

# Self-Bäcklund curves in centroaffine geometry and Lamé's equation

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## Abstract

Twenty five years ago U. Pinkall discovered that the Korteweg-de Vries equation can be realized as an evolution of curves in centroaffine geometry. Since then, a number of authors interpreted various properties of KdV and its generalizations in terms of centroaffine geometry. In particular, the Bäcklund transformation of the Korteweg-de Vries equation can be viewed as a relation between centroaffine curves.

Our paper concerns self-Bäcklund centroaffine curves. We describe general properties of these curves and provide a detailed description of them in terms of elliptic functions. Our work is a centroaffine counterpart to the study done by F. Wegner of a similar problem in Euclidean geometry, related to Ulam's problem of describing the (2-dimensional) bodies that float in equilibrium in all positions and to bicycle kinematics.

We also consider a discretization of the problem where curves are replaced by polygons. This is related to discretization of KdV and the cross-ratio dynamics on ideal polygons.

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## 1 Introduction

The motivation for this work is the interpretation of the Korteweg-de Vries equation in terms of centroaffine geometry. This growing body of work started with U. Pinkall's paper [40], see [16, 26, 27, 47] for a sampler.

In [45], the Bäcklund transformation of the KdV equation is interpreted as a relation between centroaffine curves. We start with a very brief description of this approach to KdV.

Let  $\gamma(t)$  be a parametrized smooth curve in the affine plane with a fixed area form. The curve is *centroaffine* if the Wronski determinant is constant:  $[\gamma(t), \gamma'(t)] = 1$  for all  $t \in \mathbb{R}$ . The group  $SL_2(\mathbb{R})$  acts on centroaffine curves, and we shall also consider the moduli space of such curves. Unless specified otherwise, we assume that the curves are  $\pi$ -anti-periodic:  $\gamma(t + \pi) = -\gamma(t)$  for all  $t$ . That is, the curve is closed, centrally symmetric and  $2\pi$ -periodic (the last condition can be arranged by an appropriate rescaling.)

The rationale for assuming that the curves are centrally symmetric is as follows. An orientation preserving diffeomorphism of  $\mathbb{RP}^1$  admits a unique area

preserving and homogeneous of degree 1 lifting to a diffeomorphism of the punctured plane. The image of the unit circle under such a diffeomorphism is a centrally symmetric star-shaped curve, and projectively equivalent diffeomorphisms correspond to  $\mathrm{SL}_2(\mathbb{R})$ -equivalent curves. See [38] for details.

Our results can be extended to non-centrally symmetric curves, but we do not dwell on it in this paper.

Given a centroaffine curve, one has  $\gamma''(t) = p(t)\gamma(t)$  where  $p$  is a  $\pi$ -periodic potential function of the Hill operator  $-d^2/dt^2 + p(t)$ . In the language of centroaffine geometry,  $p$  is the *centroaffine curvature* of the curve  $\gamma$  (alternatively, some authors call  $-p$  the *centroaffine curvature*, but we shall adopt the plus sign convention).

For example,  $\gamma(t) = (\cos t, \sin t)$  has  $p(t) = -1$ . This unit circle, and its  $\mathrm{SL}_2(\mathbb{R})$  images, are trivial examples of centroaffine curves. We refer to these curves as centroaffine conics.

A tangent vector to a centroaffine curve  $\gamma(t)$  (in the space of centro affine curves) is given by a vector field along it of the form  $g(t)\gamma(t) + f(t)\gamma'(t)$ . Taking the derivative of the centroaffine condition  $[\gamma, \gamma'] = 1$  with respect to this vector field we obtain  $f' + 2g = 0$ . Thus such a vector field has the form

$$V_f := -\frac{1}{2}f'(t)\gamma(t) + f(t)\gamma'(t), \quad (1)$$

where  $f$  is a  $\pi$ -periodic function. Pinkall observed in [40] that the evolution of the curves  $\gamma(t)$  with the potential function  $p(t)$  under the vector field  $V_p$  is a centroaffine version of the Korteweg-de Vries equation: the potential evolves according to the equation

$$\dot{p} = -\frac{1}{2}p''' + 3p'p$$

(where dot is the time derivative).

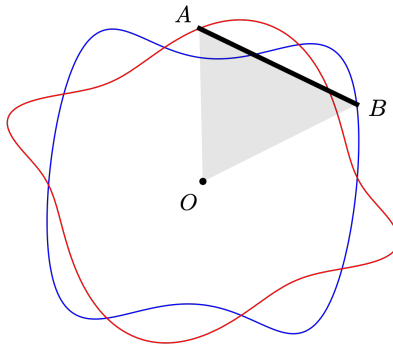


Figure 1: Bäcklund transformation: as the end points of the line segment  $AB$  trace the two curves,  $OA$  and  $OB$  sweep area which the same rate and the area of the shaded triangle  $OAB$  remains constant.

We say that two centroaffine curves,  $\gamma(t)$  and  $\delta(t)$ , are *c-related* if  $[\gamma(t), \delta(t)] =$

$c$  for all  $t$ . See Figure 1. It is shown in [45] that this relation is a geometric realization of the Bäcklund transformation for the KdV equation.

In this paper we are mostly interested in *self-Bäcklund* centroaffine curves, the curves  $\gamma(t)$  for which there exists  $\alpha \in (0, \pi)$  and a constant  $c$  such that

$$[\gamma(t), \gamma(t + \alpha)] = c \text{ for all } t. \quad (2)$$

For example, the centroaffine conics are self-Bäcklund for every choice of  $\alpha$  with  $c = \sin \alpha$ . To exclude trivial cases, we assume that  $c \neq 0$ . We call  $\alpha$  in equation (2) the *rotation number* of a self-Bäcklund curve. See Figure 2 for examples of self-Bäcklund curves.

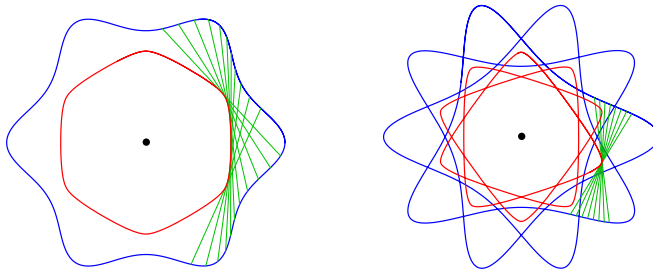


Figure 2: Self-Bäcklund curves (blue), with winding numbers 1 (left) and 3 (right). A line segment (green) moves with its endpoints sliding along the curve, forming a constant area triangle with the origin, while the midpoint of the line segment traces a curve (red), always tangent to the line segment at its midpoint. The two curves depicted here are members of an infinite family of self-Bäcklund curves described explicitly in Section 4 in terms of the Weierstrass  $\wp$ -function.

An analogous problem in Euclidean geometry was thoroughly studied relatively recently. The problem is to describe the closed smooth arc length parametrized curves  $\gamma(t) \subset \mathbb{R}^2$  for which there exist constants  $s$  and  $\ell$  such that  $|\gamma(t + s) - \gamma(t)| = \ell$  for all  $t$ . For example, a circle is a trivial solution to this problem.

Although the full solution of this problem is not available yet, there is a wealth of results, including many non-trivial examples of such curves. See [12, 42, 44, 49, 50, 51] for a sampler.

This problem originated in two seemingly unrelated theories. First, such curves are the boundaries of 2-dimensional bodies that float in equilibrium in all positions – to describe such bodies (in all dimensions) is S. Ulam’s problem in flotation theory, see [35], problem 19.

Second, an interesting problem in the study of a bicycle kinematics is to describe the pairs of front and rear bicycle tracks for which one cannot determine the direction of the bicycle motion. The above mentioned curves appear in this problem as the front tracks in such ambiguous pairs; they are referred to as *bicycle curves*. See [32] for a survey of this approach to bicycle kinematics.

This geometric problem is intimately related to another completely integrable equation of soliton type, the filament – or binormal, or smoke ring, or local induction – equation; more precisely, to the planar filament equation.

Two arc length parametrized curves,  $\gamma(t)$  and  $\delta(t)$ , are in bicycle correspondence if the length of the segment  $\gamma(t)\delta(t)$  is constant and the velocity of its midpoint is aligned with the segment for all  $t$ . This correspondence is a geometric realization of the Bäcklund transformation of the planar filament equation, and in this sense, bicycle curves are self-Bäcklund.

We must say more about the work of Franz Wegner, cited above. He discovered a large variety of bicycle curves (or solutions to the 2-dimensional Ulam’s problem), explicitly described in terms of elliptic functions. Wegner made his discovery by assuming that the desired solutions have a certain geometrical property, resulting in a differential equation on their curvature, that was solved in elliptic functions. Then he proved that indeed, for a proper choice of parameters, these curves solved the problem.

It is shown in [12] that Wegner’s curves are solutions to a variational problem: they are buckled rings (the relative extrema of the elastic – or bending – energy, subject to the length and area constraints), and they are solitons: under the planar filament flow, they evolve by isometries.

Our main goal in this paper is to obtain centroaffine analogs of these results.

