponding genus-3 curve with four real components. Using $x \mapsto (1 - tx)/(t + x)$, we bring it into the normal form

$$y^2 = \big(x^2 - t^2\big)\left(x^2 - \frac{1}{t^2}\right)\left(x^2 - \left(\frac{1+t}{1-t}\right)^2\right)\left(x^2 - \left(\frac{1-t}{1+t}\right)^2\right), \tag{6.14}$$

with $t = \sqrt{4 + 2\sqrt{2}} - (\sqrt{2} + 1)$. As in (5.9)–(5.10) we take the genus-2 quotient under $(x, y) \mapsto (-x, -y)$ and obtain the normal form

$$y^2 = \left(x^2 - \left(\frac{1}{4}\sqrt{2} + \frac{1}{2}\right)\right)\big(x^2 - 1\big)\Big(x^2 - \big(1 + \sqrt{2}\big)^2\Big).$$

Since the lengths of the sides of the octagon are $\operatorname{arccosh}(1 + \sqrt{2})$, we have the correspondence

$$\begin{gathered}\cosh(l_1) = 1 + \sqrt{2}, \qquad \cosh(l_2) = \sqrt{1 + \frac{1}{2}\sqrt{2}},\\ a = \frac{1}{4}\sqrt{2} + \frac{1}{2}, \qquad b = 3 + 2\sqrt{2}.\end{gathered} \tag{6.15}$$

We can do a similar construction with the regular dodecagon, defining a genus-5 curve. After some computations we find that a fourfold genus-2 quotient has equation

$$y^2 = x\big(x^2-1\big)\big(x-\big(7+4\sqrt{3}\big)\big)\big(x-\big(7-4\sqrt{3}\big)\big) = x\big(x^2-1\big)\big(x^2-14x+1\big) \quad (6.16)$$

and corresponds to $\cosh(l_1) = (1+\sqrt{3})/2$, $\cosh(l_2) = 1+\sqrt{3}$.

Remarks 6.17
(i) The values of a and b for the curve in (6.16) (see Table 8.4) are not particularly "nice" but the curve has a nice period matrix

$$\frac{i}{3}\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

that can be computed using the genus-1 quotients.

(ii) The curve in (6.16) should not be confused with the curve in Lemma 6.6 which has equation $y^2 = (x^2-1)(x^4-14x^2+1)$ nor with the curve $y^2 = x(x^4-14x^2+1)$, quotient of the special genus-3 curve $y^2 = x^8 - 14x^4 + 1$ (see [RG]). The latter is isomorphic to the curve with equation $y^2 = (x^2-1/3)(x^2-1)(x^2-3)$, and we give its explicit uniformization later (see Table 8.2).

The intersection of (6.3) and (6.6) yields another curve that we describe in Section 8.5, and of course we also have all the transforms of the ones we have obtained under the D_5-action. We list these in Section 8, but beforehand we want to show that there are still other natural transformations not covered by the D_5-action.

PROPOSITION 6.18
Let C be a real genus-2 M-curve satisfying the length conditions $l_1 = 2l_2 = 2l_3$. Let $l_1' = \operatorname{arccosh}((\cosh(l_1)+1)/(\cosh(l_1)-1))$, and let $l_2' = l_3' = l_1'/2$. Let C' be associated to $\{l_1', l_2', l_3'\}$. Then if

$$y^2 = \big(x^2-\alpha^2\big)\big(x^2-1\big)\big(x^2-\beta^2\big), \quad 0 < \alpha < 1 < \beta,$$

is an equation defining C,

$$y^2 = \Big(x^2-\alpha'^2\Big)\big(x^2-1\big)\Big(x^2-\beta'^2\Big), \quad 0 < \alpha' < 1 < \beta',$$

with

$$\alpha'^2 = \frac{\beta'^2}{\beta'^2+1} \qquad \text{and} \qquad \beta' = \frac{\beta+1}{\beta-1},$$

is an equation defining C'.

Proof
We consider the genus-3 curves Γ defined by an equation of the form in (6.14) with $0 < t < \sqrt{2} - 1$. For such curves we have $D_4 \subset \mathrm{Aut}^+(\Gamma)$, the group of automorphisms being generated by $x \mapsto 1/x$ and $x \mapsto (x-1)/(x+1)$. Looking at this action, it is easily checked that the lengths of the four real components of Γ are all equal. This implies that the quotient C of Γ by $(x, y) \mapsto (-x, -y)$ satisfies the condition $l_1 = 2l_2 = 2l_3$. Hence C is in the family in (6.2).

Using (5.10), the normalized equation for C becomes

$$y^2 = \left(x^2 - \alpha^2\right)\left(x^2 - 1\right)\left(x^2 - \beta^2\right),$$

with

$$\alpha = \frac{1 - t^2}{1 + t^2} \qquad \text{and} \qquad \beta = \frac{1 - t^2}{2t}. \tag{6.19}$$

Let $\widehat{\Gamma}$ be the dual curve to Γ—that is, defined by $y^2 = -P(x)$. An equation for this curve in the form in (6.14) is obtained by replacing t by $(1 - (1 + \sqrt{2})t)/((1 + \sqrt{2}) + t)$, and it can be checked that the equation of the quotient C' of $\widehat{\Gamma}$ is

$$y^2 = \left(x^2 - \alpha'^2\right)\left(x^2 - 1\right)\left(x^2 - \beta'^2\right),$$

with

$$\alpha'^2 = \frac{\beta'^2}{\beta'^2 + 1} \qquad \text{and} \qquad \beta' = \frac{\beta + 1}{\beta - 1}.$$

In terms of lengths of geodesics, passing from C to C' can be described as replacing l_1 by $\widehat{l_2} = \operatorname{arccosh}((\cosh(l_1) + 1)/(\cosh(l_1) - 1))$, and this ends the proof. □

The transformation described in (6.18) has one fixed point that corresponds to the curve in (6.15).