In the spirit of discrete differential geometry, we also consider centroaffine polygons, a discretization of centroaffine curves. These are centrally symmetric  $2n$ -gons  $P_1, \dots, P_{2n}$  such that  $[P_i, P_{i+1}] = 1$  and  $P_{i+n} = -P_i$  for all  $i$  (the index is understood cyclically). A centroaffine  $2n$ -gon is a *self-Bäcklund*  $(n, k)$ -gon if there exists a constant  $c$  such that  $[P_i, P_{i+k}] = c$  for all  $i$ . A trivial example is an affine-regular  $2n$ -gon which is a self-Bäcklund  $(n, k)$ -gon for all  $k$ . The problem is to describe non-trivial self-Bäcklund  $(n, k)$ -gons.

These polygons are centroaffine analogs of the discretization of the bicycle curves, the bicycle polygons, studied in [42, 46]. Some of the results on self-Bäcklund  $(n, k)$ -gons in Section 5 were included in Section 7.3 of the original (but not the final) version of [7], and were motivated by the study of the cross-ratio dynamics on ideal polygons in the hyperbolic plane and hyperbolic space therein.

Centroaffine polygons are closely related to linear second-order difference equations with periodic solutions and with Coxeter’s frieze patterns, see [37]. In particular, given a simple centroaffine  $2n$ -gon, the determinants  $[P_i, P_j]$  with  $|i - j| < n$  form the entries, all positive, of a frieze pattern of width  $n - 3$ . In these terms, we are interested in frieze patterns that have a row consisting of the same numbers, but not every row being constant.

A word about the terminology that we use. We call a closed smooth curve *star-shaped* if every ray emanating from the origin intersects the curve transversely and only once. A curve is *locally star-shaped* if the above property holds locally, near every point. Equivalently,  $[\gamma(t), \gamma'(t)] \neq 0$  for all  $t$ . Star-shaped curves have winding number 1, but locally star-shaped curves can go around the origin several times.

The contents of this paper are as follows.

Section 2 concerns Bäcklund transformations of centroaffine curves. We describe a centroaffine analog of the rear track curve (in the above mentioned bicycle setting). We also interpret the Miura transformation in terms of centroaffine geometry.

Section 2.4 is devoted to the following problem: given a centroaffine curve  $\gamma$ , for which  $c$  do  $c$ -related curves exist? We provide a complete answer to this question. This result is a centroaffine analog of Menzin’s conjecture – now a theorem, originally formulated for hatchet planimeters, but it also applies to the bicycle model, see [32] or [24].

Section 3 comprises a variety of results on self-Bäcklund curves. In Theorem 3 we prove that a non-trivial infinitesimal deformation of a central conic as a self-Bäcklund curve exists if and only if either  $\alpha = \pi/2$  or  $\alpha$  satisfies the equation

$$\tan(k\alpha) = k \tan \alpha$$

for some integer  $k \geq 4$ . A similar result is known for bicycle curves, see [42].

We show that the cases of  $\alpha = \pi/3$  and  $\pi/4$  are rigid: only the central ellipses are self-Bäcklund. In contrast, if  $\alpha = \pi/2$ , one has a family of self-Bäcklund centroaffine curves with functional parameters. Example 4.11 provides families of analytic curves with rotation number  $\pi/2$  and, at the same time, examples of analytic Radon curves.

Sections 3.4 and 3.5 concern *centroaffine carrousel*s, self-Bäcklund curves with a rational rotation number (we call them carrousel because this term was used in [14, 15] in the study of a similar problem in Euclidean geometry). We describe centroaffine carrousel as closed trajectories of a certain Hamiltonian vector field on the space of centroaffine  $2n$ -gons. We provide details in the first non-trivial case,  $n = 5$ . A similar approach to bicycle curves was developed in [14, 15]. We also study the centroaffine curves that are  $c$ -related to central ellipses.

Section 4 is the core part of the paper. We start by developing a centroaffine analog of Wegner’s ansatz, that is, we guess what geometric properties self-Bäcklund curves may possess. This leads to the assumption that these curves correspond to the traveling wave solutions of the KdV equation, that is, their centroaffine curvature is an elliptic function.

Thus we assume that the coordinates of our self-Bäcklund curves satisfy the Lamé equation, the Hill equation whose potential is an elliptic function. In Section 4.2 we construct these curves and describe the conditions on the parameters for which the curves are self-Bäcklund. This work is analogous to the one done by F. Wegner. In Section 4.3 we show that central conics indeed admit a deformation into self-Bäcklund centroaffine curves for each  $\alpha$  appearing in Theorem 3.

Section 5 concerns self-Bäcklund centroaffine polygons. We describe a discrete version of Bäcklund transformation on centroaffine polygons. Theorem 11 lists some pairs  $(n, k)$  for which non-trivial self-Bäcklund polygons do not exist, and some pairs for which they do. We also describe necessary and sufficient

conditions for the existence of non-trivial infinitesimal deformations of regular centroaffine  $n$ -gons in the class of self-Bäcklund polygons. Similar results were known for bicycle polygons, see [42].

In Appendix A we connect centroaffine geometry with another geometry associated with the group  $\mathrm{SL}_2(\mathbb{R})$ , two-dimensional hyperbolic geometry. We assign to a centroaffine curve a curve in the hyperbolic plane, its dual. The centroaffine curvature  $p$  of a curve and the curvature  $\kappa$  of its dual are related by the equation  $(1+p)(1+\kappa) = 2$ .

Appendix B is a compendium of the formulas involving Weierstrass elliptic functions that we use in the body of the paper.

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## 2 Bäcklund transformations of centroaffine curves

### 2.1 The middle curve

Let  $\gamma(t)$  be a centroaffine curve satisfying  $\gamma''(t) = p(t)\gamma(t)$ . Construct a new centroaffine curve  $\delta(t) = f(t)\gamma(t) + g(t)\gamma'(t)$ , where  $f(t)$  and  $g(t)$  are  $\pi$ -periodic functions. The next lemma repeats Lemma 1.2 of [45].

**Lemma 2.1.** *The curves  $\gamma$  and  $\delta$  are  $c$ -related if and only if  $g(t) = c$  and*

$$cf'(t) - f^2(t) + c^2p(t) + 1 = 0. \quad (3)$$

*Proof.* One has

$$c = [\gamma(t), \delta(t)] = g(t)[\gamma(t), \gamma'(t)] = g(t),$$

and therefore  $g'(t) = 0$ . Next,

$$\delta'(t) = (f'(t) + p(t)g(t))\gamma(t) + (f(t) + g'(t))\gamma'(t),$$

hence

$$1 = [\delta(t), \delta'(t)] = f^2(t) - c(f'(t) + cp(t)).$$

This implies equation (3). □

Note that equation (3) is a Riccati equation for the unknown function  $f(t)$ .

**Lemma 2.2.** *Let  $\gamma$  and  $\delta$  be  $c$ -related and let  $\Gamma(t)$  be the midpoint of the segment  $\gamma(t)\delta(t)$ . Then the velocity of  $\Gamma$  is aligned with this segment:*

$$\Gamma'(t) \sim \delta(t) - \gamma(t)$$

for all  $t$ . In addition,  $\Gamma$  is locally star-shaped, that is,  $[\Gamma(t), \Gamma'(t)] \neq 0$  for all  $t$ .

*Proof.* Since  $[\gamma, \gamma'] = [\delta, \delta'] = 1$  and  $[\gamma, \delta] = c$ , one has

$$[\gamma' + \delta', \delta - \gamma] = [\gamma', \delta] - [\delta', \gamma] = [\gamma, \delta]' = 0,$$

as needed.

For the second statement, if  $[\Gamma(t), \Gamma'(t)] = 0$  then the line connecting  $\gamma(t)$  and  $\delta(t)$  passes through the origin, and then  $c = 0$ .  $\square$

**Remark 2.3.** The locus of midpoints in the previous lemma plays the role of the rear bicycle track in the analogous problem mentioned in the Introduction. This middle curve may have cusps.

We describe a method of constructing pairs of  $c$ -related curves. Start with a locally star shaped curve  $\Gamma$ , with a centroaffine parameter  $s$  and curvature  $p(s)$ , so that  $[\Gamma, \Gamma_s] = 1$ ,  $\Gamma_{ss} = p\Gamma$ . Let  $\gamma_{\pm} := \Gamma \pm (c/2)\Gamma_s$ . The condition  $[\gamma_-, \gamma_+] = c$  is immediate; however, in general,  $s$  is not a centroaffine parameter for  $\gamma_{\pm}$ .

**Proposition 2.4.** *If  $c^2p \neq 4$  along  $\Gamma$  (for example, if  $\Gamma$  is locally convex, that is,  $p < 0$ ), then  $\gamma_{\pm}$  can be simultaneously reparametrized by a centroaffine parameter  $t$ , so that  $[\gamma_{\pm}, (\gamma_{\pm})_t] = 1$ .*

*Proof.* We calculate that  $[\gamma_{\pm}, (\gamma_{\pm})_s] = 1 - (c^2/4)p$ . If this does not vanish, then the desired parameter  $t$  is defined by

$$\frac{dt}{ds} = 1 - \frac{c^2p(s)}{4}.$$

With this new parameter one has  $[\gamma_{\pm}, (\gamma_{\pm})_t] = 1$ , as needed.  $\square$

**Remark 2.5.** As we mentioned, and as is seen from illustrations in this paper, the middle curve  $\Gamma$  may have cusps. The above construction of the curves  $\gamma_{\pm}$  from  $\Gamma$  extends to the case when  $\Gamma$  has cusps and the curves  $\gamma_{\pm}$  remain smooth. Without going into details, we illustrate this with an example.

Let  $\Gamma(x) = (x^2, x^3 + 1)$  be a cusp, and let  $s$  be a centroaffine parameter. Then  $\Gamma_x = (2x, 3x^2)$  and

$$\frac{ds}{dx} = [\Gamma, \Gamma_x] = x^4 - 2x.$$

It follows that

$$\gamma_{\pm} = \Gamma \pm \frac{c}{2}\Gamma_s = \left(\mp\frac{c}{2}, 1\right) + \left(0, \mp\frac{3c}{4}\right)x + O(x^2),$$

which, for  $c \neq 0$  and  $x$  close to zero, are smooth curves.

**Remark 2.6.** Consider an oriented smooth closed strictly convex plane curve  $\Gamma$ . The outer billiard transformation  $T$  is a map of its exterior, defined as follows: given a point  $x$ , draw the oriented tangent line from  $x$  to  $\Gamma$ , and reflect  $x$  in the tangency point to obtain the point  $T(x)$ . See [21] for a survey.

The relation of our topic to outer billiards is as follows: if  $\gamma$  is a self-Bäcklund curve and the respective middle curve  $\Gamma$  is convex, then  $\gamma$  is an invariant curve of the outer billiard map about  $\Gamma$ .



## 2.2 Curves $c$ -related to centroaffine conics

In this section we consider the curves that are  $c$ -related to centroaffine conics and attempt to find self-Bäcklund curves among them. These curves will have points at infinity.

Let  $\gamma(t) = (\cos t, \sin t)$ , and let us construct a  $c$ -related curve as in Lemma 2.1:  $\delta(t) = f(t)\gamma(t) + c\gamma'(t)$ . The respective Riccati equation for the function  $f$  is

$$cf'(t) = f^2(t) + c^2 - 1. \quad (4)$$

Assume that  $c > 1$ . This differential equation is easily solved:

$$f(t) = a \tan\left(\frac{at}{c}\right), \quad \text{where } a = \sqrt{c^2 - 1} \quad (5)$$

and a choice of the constant of integration has been made so that  $f(0) = 0$  (any other solution is obtained by a parameter shift).

The function  $f$  has poles (the same is true for the solutions with  $c < 1$  and  $c = 1$ ), and the respective centroaffine curve goes to infinity, having there an inflection point.

For example, let  $c = 5/3, a = 4/3$ , see Figure 3. This curve is periodic with period  $10\pi$ .

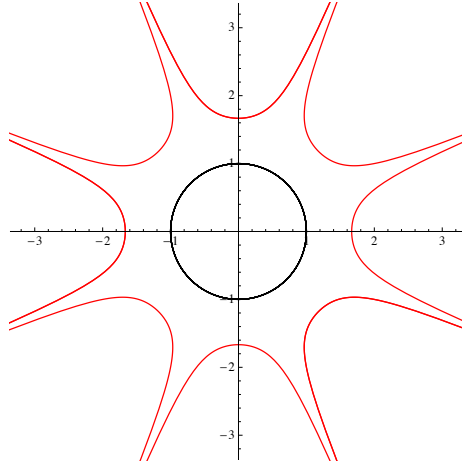


Figure 3: The curve  $\delta(t) = \left(\frac{4}{3} \tan\left(\frac{4t}{5}\right) \cos t - \frac{5}{3} \sin t, \frac{4}{3} \tan\left(\frac{4t}{5}\right) \sin t + \frac{5}{3} \cos t\right)$ .

Let us look for Bäcklund curves among the above curves  $\delta$ .

**Lemma 2.7.** *Let  $\delta$  be the centroaffine curve  $c$ -related to the unit circle  $\gamma(t) = (\cos t, \sin t)$ , where  $c > 1$ . Then  $\delta$  is self-Bäcklund with rotation number  $\alpha$ , that is,  $[\delta(t), \delta(t + \alpha)] = \text{const}$ , if and only if  $\alpha$  satisfies*

$$\tan(u\alpha) = u \tan \alpha, \quad \text{where } u = \frac{\sqrt{c^2 - 1}}{c}. \quad (6)$$

Furthermore, given such an  $\alpha$ , one has  $[\delta(t), \delta(t + \alpha)] = \sin \alpha$ .

*Proof.* The statement is invariant under parameter shift so it is enough to consider  $\delta = f\gamma + c\gamma'$ , where  $f$  is given by formula (5). Next, by a straightforward calculation, the derivative of  $[\delta(t), \delta(t + \alpha)]$  with respect to  $t$  is some non-zero function times  $\tan(u\alpha) - u \tan \alpha$ . It follows that  $[\delta(t), \delta(t + \alpha)]$  is constant if and only if  $\tan(u\alpha) = u \tan \alpha$ . Using this equation for  $\alpha$ , we calculate that  $[\delta(t), \delta(t + \alpha)] = \sin \alpha$ .  $\square$

In general, for a fixed  $u \in (0, 1)$ , equation (6) has infinitely many solutions. See Figure 4. If  $u$  is rational then  $\delta$  is periodic and there are finitely many solutions  $\alpha$  within a period.

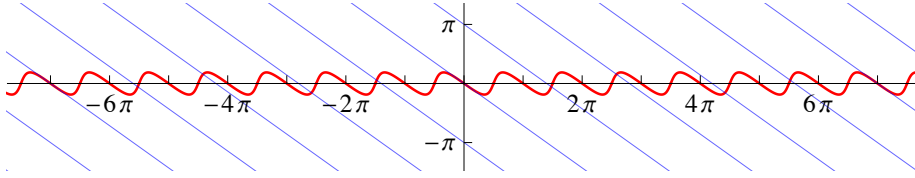


Figure 4: Solutions to equation (6),  $u \tan \alpha = \tan(u\alpha)$ ,  $u \in (0, 1)$ , are given by the intersection points of the (red) graph of the  $\pi$ -periodic function  $y = \tan^{-1}(u \tan \alpha) - \alpha + \pi n$ ,  $\pi n - \frac{\pi}{2} \leq \alpha \leq \pi n + \frac{\pi}{2}$ ,  $n \in \mathbb{Z}$ , and any of the (blue) lines  $y = (u - 1)\alpha + n\pi$ ,  $n \in \mathbb{Z}$ . If  $u$  is rational then  $f = a \tan(ut)$  is periodic and  $\delta$  is closed, self-Bäcklund with rotation numbers  $\alpha$  given by the intersection points within a period of  $f$ . In the figure above,  $u = 2/7$ ,  $f$  is  $7\pi$ -periodic,  $\delta$  is  $14\pi$ -periodic, and there are 8 solutions  $\alpha \in (0, 14\pi)$  with  $\sin \alpha \neq 0$ .

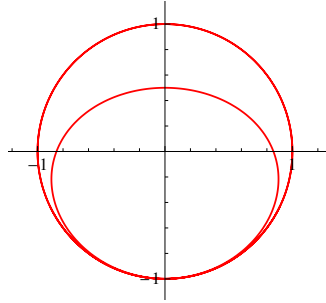


Figure 5: The curve  $\delta(t) = \left(-\frac{4}{5} \tanh\left(\frac{4t}{3}\right) \cos t - \frac{3}{5} \sin t, -\frac{4}{5} \tanh\left(\frac{4t}{3}\right) \sin t + \frac{3}{5} \cos t\right)$ .

A solution of equation (4) for  $c < 1$  is similar:

$$f(t) = -a \tanh\left(\frac{at}{c}\right),$$

where  $a^2 = 1 - c^2$ . The associated  $c$ -related curve  $\delta = f\gamma + c\gamma'$  is non-periodic

and stays bounded; it is self-Bäcklund with a parameter shift  $\alpha$  satisfying

$$\tanh(u\alpha) = u \tan \alpha, \quad \text{where } u = \frac{\sqrt{1-c^2}}{c}$$

and the constant determinant is  $\sin \alpha$ . This equation admits infinitely many solutions  $\pm\alpha_1, \pm\alpha_2, \dots$ , with  $\alpha_n \in (n\pi, n\pi + \pi/2)$ . For  $t \rightarrow \pm\infty$ , the curve approaches the unit circle, see Figure 5.

Another solution of (4) for  $c < 1$  is

$$f(t) = -a \coth\left(\frac{at}{c}\right),$$

with the respective value of  $\alpha$  given by

$$\coth(u\alpha) = u \tan \alpha, \quad \text{where } u = \frac{\sqrt{1-c^2}}{c}$$

and the constant determinant is  $\sin \alpha$ . There are infinitely many solutions here as well,  $\pm\alpha_0, \pm\alpha_1, \dots$ , with  $\alpha_n \in (n\pi, n\pi + \pi/2)$ . This curve approaches the unit circle as  $t \rightarrow \pm\infty$  and goes to infinity as  $t \rightarrow 0$ . See Figure 6.