We can also transport this transformation to the other fixed spaces of the $\varphi\psi^k$'s in equations (6.1)–(6.5). The description is rather lengthy, and we only note that on the subspace (6.1), defined by $l_1 = \widehat{l_1}$, the transformation is also described by $l_1 \mapsto \operatorname{arccosh}((\cosh(l_1) + 1)/(\cosh(l_1) - 1))$ and $\beta = \sqrt{b} \mapsto (\beta + 1)/(\beta - 1)$.

7. Curves with one half-twist

To describe the action of D_5 we have used the correspondence between two different quotients of a genus-3 curve C defined by an equation of the form

$$y^2 = \left(x^2 - x_1^2\right)\left(x^2 - x_2^2\right)\left(x^2 - \frac{1}{x_1^2}\right)\left(x^2 - \frac{1}{x_2^2}\right). \tag{7.1}$$

We have used the involutions induced by $x \mapsto -x$ and $x \mapsto 1/x$, but there is a third induced by $x \mapsto -1/x$. An equation for the quotient Γ_3 of C by this last involution is

$$y^2 = (x^2+4)\left(x^2 - \left(x_1 - \frac{1}{x_1}\right)^2\right)\left(x^2 - \left(x_2 - \frac{1}{x_2}\right)^2\right), \tag{7.2}$$

the quotient map being $(x, y) \mapsto (x - 1/x, (x^2+1)y/x^3)$.

To find Fenchel-Nielsen coordinates for this new curve Γ_3, we use the notation introduced in (5.13) and (5.15).

If we call ψ_3 the quotient map $C \to \Gamma_3$ and call π_3 the projection $\Gamma_3 \to \mathbb{P}^1$, we have (notation as in Section 5)

$$\begin{aligned}
&\pi_3^{-1}\left(\left]-\infty, x_1 - \frac{1}{x_1}\right] \cup \left[\frac{1}{x_1} - x_1, \infty\right[\right) = \psi_3(\lambda_2) = \psi_3(\lambda_2'),\\
&\pi_3^{-1}\left(\left[x_1 - \frac{1}{x_1}, x_2 - \frac{1}{x_2}\right]\right) = \psi_3(\mu_1) = \psi_3(\mu_2'),\\
&\pi_3^{-1}\left(\left[x_2 - \frac{1}{x_2}, \frac{1}{x_2} - x_2\right]\right) = \psi_3(\lambda_1) = \psi_3(\lambda_1'),\\
&\pi_3^{-1}\left(\left[\frac{1}{x_2} - x_2, \frac{1}{x_1} - x_1\right]\right) = \psi_3(\mu_2) = \psi_3(\mu_1'),\\
&\pi_3^{-1}\left(]-\infty, -2i] \cup [2i, \infty[\right) = \psi_3(\mu_3) = \psi_3(\mu_3'),\\
&\pi_3^{-1}([-2i, 2i]) = \psi_3(\nu) = \psi_3(\nu').
\end{aligned} \tag{7.3}$$

From this it is easy to check on which cycles the ψ_i's are one-to-one or two-to-one and compare lengths.

We note also that the simple closed geodesics $\pi_3^{-1}(]-\infty, -2i] \cup [2i, \infty[)$, $\pi_3^{-1}([x_1 - 1/x_1, x_2 - 1/x_2])$, and $\pi_3^{-1}([1/x_2 - x_2, 1/x_1 - x_1])$ define a pants decomposition of Γ_3. Moreover, looking at the action of $(x, y) \mapsto (\bar{x}, -\bar{y})$ on this decomposition, we see that we can take the twist parameters to be 0 on the last two and $1/2$ on the first.

PROPOSITION 7.4
Let P be a pair of pants with lengths of boundary components $2l_1$, $2l_2$, and $2l_2$.

Let C_1 be the genus-2 curve obtained by gluing two copies of P with zero twists, and let C_2 be the curve obtained by gluing two copies of P with zero twists along the sides of length $2l_2$ and a half-twist along the side of length $2l_1$. Then if C_1 has equation

$$y^2 = (x^2 - a)(x^2 - 1)(x^2 - b), \quad 0 < a < 1 < b,$$

C_2 has equation

$$y^2 = -(x^2+1)\left(x^2 - \frac{a}{b-a}\right)\left(x^2 - \frac{1}{b-1}\right).$$

Proof
Let $\widehat{l_1}$, $\widehat{l_2}$, and $\widehat{l_2}$ be the lengths of the common perpendiculars to two boundary components of P, and let $\widehat{P}$ be the pair of pants with boundary lengths $2\widehat{l_1}$, $2\widehat{l_2}$, and $2\widehat{l_2}$.

By the construction given at the beginning of this section, we see that if we identify C_1 with Γ_1, then Γ_3 is obtained by gluing two copies of $\widehat{P}$ with zero twist on the sides of length $2\widehat{l_2}$ and a half-twist along the side of length $2\widehat{l_1}$.