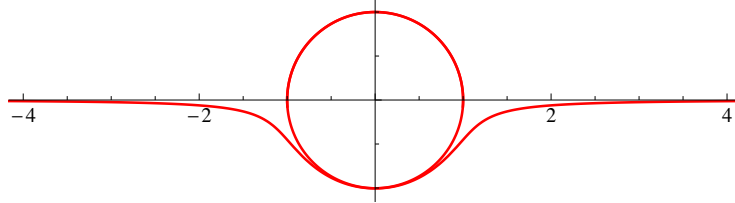


Figure 6: The curve  $\delta(t) = \left(-\frac{4}{5} \coth\left(\frac{4t}{3}\right) \cos t - \frac{3}{5} \sin t, -\frac{4}{5} \coth\left(\frac{4t}{3}\right) \sin t + \frac{3}{5} \cos t\right)$ .

If  $c = 1$ , a solution of equation (4) is  $f(t) = -1/t$ . This curve is self-Bäcklund with a parameter shift  $\alpha$  satisfying  $\tan \alpha = \alpha$  and the constant determinant is  $\sin \alpha$ . There are infinitely many solutions  $\pm\alpha_1, \pm\alpha_2, \dots$ , with  $\alpha_n \in (n\pi, n\pi + \pi/2)$ . Its asymptotic behavior is the same as in the previous example, see Figure 7.

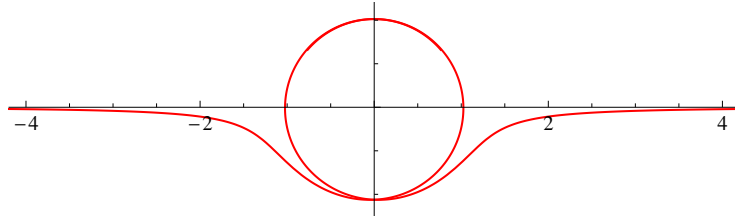


Figure 7: The curve  $\delta(t) = \left(-\frac{1}{t} \cos t - \sin t, -\frac{1}{t} \sin t + \cos t\right)$ .

For completeness, consider the case of a straight line  $\gamma(t) = (t, -1)$ . This centroaffine curve is self-Bäcklund for an arbitrary parameter shift. A  $c$ -related

curve  $f\gamma + c\gamma'$  has  $f(t) = -\tanh(t/c)$ , see Figure 8. This curve is not self-Bäcklund: the respective equation on the parameter shifts  $b$  is

$$\tanh\left(\frac{b}{c}\right) = \frac{b}{c},$$

and the only solution is  $b = 0$ .

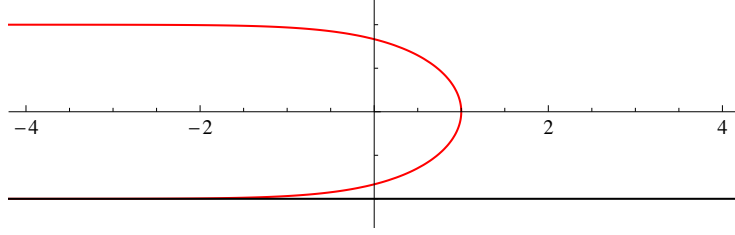


Figure 8: The curve  $\delta(t) = (1 - t \tanh t, \tanh t)$  (red), a Bäcklund transform of the line  $y = -1$  (black).

### 2.3 $c$ -related curves and Miura transformation

The Miura transformation connects the Korteweg-de Vries equation  $\dot{u} = u''' + 6uu'$  and the modified Korteweg-de Vries equation  $\dot{v} = v''' - 6v^2v'$ : if  $v$  satisfies mKdV then  $u = -v' - v^2$  satisfies KdV. More generally, if

$$u = -v' - v^2 + \lambda, \quad (7)$$

and  $v$  satisfies

$$\dot{v} = v''' - 6v^2v' - 6\lambda v', \quad (8)$$

then  $u$  satisfies KdV. See [25].

Given  $u$ , equation (7) is a Riccati equation on  $v$ , just like equation (3) on the function  $f(t)$  that describes the curves,  $c$ -related to a centroaffine curve with curvature  $p(t)$ . This provides a geometrical interpretation of the Miura transformation in centroaffine geometry.

The details are described by the next theorem.

**Theorem 1.** *Let  $\gamma$  be a centroaffine curve, and  $\delta = f\gamma + c\gamma'$  be a  $c$ -related curve. Let the curves  $\gamma$  and  $\delta$  evolve by the KdV flow. Then they remain  $c$ -related, and the function  $f$  evolves according to a version of mKdV:*

$$\dot{f} = -\frac{1}{2}f''' + \frac{3}{c^2}(f^2 - 1)f'.$$

*Proof.* Let  $q$  be the centroaffine curvature of  $\delta$ , that is,  $\delta''(t) = q(t)\delta(t)$ . Then  $\dot{\gamma} = V_p$ ,  $\dot{\delta} = V_q$ , where we use the notation as in equation (1).

We start with the observation that  $\gamma = f\delta - c\delta'$ , and then we express the curvatures  $p$  and  $q$  from equations (3) as follows

$$p = \frac{1}{c^2}(f^2 - 1 - cf'), \quad q = \frac{1}{c^2}(f^2 - 1 + cf') \quad (9)$$

(compare with Lemma 3.1 in [45]). It follows that

$$q - p = \frac{2}{c}f', \quad p' + q' = \frac{4}{c^2}ff'. \quad (10)$$

That  $\gamma$  and  $\delta$  remain  $c$ -related under the KdV flow follows from the fact the  $c$ -relation commutes with the KdV flow, see [45]. Here is an independent verification.

We have:  $\delta' = (f' + cp)\gamma + f\gamma'$ , and

$$\begin{aligned} [\gamma, \delta]' &= [V_p, \delta] + [\gamma, V_q] = [-0.5p'\gamma + p\gamma', \delta] + [\gamma, -0.5q'\delta + q\delta'] = \\ &= -0.5c(p' + q') + f(q - f) = 0, \end{aligned}$$

the last equality due to equation (10).

To calculate  $\dot{f}$ , note that  $f = [\delta, \gamma']$ . Then

$$\dot{f} = [\dot{\delta}, \gamma'] + [\delta, \dot{\gamma}'] = [V_q, \gamma'] + [\delta, V_p'] = [-0.5q'\delta + q\delta', \gamma'] + [\delta, (-0.5p'\gamma + p\gamma')'].$$

After substituting the values of  $p$  and  $q$  and their derivatives in terms of  $f$  from equation (9) and collecting terms we obtain the stated equality.  $\square$

One can expand a periodic solution of equation (3) in a power series in  $c$ :

$$\begin{aligned} f &= 1 + \frac{c^2}{2}p + \frac{c^3}{4}p' + \frac{c^4}{8}(p'' - p^2) + \frac{c^5}{16}(p''' - 8pp') \\ &\quad + \frac{c^6}{32}[p'''' - 10pp'' - 9(p')^2 + 2p^3] + \dots \end{aligned}$$

Given the relation of  $f$  with the Miura transformation, one has the next statement; see Section 1.1 of [25].

**Corollary 2.8.** *The integrals of the odd terms of this series vanish, and the integrals of the even terms are integrals of the KdV equation:*

$$\int_0^\pi p \, dt, \quad \int_0^\pi p^2 \, dt, \quad \int_0^\pi \left( p^3 + \frac{1}{2}(p')^2 \right) \, dt, \dots$$

See [12], Section 3.3 for a similar statement about the bicycle transformation and the filament equation.

## 2.4 Range of the parameter $c$

The aim of this section is to describe, for a given centroaffine closed  $\pi$ -anti-symmetric curve  $\gamma(t)$ , the range of the parameter  $c$  for which  $\gamma$  admits closed centroaffine  $c$ -related curves. The main result is Theorem 2 below, describing this range (a closed interval) in terms of the lowest eigenvalue of a Hill equation associated with  $\gamma$ . For a convex  $\gamma$  we obtain as a corollary an upper bound on  $c$  in terms of the area enclosed by its dual curve  $\gamma^*$ . This result can be viewed as a centroaffine analog of Menzin's conjecture for hatchet planimeters (equivalently, bicycle monodromy), discussed and proved in [32].

As we saw in Lemma 2.1, finding a centroaffine curve  $c$ -related to a given curve  $\gamma$  amounts to finding a solution  $f(t)$  to the Riccati equation

$$cf' - f^2 + c^2p(t) + 1 = 0, \quad (11)$$

where  $p = [\gamma'', \gamma']$  (the centroaffine curvature of  $\gamma$ ). The corresponding  $c$ -related centroaffine curve is  $\delta = f\gamma + c\gamma'$ . If  $\gamma$  is  $\pi$ -anti-symmetric then  $p$  in equation (11) is  $\pi$ -periodic and we are looking for the values of the parameter  $c$  for which the equation admits a  $\pi$ -periodic solution, so that  $\delta$  is  $\pi$ -anti-symmetric as well. Note that for  $c = 0$  the equation admits the trivial solution  $f \equiv 1$ .

Our study of the Riccati equation (11) is based on its relation with the Hill equation

$$y'' + (\lambda - p(t))y = 0. \quad (12)$$

To state this relation we recall first that a solution  $y(t)$  of (12) is called  $\pi$ -quasiperiodic if  $y(t + \pi) = \mu y(t)$  for all  $t$  and some  $\mu \in \mathbb{R}$ ,  $\mu \neq 0$ , called the Floquet multiplier of  $y(t)$ . If  $\mu = 1$  then the solution is  $\pi$ -periodic and if  $\mu = -1$  it is  $\pi$ -antiperiodic.

**Proposition 2.9.** *The Riccati equation (11) with a  $\pi$ -periodic  $p(t)$  admits a  $\pi$ -periodic solution  $f(t)$  for a parameter value  $c \neq 0$  if and only if the Hill equation (12) admits a positive  $\pi$ -quasiperiodic solution  $y(t)$  for  $\lambda = -1/c^2$ .*

*Proof.* Indeed, if there exists such  $y(t)$ , then  $f := -cy'/y$  is a periodic solution of equation (11). In the opposite direction: if  $f$  is a periodic solution of equation (11) and  $F$  is a primitive of  $f$  then  $y := e^{-F/c}$  is the required solution of equation (12).  $\square$

We now borrow a well-known result from the general theory of the Hill equation, due to Lyapunov and Haupt (ca. 1910, see Theorem 2.1 on page 11 of [33]).

**Theorem** (Spectrum of the Hill operator). *Consider equation (12),*

$$y'' + (\lambda - p(t))y = 0,$$

*where  $y(t)$  is an unknown real function,  $p(t)$  is a real  $\pi$ -periodic function and  $\lambda$  a real parameter. Then there exist two unbounded sequences of real numbers*

$$\begin{aligned} \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots, \\ \mu_0 \leq \mu_1 < \mu_2 \leq \mu_3 < \mu_4 \leq \dots, \end{aligned}$$

satisfying

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_4 < \dots, \quad (13)$$

such that equation (12) has a non-trivial  $\pi$ -periodic solution if and only if  $\lambda = \lambda_k$ , and a  $\pi$ -anti-periodic non-trivial solution if and only if  $\lambda = \mu_k$ ,  $k = 0, 1, \dots$ . The number of zeros on  $[0, \pi)$  of a solution corresponding to  $\lambda_{2k-1}$  or  $\lambda_{2k}$  is  $2k$ . In particular, if a  $\pi$ -periodic solution has no zeros, then  $\lambda = \lambda_0$ . Similarly, the number of zeros on  $[0, \pi)$  of a non-trivial solution corresponding to  $\mu_{2k}$  or  $\mu_{2k+1}$  is  $2k+1$ . Moreover, a solution to equation (12) is unstable (that is, unbounded) if and only if  $\lambda$  belongs to one of the intervals  $(-\infty, \lambda_0)$ ,  $(\mu_0, \mu_1)$ ,  $(\lambda_1, \lambda_2), \dots$  (called instability intervals, or ‘gaps’). See Figure 9.

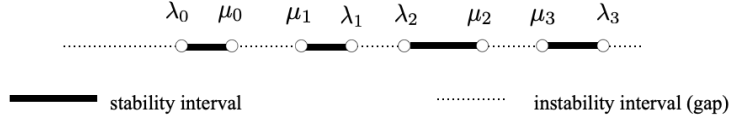


Figure 9: The spectrum of Hill’s equation (12), stability and instability intervals.

Concerning the lowest eigenvalue  $\lambda_0$ , we have the following.

**Lemma 2.10.** *Let  $\lambda_0$  be the first eigenvalue of the spectrum (13) of the Hill equation (12) associated with a  $\pi$ -antisymmetric centroaffine curve  $\gamma$ . Then*

$$\lambda_0 < 0, \quad \lambda_0 \leq -P,$$

where

$$P := -\frac{1}{\pi} \int_0^\pi p(t) dt. \quad (14)$$

*Proof.* Each of the two coordinate components of  $\gamma$  is a non-trivial  $\pi$ -anti-periodic solution of equation (12) for  $\lambda = 0$ . This implies that  $\mu_k = 0$  for some  $k \geq 1$ , hence  $\lambda_0 < 0$ .

The inequality  $\lambda_0 \leq -P$  is due to Borg (see Theorem 3.3.1 of [23]). The following argument is due to Ungar: Take a positive periodic solution  $y(t)$  of equation (12) corresponding to  $\lambda_0$ . Then  $h(t) = y'(t)/y(t)$  is a periodic solution of the Riccati equation  $h' + h^2 + (\lambda_0 - p(t)) = 0$ . Integrating this equation over the period gives:

$$\int_0^\pi (\lambda_0 - p(t)) dt \leq 0.$$

This yields the result.  $\square$

**Remark 2.11.** If  $\gamma$  is locally convex, so that  $p(t)$  is strictly negative, then  $P > 0$  and we have  $\lambda_0 \leq -P < 0$ . The geometric meaning of  $P$  is the area bounded by the dual curve  $\gamma^*$  (we refer to [29] and [43] for this and related facts).

**Theorem 2.** *Let  $\gamma$  be a centroaffine  $\pi$ -anti-symmetric curve and  $\lambda_0 < 0$  the lowest  $\pi$ -periodic eigenvalue of the associated Hill equation (12). Then  $\gamma$  admits a  $c$ -related closed curve if and only if  $|c| \leq 1/\sqrt{-\lambda_0}$ .*

An immediate consequence of this theorem and Lemma 2.10 is the following.

**Corollary 2.12.** *Suppose  $P > 0$  (for example  $\gamma$  is locally convex) and  $\gamma$  admits a  $c$ -related  $\pi$ -anti-periodic closed curve. Then  $|c| \leq 1/\sqrt{P}$ .*

*Proof of Theorem 2.* By Proposition 2.9, we need to show that equation (12) admits a  $\pi$ -quasiperiodic positive solution if and only if  $\lambda \leq \lambda_0$ .

Consider first the “if” part. If  $\lambda = \lambda_0$  then equation (12) has a positive periodic solution, hence quasi-periodic. So we shall assume now that  $\lambda < \lambda_0$ . In this case equation (12) has no conjugate points, that is, a non-trivial solutions vanishing at two distinct points  $t_1, t_2$  because, by the Sturm Comparison Theorem, any solution for every larger  $\lambda$  must have a zero between  $t_1, t_2$ . However for  $\lambda_0$  there is a positive periodic solution. To complete the proof of the “if” part we make use of the following lemma.

**Lemma 2.13.** *The equation  $y'' + q(t)y = 0$ , where  $q(t + \pi) = q(t)$ , has no conjugate points if and only if it admits a positive  $\pi$ -quasiperiodic solution.*

As far as we know, this lemma is due to E. Hopf [30]. For completeness, we give its proof below.

Now we prove Theorem 2 in the opposite direction. We need to show that equation (12) admits no positive  $\pi$ -quasiperiodic solution for  $\lambda > \lambda_0$ . Assume  $y(t)$  is such a solution,  $y(t + \pi) = \mu y(t)$ , where  $\mu > 0$ . There are two cases:

- If  $\mu = 1$  then  $y(t)$  is a positive periodic solution. But this is possible only for  $\lambda = \lambda_0$ , a contradiction.
- If  $\mu \neq 1$  then the solution  $y(t)$  is unbounded, and hence  $\lambda$  belongs to one of the instability zones. In particular,  $\lambda > \mu_0$ . But then, by the Sturm Comparison Theorem,  $y(t)$  cannot be positive since solutions for  $\mu_0$  have zeroes.

This completes the proof of Theorem 2. □

*Proof of Lemma 2.13* (after E. Hopf). If a Hill equation  $y'' + q(t)y = 0$  has no conjugate points then for every two distinct  $a, b \in \mathbb{R}$  there exist a unique solution  $y(t; a, b)$  satisfying

$$y(a; a, b) = 1, \quad y(b; a, b) = 0.$$

By uniqueness, one has the relation for distinct  $a, a'$ :

$$y(t; a, b) = y(a'; a, b)y(t; a', b). \tag{15}$$

Using disconjugacy, one can show that a limiting solution exists and is positive everywhere:

$$y(t; a) := \lim_{b \rightarrow +\infty} y(t; a, b).$$



These positive solutions are  $\pi$ -quasiperiodic. Indeed, setting  $a' \mapsto a + \pi$ ,  $t \mapsto t + \pi$  in equation (15)) and passing to the limit  $b \rightarrow +\infty$ , we get

$$y(t + \pi; a) = y(a + \pi; a)y(t + \pi; a + \pi) = y(a + \pi; a)y(t; a), \quad (16)$$

where the last equality is due to the  $\pi$ -periodicity of  $q(t)$ . Thus,  $y(t; a)$  is  $\pi$ -quasiperiodic with multiplier  $\mu = y(a + \pi; a)$ , as needed.

In the opposite direction the claim is obvious: if  $y'' + q(t)y = 0$  admits a positive solution then, by the Sturm Oscillation Theorem, any non-trivial solution has no conjugate points.  $\square$

### 3 Self-Bäcklund curves: first study

#### 3.1 Infinitesimal deformations of centroaffine conics

In this section we study infinitesimal deformations of centroaffine conics in the class of self-Bäcklund centroaffine curves. Later, in Section 4.3, we shall show that these infinitesimal deformations correspond to actual ones.