Renormalizing as before equation (7.2), we find that if Γ_1 has equation (5.5), then Γ_3 has equation

$$y^2 = -(x^2+1)\left(x^2 - \frac{a}{b-a}\right)\left(x^2 - \frac{a}{1-a}\right).$$

Recalling that to obtain the equation of the curve obtained by gluing two copies of $\widehat{P}$ with zero twists we only need to replace (a, b) by $(1/b, 1/a)$, we have the proposition. □

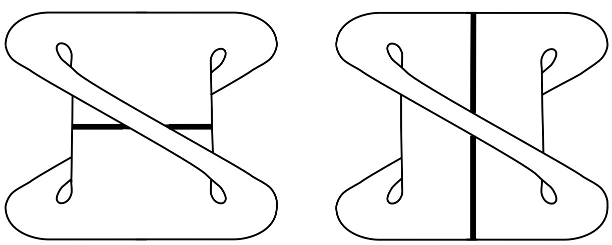

Figure 6

Remarks 7.5
(i) We have taken the half-twist along $\pi_3^{-1}(]-\infty, -2i] \cup [2i, \infty[)$. But replacing this cycle by $\pi_3^{-1}([-2i, 2i])$ would also have yielded a pants decomposition with one half-twist. In terms of lengths in Γ_1, this corresponds to replacing l_1 by $2h_1$ (see Figure 6).

(ii) The preceding construction allows us also to find the Fenchel-Nielsen coordinates of the curve $\widehat{C}_2$ with equation $y^2 = (x^2+1)(x^2 - a/(b-a))(x^2 - 1/(b-1))$.

For this let $\widehat{l_1}$ and $\widehat{l_2}$ be the dual lengths of the hexagon defining C_1 in Proposition 7.4. Let $l_2' = \widehat{l_1}$, $l_3' = 2\widehat{l_2}$, and $l_1' = \operatorname{arccosh}(\cosh(l_2') + \cosh(l_3') + 1)$. Let P' be the pair of pants with boundary components of lengths $2l_1'$, $2l_2'$, and $2l_3'$; then the curve obtained by gluing two copies of P' with a half-twist along the first boundary component and zero twists along the last two is $\widehat{C}_2$.

Via Proposition 7.4 we can transport the D_5 action of Section 5 onto the space of real genus-2 curves with two real components and nontrivial automorphisms. The images of the fixed spaces under the involutions of the D_5 action are also algebraically defined subspaces of $\mathscr{M}_{\mathbb{R}}^{(2,2,1)}(\mathbb{Z}/2 \times \mathbb{Z}/2)$. In particular, if we start with the fixed part under $\varphi\psi^4$, defined by the condition $\cosh(l_1) = 2\cosh(l_2) + 1$, then we recover the family of curves with equations of the form $y^2 = -(x^2+1)(x^2-a)(x^2-1/a)$ (since for this family we have $b = a + 1$).

We can also transport the curves for which we know exact solutions to obtain new ones. We do this in Section 8. Here we give only one example. Let $\cosh(l_1) = 3+2\sqrt{2}$, and let $\cosh(l_2) = 1+\sqrt{2}$; then the corresponding curve with one half-twist obtained as in Proposition 7.4 has equation

$$y^2 = (x^2+1)\Big(x^2 - \big(1+\sqrt{2}\big)^2\Big)\Big(x^2 - \big(\sqrt{2}-1\big)^2\Big) = (x^2+1)(x^4 - 6x^2 + 1).$$

It is easily checked that this curve is isomorphic to the curve $y^2 = x(x^4-1)$ with reduced automorphism group isomorphic to the symmetric group $\mathfrak{S}_4$.

Up until now we have made all our constructions using double covers, but we can use coverings of other orders. We indicate here one construction.

LEMMA 7.6

Let P_t be the pair of pants with boundary lengths $2t$, $6t$, and ω_t, where $\omega_t = 2\operatorname{arccosh}(2\cosh^2(t))$. Let C_t be the curve obtained by gluing two copies of P_t with zero twists along the first two boundary components and a half-twist along the last. Then when t varies from zero to ∞, the family of curves C_t describes an algebraic subspace of $\mathscr{M}_{\mathbb{R}}^{(2,2,1)}$, the moduli space of real genus-2 curves with two real components.

Proof

Start with a sphere with eight disks removed on which the full group of the cube operates (i.e., including the orientation-reversing symmetries). One can endow such a sphere with a hyperbolic metric such that the boundary consists of closed geodesics and the group acts by isometries.

One way to do this is to start with a hexagon defined by $l_1 = 2t$, $l_2 = l_3 = t$. Observe that we also have $2h_1 = \widehat{l_1}$. Glue two copies of the hexagon along $\widehat{l_1}$ to obtain an octagon with sides equal 4 by 4; then glue six copies of the octagon to obtain the sphere with eight disks removed. In order to describe the symmetries we view this sphere as a cube with holes at the vertices.

Taking the Schottky double, we obtain a real genus-7 curve C with an orientation-reversing symmetry σ interchanging the two sheets. This defines a real structure on C which has eight real components all of the same length $l = 6t$. We obtain in this way a family of dimension 1 parametrized by t.

The real automorphism group of C contains all the automorphisms induced by the orientation-preserving isometries of the cube (i.e., S_4) plus the ones induced by the orientation-reversing ones composed with σ. In particular, it contains a fixed-point free automorphism of order 3, f (rotation of angle $2\pi/3$ and axis one of the diagonals of the cube). The quotient C' of C by f is of genus 3, and it has four real components, $\beta_1, \ldots, \beta_4$. By construction we have $6t = \text{length}(\beta_1) = \text{length}(\beta_2) = 3\,\text{length}(\beta_3) = 3\,\text{length}(\beta_4)$.

Let $\widetilde{g}$ be a symmetry of C through a plane containing the axis of f, and let $g = \widetilde{g} \circ \sigma$. Then it is easily checked that g induces an involution on C' with eight fixed points; hence C' is hyperelliptic. Also, $\widetilde{g}$ defines a second real structure on C for which the half-lengths of the real components are $\widehat{l_1}, \widehat{l_2}, \widehat{l_1}, \widehat{l_2}$.

Consider C' as the Schottky double of a sphere with four disks removed. Composing the central symmetry with the involution exchanging the sheets of the Schottky double yields a fixed-point free involution on C'. The quotient C'' of C' is real of genus 2 and has two real components; hence it is obtained by gluing two copies of a pair of pants with zero twist along two boundaries and a half-twist along the third. The lengths of the two which are glued with zero twists are by construction $2t$ and $6t$. To compute the length of the third we note that the length of the common perpendicular to the first two is $\widehat{l_1}$. The result then follows from (5.1) and the formulas in [Bu, Theorem 2.4.1]. □

We can make similar constructions with the other regular polyhedrons. In the case of the tetrahedron and the octahedron this yields families obtained earlier.

On the other hand, the icosahedron gives new families. We do a construction completely similar to the one made for the cube. (The hyperbolic structure is this time obtained by gluing twenty copies of a hexagon with three opposite sides equal.) This gives a real genus-11 curve C with twelve real components all of the same length l and a real automorphism of order 5. The quotient is a genus-3 curve C' with four real components of lengths $l, l, l/5, l/5$. To see that C' is again hyperelliptic and that the lengths of the second real structure are equal in pairs, we note that on C we again have an involution with eight fixed points obtained by taking reflection through a plane containing the axis of the order-5 automorphism and two edges of the icosahedron. C' has again a real genus-2 quotient with two real components.

Finally, we can also use the considerations of this section to compute numerically period matrices and equations for general genus-2 surfaces with "one half-twist." For this we start with a genus-3 curve C, with equation of the form

$$y^2 = \prod_{k=1}^{4}(x - x_k)\left(x + \frac{1}{x_k}\right). \tag{7.7}$$

Such a curve has a genus-2 quotient Γ under one of the involutions induced by $x \mapsto -1/x$. An equation for Γ is

$$y^2 = -\left(x^2 + 4\right)\prod_{k=1}^{4}\left(x - x_k + \frac{1}{x_k}\right). \tag{7.8}$$

By (5.13) and (7.3) the lengths of the octagon defining C are completely determined by the lengths defining Γ. Since using the methods of Sections 3 and 4 we can compute numerically an equation for C in terms of the lengths of the octagon, this provides a practical way of computing an equation for Γ in terms of its Fenchel-Nielsen coordinates. Explicitly we can proceed as follows. Let H be the hexagon defined by $\{l_1, l_2, l_3\}$, and let H' be the hexagon defined by $\{l_1, l_3, l_2\}$ (i.e., obtained by taking the symmetric image with respect to the common orthogonal to l_1 and $\widehat{l_1}$). Glue H and H' along the sides of lengths l_1 to obtain an octagon (see Figure 7).

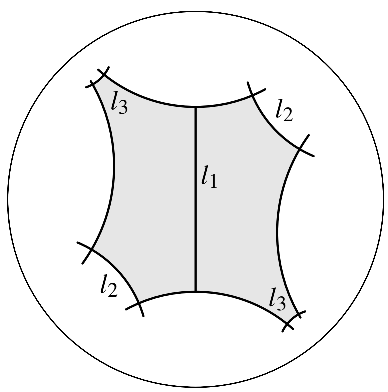

Figure 7

Compute the capacities of this octagon and the period matrix using the methods of Section 3. Because of the relations between the lengths we can, using Theta characteristics, find an equation for the corresponding genus-3 curve of the form in (7.7). The curve corresponding to $\{l_1, l_2, l_3\}$, with one half-twist along the side of length l_1 then has equation (7.8).

An alternative way to obtain an equation for Γ is to compute from the period matrix of C a period matrix of Γ and then to compute an equation for Γ using Theta characteristics. For this, let

$$Z = i\begin{pmatrix} y_1 & y_{12} & y_{13} \\ y_{12} & y_2 & y_{23} \\ y_{13} & y_{23} & y_3 \end{pmatrix}$$

be the standard period matrix associated to equation (7.7). The automorphism induced by $x \mapsto -1/x$ imposes nontrivial relations between the coefficients of Z. Namely, we have

$$y_1 - y_{12} = y_3 - y_{23} \qquad \text{and} \qquad y_2 = y_{12} + y_{23}.$$

From this, using the explicit description of the covering given above, it is straightforward to show that

$$\begin{pmatrix} \frac{1}{2} + i\frac{y_2}{2} & \frac{1}{2} + i\left(\frac{y_2}{2} - y_{12}\right) \\ \frac{1}{2} + i\left(\frac{y_2}{2} - y_{12}\right) & \frac{1}{2} + i\left(y_1 - y_{13} + \frac{y_2}{2} - y_{12}\right) \end{pmatrix}$$

is a period matrix for the curve with equation

$$y^2 = \left(x^2 + 4\right)\prod_{k=1}^{4}\left(x - x_k + \frac{1}{x_k}\right).$$

8. Tests and examples

We have written various programs using the considerations of the preceding sections. The programs have been written in C, using double precision. They are quite compact and totally portable.