Here is a brief reminder about deformations. Let  $\gamma(t)$  be a self-Bäcklund centroaffine curve,

$$[\gamma, \gamma'] = 1, \quad [\gamma(t), \gamma(t + \alpha)] = c, \quad (17)$$

for some  $\alpha, c$ . A deformation of such a curve, within the class of self-Bäcklund centroaffine curves with rotation number  $\alpha$ , is a function  $\Gamma(t, \varepsilon)$ , defined on  $\mathbb{R} \times (-\varepsilon_0, \varepsilon_0)$  for some  $\varepsilon_0 > 0$ , satisfying equation (17) for each fixed  $\varepsilon$  for some  $c$ , and such that  $\gamma = \Gamma(\cdot, 0)$ . (Note:  $c$  may vary with  $\varepsilon$ , but not  $\alpha$ .)

An *infinitesimal deformation* of  $\gamma$  is a formal expression  $\Gamma = \gamma(t) + \varepsilon\delta(t)$ , satisfying equation (17) for each  $\varepsilon$ , modulo  $\varepsilon^2$ . Clearly, if  $\Gamma$  is a deformation of  $\gamma$ , than its first jet  $\gamma + \varepsilon \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \Gamma$ , is an infinitesimal deformation of  $\gamma$ . However, the converse is not necessarily true, that is, given an infinitesimal deformation  $\gamma + \varepsilon\delta$ , it is not clear *a priori* that there exists an ‘actual’ deformation  $\Gamma$  of  $\gamma$  such that  $\delta = \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \Gamma$ .

An infinitesimal deformation is *trivial* if it is induced by a shift of the argument,  $\Gamma(t, \varepsilon) = \gamma(t + a\varepsilon)$ , or by the action of  $\mathrm{SL}_2(\mathbb{R})$ ,  $\Gamma(t, \varepsilon) = e^{\varepsilon A} \gamma(t)$ ,  $A \in \mathfrak{sl}_2(\mathbb{R})$ .

Now let  $\gamma(t) = (\cos t, \sin t)$  and let  $\Gamma = \gamma(t) + \varepsilon\delta(t)$  be an infinitesimal deformation of  $\gamma$  within the class of self-Bäcklund centroaffine curves with rotation number  $\alpha$ .

**Theorem 3.** *1. A non-trivial infinitesimal deformation of  $\gamma$  exists within the class of self-Bäcklund curves with rotation number  $\alpha$  if and only if either  $\alpha = \pi/2$ , or  $\alpha \neq \pi/2$  and  $\alpha$  satisfies the equation*

$$\tan(k\alpha) = k \tan \alpha \quad (18)$$

*for some integer  $k \geq 4$ .*

2. For  $k \geq 2$ , there are exactly  $k-2$  solutions of equation (18) in the interval  $(0, \pi)$ , counting also  $\alpha = \pi/2$  as a solution for  $k$  odd.

*Proof.* 1. Note that

$$[\gamma'(t), \gamma(t + \alpha)] = -\cos \alpha, \quad [\gamma(t), \gamma'(t + \alpha)] = \cos \alpha. \quad (19)$$

We make calculations mod  $\varepsilon^2$ .

The first equation (17) means that  $\delta$  is a vector field along  $\gamma$ , hence  $\delta = -(1/2)g'\gamma + g\gamma'$  for a  $\pi$ -periodic function  $g(t)$ , see equation (1).

The second equation (17) implies

$$[\delta(t), \gamma(t + \alpha)] + [\gamma(t), \delta(t + \alpha)] = \text{const},$$

hence

$$\begin{aligned} & [-(1/2)g'(t)\gamma(t) + g(t)\gamma'(t), \gamma(t + \alpha)] + \\ & + [\gamma(t), -(1/2)g'(t + \alpha)\gamma(t + \alpha) + g(t + \alpha)\gamma'(t + \alpha)] = \text{const}. \end{aligned}$$

In view of equation (19), this implies

$$(1/2)(g'(t) + g'(t + \alpha)) \sin \alpha - (g(t + \alpha) - g(t)) \cos \alpha = \text{const}. \quad (20)$$

Since the integral of the left hand side over the period is zero, the right hand side is also zero.

Recall that  $g$  is a  $\pi$ -periodic function and let

$$g(t) = \sum_{k=-\infty}^{\infty} a_k e^{2ikt}$$

be its Fourier expansion, with  $a_{-k} = \bar{a}_k$ . Then

$$\begin{aligned} g'(t) &= 2i \sum_{k=-\infty}^{\infty} k a_k e^{2ikt}, \quad g(t + \alpha) = e^{2in\alpha} \sum_{k=-\infty}^{\infty} a_k e^{2ikt}, \\ g'(t + \alpha) &= 2ie^{2ik\alpha} \sum_{k=-\infty}^{\infty} k a_k e^{2ikt}. \end{aligned}$$

Substitute in equation (20) to conclude that

$$a_k [i(1 + e^{2ik\alpha})k \sin \alpha - (e^{2ik\alpha} - 1) \cos \alpha] = 0$$

for each  $k$ . Hence  $a_k = 0$ , unless

$$i(1 + e^{2ik\alpha})k \sin \alpha = (e^{2ik\alpha} - 1) \cos \alpha,$$

or

$$k \frac{e^{ik\alpha} + e^{-ik\alpha}}{2} \sin \alpha = \frac{e^{ik\alpha} - e^{-ik\alpha}}{2i} \cos \alpha,$$

that is,  $k \tan \alpha = \tan(k\alpha)$ .

Conversely, if equation (18) holds, then one can choose  $g(t)$  to be a pure harmonic of order  $2k$ , and then equation (17) holds modulo  $\varepsilon^2$ . Likewise, if  $\alpha = \pi/2$ , one can choose  $g(t)$  to be a pure harmonic of order  $2k$  with odd  $k \geq 3$  or a linear combination of such harmonics.

Note that equation (18) holds trivially for  $k = 0$  and  $k = 1$ . The former case corresponds to  $g(t)$  being constant, a shift of the argument of  $\gamma(t)$ . The latter case corresponds to the action of  $\mathfrak{sl}(2, \mathbb{R})$ , a stretching of the unit circle to an ellipse bounding area  $\pi$ .

2. See Proposition 2 of [28], or Lemma 4.8 of [12]. □

**Remark 3.1.** Equation (18) appeared in the context of bicycle kinematics in [42, 12] and in the papers by Wegner, summarized in [49]. It also appeared in [28] in the context of billiards and flotation problems, and in [9], [10], [11] for magnetic, outer and wire billiards. This ubiquitous equation has a countable number of solutions but, except for  $\pi/2$ , there are no  $\pi$ -rational solutions [19].

### 3.2 Rigidity: periods 3 and 4

**Theorem 4.** *Let  $\gamma(t)$  be a  $\pi$ -antisymmetric self-Bäcklund centroaffine curve, that is,  $[\gamma(t), \gamma(t + \alpha)] = c \neq 0$ . If  $\alpha = \pi/3$  or  $\alpha = \pi/4$  then  $\gamma$  is a centroaffine ellipse.*

*Proof.* Consider the case of  $\alpha = \pi/3$ . Let us use the shorthand notation

$$\gamma(t) = \gamma_0, \quad \gamma\left(t + \frac{\pi}{3}\right) = \gamma_1, \quad \gamma\left(t + \frac{2\pi}{3}\right) = \gamma_2.$$

Then

$$[\gamma_0, \gamma_1] = [\gamma_1, \gamma_2] = [\gamma_2, -\gamma_0] = c,$$

hence  $[\gamma_0, \gamma_2] = [\gamma_0, \gamma_1]$ , and the vector  $\gamma_1 - \gamma_2$  is collinear with  $\gamma_0$ . Likewise,  $\gamma_2 + \gamma_0$  is collinear with  $\gamma_1$ , and  $\gamma_1 - \gamma_0$  with  $\gamma_2$ . We write

$$\gamma_1 - \gamma_2 = \varphi_0 \gamma_0, \quad \gamma_2 + \gamma_0 = \varphi_1 \gamma_1, \quad \gamma_1 - \gamma_0 = \varphi_2 \gamma_2.$$

Since  $[\gamma_0, \gamma_1] \neq 0$ , the linear map  $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $(x_0, x_1, x_2) \mapsto \sum x_i \gamma_i$ , has rank 2, hence nullity 1. It follows that the matrix

$$\begin{bmatrix} -\varphi_0 & 1 & -1 \\ 1 & -\varphi_1 & 1 \\ -1 & 1 & -\varphi_2 \end{bmatrix}$$

has rank 1, hence  $\varphi_0 = \varphi_1 = \varphi_2 = 1$ . Thus  $\gamma_2 = \gamma_1 - \gamma_0$ .

It follows that  $\gamma'_2 = \gamma'_1 - \gamma'_0$ , and hence

$$1 = [\gamma_2, \gamma'_2] = [\gamma_1 - \gamma_0, \gamma'_1 - \gamma'_0] = 2 - [\gamma_0, \gamma'_1] + [\gamma'_0, \gamma_1].$$

Since  $[\gamma_0, \gamma_1] = c$ , one has  $[\gamma'_0, \gamma_1] + [\gamma_0, \gamma'_1] = 0$ . This implies that

$$[\gamma_0, \gamma'_1] = \frac{1}{2}, \quad [\gamma'_0, \gamma_1] = -\frac{1}{2},$$

and hence  $\gamma_1 = (1/2)\gamma_0 + c\gamma'_0$ .

It follows that in equation (3) one has  $f = 1/2$ , and hence, by Lemma 2.1,  $c^2p = -3/4$ . That is,  $p$  is constant, which implies  $p = -1$  and  $c = \sqrt{3}/2$ , and therefore the curve is a centroaffine conic.