We have tested various values for the number of points per arc and the degree of the harmonic polynomials. Experimentally we have found that taking 40 to 50 points per arc and degree of the polynomial between 80 and 120 is sufficient to obtain an accuracy of at least $\pm 10^{-10}$ in all examples listed below and even $\pm 10^{-12}$ in most cases. The time needed for the computations depends on the machine used but is never more than a few seconds.

We have mostly concentrated on genus 2, but we have also made tests for genus 1. In this case we have one example for which we have no explanation but which is nevertheless worth mentioning. The curve defined by the condition $l = \operatorname{arcsinh}(\sqrt{2})$ (half-length) and zero twist seems to have the equation $y^2 = x(x-1)(x-9/8)$ or, equivalently, $y^2 = (x^2-1)(x^2-4)$; the period is approximately $0.6396307855855\cdots i$.

For the related curve defined by the condition $l = \operatorname{arcsinh}(1/\sqrt{2})$, we find $y^2 = x(x-1)(x-9)$—and this again with a very good approximation.

For genus 2 we have many examples, including many exact examples, obtained using Table 5.17 and Propositions 6.18 and 7.4. We list these below and emphasize the point that no two surfaces in these lists are isometric (i.e., the curves are not isomorphic over $\mathbb{C}$). That the curves are not isomorphic over $\mathbb{R}$ is obvious from the length conditions, and that they are not isomorphic over $\mathbb{C}$ can be deduced from the results of S. Natanzon [Na, p. 70] (see also [BEGG]).

Examples with zero twists

Here $l_1, l_2, h_1, \widehat{l_1}, \widehat{l_2}$, and (a, b) have the same meaning as in Table 5.17. In the following tables the symbol —∗— means that the term does not have a simple expression, although it can be computed using the formulas in Table 5.17 and in Proposition 6.18.

To obtain a real equation for the curve corresponding to $(\widehat{l_1}, \widehat{l_2}, h_1, l_1, l_2)$, one only needs to replace (a, b) by $(1/b, 1/a)$.

Table 8.1. Obtained from the regular pentagon

$(\cosh l_1, \cosh l_2, \cosh h_1, \cosh \widehat{l_1}, \cosh \widehat{l_2})$	(a, b)
$\left(2+\sqrt{5}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, 2+\sqrt{5}, \frac{1+\sqrt{5}}{2}\right)$	$\left(\frac{\sqrt{5}-1}{2}, \frac{1+\sqrt{5}}{2}\right)$
$\left(\frac{1+\sqrt{5}}{2}, \frac{\sqrt{2}+\sqrt{10}}{4}, \frac{\sqrt{2}+\sqrt{10}}{2}, 5+2\sqrt{5}, 2+\sqrt{5}\right)$	$\left(\frac{\left(\sqrt{1+\sqrt{5}}+\sqrt{2}\right)^2}{2(3+\sqrt{5})}, \left(\frac{\sqrt{1+\sqrt{5}}+\sqrt{2}}{\sqrt{1+\sqrt{5}}-\sqrt{2}}\right)^2\right)$
$\left(5+2\sqrt{5}, \frac{\sqrt{2}+\sqrt{10}}{4}, \frac{\sqrt{2}+\sqrt{10}}{4}, 17+8\sqrt{5}, \frac{\sqrt{2}+\sqrt{10}}{2}\right)$	(—∗—, —∗—)
$\left(\frac{1+\sqrt{5}}{2}, \frac{\sqrt{2}+\sqrt{10}}{2}, 2+\sqrt{5}, \frac{1+\sqrt{5}}{2}, \frac{\sqrt{2}+\sqrt{10}}{2}\right)$	$\left(\left(\frac{\sqrt{1+\sqrt{5}}-\sqrt{2}}{\sqrt{1+\sqrt{5}}+\sqrt{2}}\right)^2, \left(\frac{\sqrt{1+\sqrt{5}}+\sqrt{2}}{\sqrt{1+\sqrt{5}}-\sqrt{2}}\right)^2\right)$

Table 8.2. Obtained from the regular hexagon

$\left(\cosh l_1, \cosh l_2, \cosh h_1, \cosh \widehat{l_1}, \cosh \widehat{l_2}\right)$	(a, b)
$(2, 2, 3, 2, 2)$	$\left(7-4\sqrt{3}, 7+4\sqrt{3}\right)$
$\left(2, \sqrt{\frac{3}{2}}, 2, 7, 3\right)$	$\left(\frac{2+\sqrt{3}}{4}, 7+4\sqrt{3}\right)$
$\left(7, \sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 17, 2\right)$	$\left(56\sqrt{3}-96, 8-4\sqrt{3}\right)$
$\left(3, \sqrt{2}, \sqrt{3}, 5, 2\right)$	$\left(\frac{3}{4}, 3\right)$
$\left(5, \sqrt{2}, \sqrt{2}, 7, \sqrt{3}\right)$	$\left(\frac{8}{9}, \frac{4}{3}\right)$
$\left(3, \sqrt{3}, 2, 3, \sqrt{3}\right)$	$\left(\frac{1}{3}, 3\right)$