The case  $\alpha = \pi/4$  is similar. In analogous notations, one has

$$[\gamma_0, \gamma_1] = [\gamma_1, \gamma_2] = [\gamma_2, \gamma_3] = [\gamma_3, -\gamma_0] = c,$$

hence

$$\gamma_0 \sim \gamma_1 - \gamma_3, \gamma_1 \sim \gamma_0 + \gamma_2, \gamma_2 \sim \gamma_1 + \gamma_3, \gamma_3 \sim -\gamma_0 + \gamma_2.$$

This implies

$$\gamma_1 = g(\gamma_0 + \gamma_2), \quad \gamma_3 = g(\gamma_2 - \gamma_0) \tag{21}$$

for some function  $g(t)$ .

Since  $[\gamma_1, \gamma'_1] = [\gamma_3, \gamma'_3] = 1$ , equation (21) implies

$$2g^2 = 1, \quad [\gamma_0, \gamma'_2] + [\gamma_2, \gamma'_0] = 0.$$

But  $[\gamma_0, \gamma_2] = c$ , hence  $[\gamma'_0, \gamma_2] + [\gamma_0, \gamma'_2] = 0$ , and therefore  $[\gamma'_0, \gamma_2] = [\gamma_0, \gamma'_2] = 0$ . In particular,  $\gamma_2 \sim \gamma'_0$ .

It follows that  $\gamma_1 = (1/\sqrt{2})\gamma_0 + c\gamma'_0$ . Then, in equation (3), one has  $f = 1/\sqrt{2}$ , and hence, by Lemma 2.1,  $c^2p = -1/2$ . Thus  $p = -1$ ,  $c = 1/\sqrt{2}$ , and the curve is a centroaffine conic.  $\square$

**Remark 3.2.** An analogous result, rigidity for periods 3 and 4, holds for bicycle curves, see [14, 15, 42].

### 3.3 Period two: flexibility and Radon curves

In this section we show that self-Bäcklund curves of period two, that is,  $\alpha = \pi/2$ , exhibit a substantial flexibility. A similar result for Ulam's flotation in equilibrium problem was known for a long time [8, 52].

Let us construct a self-Bäcklund curve of period two as a closed trajectory of a vector field  $V$  on the space of origin-centered parallelograms. Let the vertices be  $P_1, P_2, -P_1, -P_2$ , and let the vector field have the values  $V_1, V_2, -V_1, -V_2$  at these vertices, respectively.

We want the trajectories of the points  $P_1, P_2, -P_1, -P_2$  to coincide and to form a self-Bäcklund curve with  $\alpha = \pi/2$ . Let  $(P_1(t), P_2(t))$  be an integral curve of such a vector field. Then  $P_2(t) = P_1(t + \pi/2)$ . The centroaffine conditions  $[P_i, P'_i] = 1$  and the  $c$ -relation  $[P_1, P_2] = c$  amount to

$$[P_1, V_1] = [P_2, V_2] = 1, \quad [V_1, P_2] + [P_1, V_2] = 0. \tag{22}$$

Note that the area of the parallelogram  $(P_1, P_2, -P_1, -P_2)$  remains constant.

**Lemma 3.3.** *Equations (22) are satisfied if and only if*

$$V_1 = fP_1 + \frac{1}{c}P_2, \quad V_2 = -\frac{1}{c}P_1 - fP_2,$$

where  $f(P_1, P_2)$  is an odd function, in the sense that  $f(P_2, -P_1) = -f(P_1, P_2)$ .

*Proof.* Write  $V_1 = fP_1 + gP_2, V_2 = \bar{f}P_1 + \bar{g}P_2$  and substitute into equations (22), using  $[P_1, P_2] = c$ , to obtain  $f + \bar{g} = 0, g = -\bar{f} = 1/c$ . That  $f$  is odd follows from the central symmetry of the parallelogram.  $\square$

Thus one has a functional parameter  $f$  to play with. The boundary conditions

$$P_1(0) = (1, 0), \quad P_1\left(\frac{\pi}{2}\right) = P_2(0) = (0, c), \quad P_2\left(\frac{\pi}{2}\right) = -P_1(0) = (-1, 0) \quad (23)$$

impose a finite-dimensional restriction on the function  $f$ . As a result, we obtain a functional space of self-Bäcklund curves of period two.

For example, if  $f$  is identically zero and  $c = 1$ , then  $P_1'' = P_2' = -P_1$ , and the curve is a centroaffine ellipse. See Figure 10 for a non-trivial example. In Example 4.11 below (Figure 17) we construct explicitly many analytic examples.

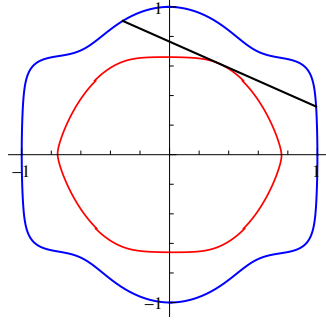


Figure 10: A self-Bäcklund curve with rotation angle  $\alpha = \pi/2$  and  $c = 1$ , using Lemma 3.3 and equation (23), where  $f(P_1, P_2) = u(P_1)u(P_2)$  and  $u(x, y) = 1.2x - 4x^3 - 4x^5$  (approximately).

**Remark 3.4.** The space of origin-centered parallelograms of a fixed area is identified with  $\mathrm{SL}_2(\mathbb{R})$ . If  $P = (p_1, p_2), Q = (q_1, q_2)$ , then the first equation (22),  $[P, U] = [Q, V]$ , means that the curve under consideration is tangent to the kernel of the 1-form  $p_1 dp_2 - p_2 dp_1 + q_2 dq_1 - q_1 dq_2$ . This form defines a contact structure on  $\mathrm{SL}_2(\mathbb{R})$ , and the curve is Legendrian.

Let  $\Gamma$  be a smooth closed convex curve, symmetric with respect to the origin. Let  $x, y \in \Gamma$ . One says that  $y$  is Birkhoff orthogonal to  $x$  if  $y$  is parallel to the tangent line to  $\Gamma$  at  $x$ . This relation is not necessarily symmetric; if it is symmetric, then  $\Gamma$  is called a *Radon curve*. Radon curves comprise a functional space, with ellipses providing a trivial example.

Radon curves have been thoroughly studied since their introduction more than 100 years ago; see [34] for a modern treatment.

Let  $\Gamma$  be a Radon curve,  $x \in \Gamma$  be a point, and  $y \in \Gamma$  be its Birkhoff orthogonal. Then the tangent lines at points  $x, y, -x, -y$  form a parallelogram circumscribed about  $\Gamma$ . As  $x$  traverses  $\Gamma$ , the vertices of the parallelogram describe a curve  $\gamma$ . The latter curve is an invariant curve of the outer billiard transformation about  $\Gamma$ , see Remark 2.6.

The relation of self-Bäcklund curves with Radon curves is as follows. Let  $\gamma$  be a self-Bäcklund curve with rotation number  $\pi/2$ , then the points  $\gamma(t), \gamma(t + \pi/2), \gamma(t + \pi), \gamma(t + 3\pi/2)$  form a parallelogram. Therefore the middle curve  $\Gamma$  is a Radon curve. Example 4.11 below provides analytic families of Radon curves.

### 3.4 Centraffine odd-gons and centraffine carrousel

In this and the next section we extend the approach of the preceding section to centraffine polygons with a greater number of sides. This material is closely related to results in chapters 6 and 7 of [7] where similar questions about polygons in  $\mathbb{RP}^1$  were studied. Some formulas obtained in [7] simplify when expressed in terms of centraffine polygons.

Denote by  $\mathcal{P}_n$  the space of centraffine  $2n$ -gons, and let  $\mathcal{Q}_n = \mathcal{P}_n/\mathrm{SL}_2(\mathbb{R})$  be the moduli space of centraffine  $2n$ -gons. We assume that  $n$  is odd.

Let  $\gamma(t)$  be a self-Bäcklund curve with  $\alpha = \pi/n$ . Then one has a centrally symmetric  $2n$ -gon  $\mathbf{P}(t)$  that revolves inside  $\gamma$  in such a way that  $[P_i, P'_i] = 1$  and  $[P_i, P_{i+1}] = c$  for all  $i$ . Rescaling the polygon by the factor  $1/c$  and also rescaling the parameter, one obtains a vector field  $(V_i)$  on  $\mathcal{P}_n$ , characterized by the equations

$$[P_i, V_i] = 1 \quad \text{and} \quad [V_i, P_{i+1}] + [P_i, V_{i+1}] = 0 \quad (24)$$

for all  $i$ . We want to construct a self-Bäcklund curve with  $\alpha = \pi/n$  as a periodic trajectory of this vector field. We call the resulting closed curves in the space of centraffine polygons *centraffine carrousel*.

The vertices of a centraffine  $2n$ -gon satisfy a discrete Hill's equation

$$P_{i+1} = a_i P_i - P_{i-1},$$

and the  $n$ -periodic coefficients  $a_i$  uniquely determine the polygon up to  $\mathrm{SL}_2(\mathbb{R})$ -equivalence. One has  $a_i = [P_{i-1}, P_{i+1}]$ . The determinants  $[P_i, P_j]$  are the entries of a frieze pattern of width  $n - 3$ , whose first row comprises the coefficients  $a_i$ , see [37].