Table 8.3. Obtained from the regular octagon

$\left(\cosh l_1, \cosh l_2, \cosh h_1, \cosh \widehat{l_1}, \cosh \widehat{l_2}\right)$	(a, b)
$\left(1+\sqrt{2}, \sqrt{1+\frac{\sqrt{2}}{2}}, \sqrt{2+\sqrt{2}}, 3+2\sqrt{2}, 1+\sqrt{2}\right)$	$\left(\frac{2+\sqrt{2}}{4}, 3+2\sqrt{2}\right)$
$\left(3+2\sqrt{2}, \sqrt{1+\frac{\sqrt{2}}{2}}, \sqrt{1+\frac{\sqrt{2}}{2}}, 5+4\sqrt{2}, \sqrt{2+\sqrt{2}}\right)$	$\left(12\sqrt{2}-16, 4-2\sqrt{2}\right)$
$\left(1+\sqrt{2}, \sqrt{2+\sqrt{2}}, 1+\sqrt{2}, 1+\sqrt{2}, \sqrt{2+\sqrt{2}}\right)$	$\left(3-2\sqrt{2}, 3+2\sqrt{2}\right)$

The curve corresponding to the regular dodecagon has equation $y^2 = x(x^2-1)(x^2-14x+1)$. The coefficients a and b of the normalized equation do not, however, have a simple expression since $a^{-1} = b = (7+4\sqrt{3}+2\sqrt{2}\sqrt{12+7\sqrt{3}})^2$.

Table 8.4. Obtained from the regular dodecagon

$\left(\cosh l_1, \cosh l_2, \cosh h_1, \cosh \widehat{l_1}, \cosh \widehat{l_2}\right)$	(a, b)
$\left(\frac{1+\sqrt{3}}{2}, 1+\sqrt{3}, 3+2\sqrt{3}, \frac{1+\sqrt{3}}{2}, 1+\sqrt{3}\right)$	(—∗—, —∗—)
$\left(7+4\sqrt{3}, \frac{\sqrt{3+\sqrt{3}}}{2}, \frac{\sqrt{3+\sqrt{3}}}{2}, 41+24\sqrt{3}, 1+\sqrt{3}\right)$	(—∗—, —∗—)
$\left(\frac{1+\sqrt{3}}{2}, \frac{\sqrt{3+\sqrt{3}}}{2}, 1+\sqrt{3}, 7+4\sqrt{3}, 3+2\sqrt{3}\right)$	(—∗—, —∗—)
$\left(3+2\sqrt{3}, \frac{\sqrt{2}+\sqrt{6}}{2}, \sqrt{\frac{3+\sqrt{3}}{2}}, 2+\sqrt{3}, \frac{1+\sqrt{3}}{2}\right)$	$\left(4-2\sqrt{3}, \frac{2\sqrt{3}}{3}\right)$
$\left(2+\sqrt{3}, \frac{\sqrt{2}+\sqrt{6}}{2}, \frac{\sqrt{2}+\sqrt{6}}{2}, 1+\sqrt{3}, \sqrt{\frac{3+\sqrt{3}}{2}}\right)$	$\left(\frac{1}{4}, 1+\frac{\sqrt{3}}{2}\right)$
$\left(3+2\sqrt{3}, \sqrt{\frac{3+\sqrt{3}}{2}}, \frac{1+\sqrt{3}}{2}, 3+2\sqrt{3}, \sqrt{\frac{3+\sqrt{3}}{2}}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{2\sqrt{3}}{3}\right)$

8.5

We also have one last exact family with zero twists; it is formed by the images of the surface at the intersection of the families defined by $l_1 = l_2 = l_3$ and $l_2 = h_1$—the latter being the fixed subspace under $\varphi\psi^3$ of Table 5.17. For this curve we find that $\cosh(l_1)$ is the root $3.214\ldots$ of the equation $x^3 - 3x^2 - x + 1 = 0$ and that $a = ((3-t)/(t^2+t))^2$, $b = ((3+t)/(t^2-t))^2$, where t is the root $2.542\ldots$ of the equation $t^6 + 22t^5 - 29t^4 - 28t^3 - 105t^2 - 90t - 27 = 0$. Note that this surface satisfies also the relations $l_1 = 2\widehat{l_1} = 2\widehat{l_2}$.

This curve and its transforms yield six new and nonisometric exact correspondences (with zero twists). Among these we find the curves defined by the relations $l_1 = \widehat{l_1} = 4l_2$ and $l_1 = 2l_2 = 2\widehat{l_1} = 8\widehat{l_2}$ (and, as always, $l_2 = l_3$).

Apart from these exact correspondences, we have found some very intriguing approximations (which are exact up to $\pm 10^{-12}$ or better).

Table 8.6

$\left(\cosh l_1, \cosh l_2, \cosh h_1, \cosh \widehat{l_1}, \cosh \widehat{l_2}\right)$	(a, b)
$\left(2, \sqrt{2}, \sqrt{5}, 4, \sqrt{6}\right)$	$\left(\frac{16}{25}, 16\right)$
$\left(9, \sqrt{6}, \frac{\sqrt{5}}{\sqrt{2}}, 3, \frac{\sqrt{3}}{\sqrt{2}}\right)$	$\left(\frac{3}{8}, \frac{25}{24}\right)$
$\left(4, \frac{\sqrt{3}}{\sqrt{2}}, \sqrt{2}, 11, \sqrt{5}\right)$	$\left(\frac{125}{128}, \frac{25}{16}\right)$
$\left(3, \sqrt{5}, \sqrt{6}, 2, \frac{\sqrt{5}}{\sqrt{2}}\right)$	$\left(\frac{1}{16}, \frac{8}{3}\right)$
$\left(11, \frac{\sqrt{5}}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}, 9, \sqrt{2}\right)$	$\left(\frac{24}{25}, \frac{128}{125}\right)$

Table 8.7

$\left(\cosh l_1, \cosh l_2, \cosh h_1, \cosh \widehat{l_1}, \cosh \widehat{l_2}\right)$	(a, b)
$\left(2, \frac{\sqrt{5}}{2}, \frac{\sqrt{7}}{\sqrt{2}}, 13, \sqrt{15}\right)$	$\left(\frac{896}{900}, \frac{35}{3}\right)$
$\left(6, \sqrt{15}, \sqrt{7}, \frac{3}{2}, \frac{\sqrt{3}}{\sqrt{2}}\right)$	$\left(\frac{5}{1029}, \frac{375}{343}\right)$
$\left(13, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{5}}{2}, 29, \frac{\sqrt{7}}{\sqrt{2}}\right)$	$\left(\frac{2400}{2401}, \frac{225}{224}\right)$
$\left(\frac{3}{2}, \frac{\sqrt{7}}{\sqrt{2}}, \sqrt{15}, 2, \sqrt{7}\right)$	$\left(\frac{3}{35}, \frac{1029}{5}\right)$
$\left(29, \sqrt{7}, \frac{\sqrt{3}}{\sqrt{2}}, 6, \frac{\sqrt{5}}{2}\right)$	$\left(\frac{343}{375}, \frac{2401}{2400}\right)$

For other examples the coefficients a and b are not as simple, but other forms of the equations can be expressed simply. We give these here but limit ourselves to one or two examples in each family.

8.8

We have $\cosh(l_1) = 3/2$, $\cosh(l_2) = \sqrt{6}$, and equation $y^2 = (x^2 - 1)(x^4 - (506/3)x^2 + 1)$. This curve and its transforms give six new ones with zero twists, among which $\cosh(l_1) = 5$, $\cosh(l_2) = \sqrt{5/2}$, with equation $y^2 = (x^2 - 1)(x^4 - (262/125)x^2 + 1)$.

8.9

We have $\cosh(l_1) = 3/2$, $\cosh(l_2) = 3/2$, and equation $y^2 = (x^2 - 1)(x^4 - 236x^2 + 100)$. This example has five distinct transforms (including this one) with zero twists. It is also isometric to the curve defined by $\cosh(l_1) = 3$, $\cosh(l_2) = 3$ with equation $y^2 = (x^2 - 1)(x^4 - (236/100)x^2 + 1/100)$.

8.10

We have $\cosh(l_1) = 2$, $\cosh(l_2) = \sqrt{7}/2$, and equation $y^2 = (x^2 - 1)(x^4 - (113/7)x^2 + 49/4)$. This example has five distinct transforms with zero twists.

Examples with one half-twist

We use here a somewhat different notation. The notation

$$\begin{cases} (\lambda_1, \lambda_2) \simeq (\lambda_1', \lambda_2) \\ (\mu_1, \mu_2, \mu_3) \end{cases} \qquad (a, b)$$

contains several pieces of information. First of all, it says that the surface is obtained by pasting two copies of the pair of pants with lengths $2l_i = 2\operatorname{arccosh}(\lambda_i)$, $l_2 = l_3$, zero twist along the boundary components of lengths $2l_2$ and $2l_3$, and a half-twist along the one of length $2l_1$; secondly, it says that this surface is isometric to the one obtained by replacing λ_1 by λ_1'. Both of these have a real structure, with two real components corresponding to the sides of lengths $2l_2$ and $2l_3$ (and these real structures are isomorphic). The corresponding equation is $y^2 = -(x^2 - a)(x^2 - b)(x^2 + 1)$. Finally, the surface is also isometric to the one obtained by pasting two copies of the pair of pants with lengths $2m_i = 2\operatorname{arccosh}(\mu_i)$, zero twist along the boundary components of lengths $2m_2$ and $2m_3$, and a half twist along the one of length $2m_1$. The real structure we consider in this case is the one with real components of lengths $2m_2$ and $2m_3$, and the real equation is this time $y^2 = (x^2 - a)(x^2 - b)(x^2 + 1)$ (see Section 7 for more details). Again the symbol —$*$— means that the coefficients do not have a simple expression but that they can be computed exactly (using Proposition 7.4 this time).

Table 8.11. Obtained from the regular pentagon

$\begin{cases} \left(2+\sqrt{5}, \frac{1+\sqrt{5}}{2}\right) \simeq \left(2+\sqrt{5}, \frac{1+\sqrt{5}}{2}\right) \\ \left(5+2\sqrt{5}, 2+\sqrt{5}, 2+\sqrt{5}\right) \end{cases}$	$\left(\frac{\sqrt{5}-1}{2}, \frac{1+\sqrt{5}}{2}\right)$
$\begin{cases} \left(\frac{1+\sqrt{5}}{2}, \frac{\sqrt{2}+\sqrt{10}}{4}\right) \simeq \left(5+2\sqrt{5}, \frac{\sqrt{2}+\sqrt{10}}{4}\right) \\ \left(23+10\sqrt{5}, 5+2\sqrt{5}, 17+8\sqrt{5}\right) \end{cases}$	(— ∗ —, — ∗ —)
$\begin{cases} \left(17+8\sqrt{5}, \frac{\sqrt{2}+\sqrt{10}}{2}\right) \simeq \left(\frac{1+\sqrt{5}}{2}, \frac{\sqrt{2}+\sqrt{10}}{2}\right) \\ \left(\frac{13+5\sqrt{5}}{2}, 5+2\sqrt{5}, \frac{1+\sqrt{5}}{2}\right) \end{cases}$	(— ∗ —, — ∗ —)

Table 8.12. Obtained from the regular hexagon

$\begin{cases} (2,2) \simeq (17,2) \\ (10,2,7) \end{cases}$	$\left(\frac{7\sqrt{3}}{24} - \frac{1}{2}, \frac{2\sqrt{3}-3}{6}\right)$
$\begin{cases} \left(2, \sqrt{\frac{3}{2}}\right) \simeq \left(7, \sqrt{\frac{3}{2}}\right) \\ (25,7,17) \end{cases}$	$\left(7-4\sqrt{3}, \frac{2\sqrt{3}-3}{6}\right)$
$\begin{cases} (5,2) \simeq (5,2) \\ (7,3,3) \end{cases}$	$\left(\frac{1}{3}, 3\right)$
$\begin{cases} (5,\sqrt{2}) \simeq (3,\sqrt{2}) \\ (13,7,5) \end{cases}$	$(2,3)$
$\begin{cases} (7,\sqrt{3}) \simeq (3,\sqrt{3}) \\ (9,5,3) \end{cases}$	$(2,8)$

Table 8.13. Obtained from the regular octagon

$\begin{cases}\left(1+\sqrt{2}, \sqrt{1+\frac{\sqrt{2}}{2}}\right) \simeq \left(3+2\sqrt{2}, \sqrt{1+\frac{\sqrt{2}}{2}}\right)\\ \left(9+6\sqrt{2}, 3+2\sqrt{2}, 5+4\sqrt{2}\right)\end{cases}$	$\left((\sqrt{2}-1)^2, \frac{\sqrt{2}-1}{2}\right)$
$\begin{cases}\left(3+2\sqrt{2}, 1+\sqrt{2}\right) \simeq \left(3+2\sqrt{2}, 1+\sqrt{2}\right)\\ \left(3+2\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\right)\end{cases}$	$\left((\sqrt{2}-1)^2, (1+\sqrt{2})^2\right)$
$\begin{cases}\left(5+4\sqrt{2}, \sqrt{2+\sqrt{2}}\right) \simeq \left(1+\sqrt{2}, \sqrt{2+\sqrt{2}}\right)\\ \left(5+3\sqrt{2}, 3+2\sqrt{2}, 1+\sqrt{2}\right)\end{cases}$	$\left(2+2\sqrt{2}, 16+12\sqrt{2}\right)$

The second curve in Table 8.13 (defined by $\cosh(l_1) = 3+2\sqrt{2}$, $\cosh(l_2) = 1+\sqrt{2}$) also has the more familiar equation $y^2 = x(x^4-1)$.

Table 8.14. Obtained from the regular dodecagon

$\begin{cases}\left(3+2\sqrt{3}, \frac{\sqrt{2}+\sqrt{6}}{2}\right) \simeq \left(2+\sqrt{3}, \frac{\sqrt{2}+\sqrt{6}}{2}\right)\\ \left(4+2\sqrt{3}, 2+\sqrt{3}, 1+\sqrt{3}\right)\end{cases}$	$\left(\frac{\sqrt{3}}{2}, 3+2\sqrt{3}\right)$
$\begin{cases}\left(2+\sqrt{3}, \frac{1+\sqrt{3}}{2}\right) \simeq \left(2+\sqrt{3}, \frac{1+\sqrt{3}}{2}\right)\\ \left(7+4\sqrt{3}, 3+2\sqrt{3}, 3+2\sqrt{3}\right)\end{cases}$	$\left(\frac{\sqrt{3}}{2}, \frac{2\sqrt{3}}{3}\right)$
$\begin{cases}\left(3+2\sqrt{3}, \sqrt{\frac{3+\sqrt{3}}{2}}\right) \simeq \left(1+\sqrt{3}, \sqrt{\frac{3+\sqrt{3}}{2}}\right)\\ \left(6+3\sqrt{3}, 3+2\sqrt{3}, 2+\sqrt{3}\right)\end{cases}$	$\left(3, 3+2\sqrt{3}\right)$
$\begin{cases}\left(\frac{1+\sqrt{3}}{2}, 1+\sqrt{3}\right) \simeq \left(41+24\sqrt{3}, 1+\sqrt{3}\right)\\ \left(\frac{17+9\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, 7+4\sqrt{3}\right)\end{cases}$	(— ∗ —, — ∗ —)
$\begin{cases}\left(\frac{1+\sqrt{3}}{2}, \frac{\sqrt{3+\sqrt{3}}}{2}\right) \simeq \left(7+4\sqrt{3}, \frac{\sqrt{3+\sqrt{3}}}{2}\right)\\ \left(49+28\sqrt{3}, 7+4\sqrt{3}, 41+24\sqrt{3}\right)\end{cases}$	(— ∗ —, — ∗ —)

8.15

For the curves obtained from the ones we mentioned in Section 8.5, we have the same difficulty as before in expressing the coefficients, but the transformation still gives five more nonisometric surfaces, for which we can in principle compute equations exactly.

8.16

For the surfaces listed in Tables 8.6 and 8.7 and in Sections 8.8–8.10 we only indicate that for each of these families we obtain five more nonisometric surfaces with one half-twist (hence a total of 25 additional ones).

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