The following lemma determines the desired vector field.

**Lemma 3.5.** *One has*

$$V_i = \frac{1}{2}(a_i - a_{i+1} + \dots + a_{i+n-1})P_i - P_{i-1},$$

where the indices are understood periodically mod  $n$ .

*Proof.* To solve equation (24), write

$$V_i = u_i P_i + \bar{u}_i P_{i+1} = v_i P_i + \bar{v}_i P_{i-1}. \quad (25)$$

Then equation (24) imply that  $\bar{u}_i = -\bar{v}_i = 1$ ,  $u_i + v_{i+1} = 0$ . Equating the two expressions for  $V_i$  yields

$$P_{i+1} = (v_i - u_i)P_i - P_{i-1},$$

hence  $v_i - u_i = a_i$ , and therefore  $v_{i+1} + v_i = a_i$ .

If  $n$  is odd, this system of linear equations has a unique solution:

$$v_i = \frac{1}{2}(a_i - a_{i+1} + \dots + a_{i+n-1}). \quad (26)$$

Substitute to equation (25) to obtain the result.  $\square$

Denote this vector field by  $\xi$ . Then  $\xi$  commutes with the action of  $\mathrm{SL}_2(\mathbb{R})$  and descends, as a field  $\bar{\xi}$ , to  $\mathcal{Q}_n$ , that is, the space of frieze patterns of width  $n - 3$ .

**Remark 3.6.** In the  $a_i$ -coordinates in  $\mathcal{Q}_n$ , one has

$$\bar{\xi} = \sum_{i=1}^n a_i (a_{i+1} - a_{i+2} + \dots - a_{n+i-1}) \frac{\partial}{\partial a_i}.$$

The space of frieze patterns of even width has a symplectic structure provided by the theory of cluster algebras, see [37]. The field  $\bar{\xi}$  is Hamiltonian with respect to this symplectic structure with the Hamiltonian function  $\sum a_i$ . The field  $\bar{\xi}$  coincides with the dressing chain of Veselov-Shabat, see formula (12) in [48]. The vector field  $\xi$  is completely integrable, see [48] and Section 5.8 of [7].

Let  $(x, y)$  be coordinates in the plane. Let  $P_i = (x_i, y_i)$ , then  $p_i = y_i/x_i$  are the projections of  $P_i$  to  $\mathbb{RP}^1$ . In [7] a closed 2-form  $\omega$  of corank 1 on the space of  $n$ -gons in  $\mathbb{RP}^1$  was constructed

$$\lambda = \frac{1}{2} \sum_{i=1}^n \frac{dp_i + dp_{i+1}}{p_{i+1} - p_i}, \quad \omega = d\lambda = \sum_{i=1}^n \frac{dp_i \wedge dp_{i+1}}{(p_i - p_{i+1})^2}.$$

We express  $\omega$  in terms of the coordinates  $x_i, y_i$ ,  $i = 1, \dots, n$ .

**Lemma 3.7.** *One has  $\omega = \sum (dx_{i+1} \wedge dy_i + dx_i \wedge dy_{i+1})$ .*

*Proof.* One has

$$x_i y_{i+1} - x_{i+1} y_i = 1, \quad i = 1, \dots, n. \quad (27)$$

therefore  $p_{i+1} - p_i = 1/(x_i x_{i+1})$ .

Hence

$$\begin{aligned}
2\lambda &= \sum x_i x_{i+1} \left( \frac{x_i dy_i - y_i dx_i}{x_i^2} + \frac{x_{i+1} dy_{i+1} - y_{i+1} dx_{i+1}}{x_{i+1}^2} \right) = \\
&= \sum \left( x_{i+1} dy_i + x_i dy_{i+1} - \frac{x_{i+1} y_i}{x_i} dx_i - \frac{x_i y_{i+1}}{x_{i+1}} dx_{i+1} \right) = \\
&= \sum \left( x_{i+1} dy_i + x_i dy_{i+1} - \frac{x_i y_{i+1} - 1}{x_i} dx_i - \frac{x_{i+1} y_i + 1}{x_{i+1}} dx_{i+1} \right) = \\
&= \sum (x_{i+1} dy_i + x_i dy_{i+1} - y_{i+1} dx_i - y_i dx_{i+1}),
\end{aligned}$$

where the third equality is due to equation (27).

It follows that  $d\lambda$  is indeed as stated in the lemma.  $\square$

Consider the three functions on  $\mathcal{P}_n$ :

$$I = \sum_i x_i x_{i+1}, \quad J = \sum_i (x_i y_{i+1} + x_{i+1} y_i), \quad K = \sum_i y_i y_{i+1}.$$

The Lie algebra  $sl(2, \mathbb{R})$  acts on  $\mathcal{P}_n$  diagonally. Let

$$e = \sum x_i \frac{\partial}{\partial y_i}, \quad h = \sum x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i}, \quad f = \sum y_i \frac{\partial}{\partial x_i}$$

be the generators.

**Lemma 3.8.** *The action of  $\mathfrak{sl}_2(\mathbb{R})$  is Hamiltonian:*

$$i_e \omega = -dI, \quad i_h \omega = dJ, \quad i_f \omega = dK.$$

The field  $\xi$  is in the kernel of  $\omega$ .

*Proof.* The first statement is a result of an obvious calculation.

For the second statement, we use the formulas from the proof of Lemma 3.5. According to equation (25), one has

$$\xi = \sum b_i \frac{\partial}{\partial x_i} + c_i \frac{\partial}{\partial y_i}$$

where

$$b_i = v_i x_i - x_{i-1} = u_i x_i + x_{i+1}, \quad c_i = v_i y_i - y_{i-1} = u_i y_i + y_{i+1}.$$

Therefore one has

$$i_\xi \omega = \sum (b_{i+1} + b_{i-1}) dy_i - (c_{i+1} + c_{i-1}) dx_i.$$

Now

$$\begin{aligned}
\sum (b_{i+1} + b_{i-1}) dy_i &= \sum (v_{i+1} x_{i+1} + u_{i-1} x_{i-1}) dy_i = \\
\sum (v_{i+1} x_{i+1} - v_i x_{i-1}) dy_i &= \sum v_{i+1} (x_{i+1} dy_i - x_i dy_{i+1}).
\end{aligned}$$



For the same reason,

$$\sum (c_{i+1} + c_{i-1})dx_i = \sum v_{i+1}(y_{i+1}dx_i - y_i dx_{i+1}).$$

Hence

$$i_\xi \omega = \sum v_{i+1}(x_{i+1}dy_i - x_i dy_{i+1} - y_{i+1}dx_i + y_i dx_{i+1}).$$

But the expression in each parentheses vanishes due to equation (27). This completes the proof.  $\square$

**Corollary 3.9.** *The functions  $I, J$  and  $K$  are integrals of the field  $\xi$ .*

*Proof.* One has  $dI(\xi) = \omega(\xi, e) = 0$ , and likewise for  $J$  and  $K$ .  $\square$

Let  $\nu = 2Ke + Jh - 2If$ . This vector field is tangent to the fiber of the projection  $\mathcal{P}_n \rightarrow \mathcal{Q}_n$ .

**Lemma 3.10.** *The functions  $I, J$  and  $K$  are integrals of the field  $\nu$ .*

*Proof.* One has

$$\begin{aligned} e(I) = 0, e(J) = 2I, e(K) = J; f(I) = J, f(J) = 2K, f(K) = 0; \\ h(I) = 2I, h(J) = 0, h(K) = -2K, \end{aligned}$$

and this implies the statement of the lemma.  $\square$

Note that  $J^2 - 4IK$  is a  $\mathrm{SL}_2(\mathbb{R})$ -invariant function that descends to  $\mathcal{Q}_n$ .

Corollary 3.9 and Lemma 3.10 imply that the fields  $\xi$  and  $\nu$  are tangent to the common level surfaces of the functions  $I, J, K$ , and since  $\xi$  also commutes with  $\mathfrak{sl}_2(\mathbb{R})$ , one has  $[\xi, \nu] = 0$ .

**Remark 3.11.** If  $n \geq 4$  is even, equations (24) still imply that  $v_i + v_{i+1} = a_i$ , and therefore

$$\sum_{i=1}^n (-1)^i a_i = 0. \quad (28)$$

Since  $a_i = [P_{i-1}, P_{i+1}]$ , the meaning of condition (28) is that the area of the polygon made of the even vertices equals the area of the polygon made of the odd ones. The space of such centroaffine polygons has dimension  $n - 1$ . If equation (28) holds, then the linear equations (24) have not a unique solution, as when  $n$  is odd, but a 1-dimensional space of solutions. Indeed, if  $V_i$  is a solution, then so is  $V_i + (-1)^i t P_i$  for all  $t$ . The respective vector field generates the kernel of the pre-symplectic form on  $\mathcal{Q}_n$  which, for even  $n$ , is an analog of the symplectic form on  $\mathcal{Q}_n$  that exists for odd  $n$ .

### 3.5 Case study: 10-periodic carrousel

In this section, we apply the machinery described in the previous section in the simplest nontrivial case  $n = 5$ .

Recall that we have a principle  $\mathrm{SL}_2(\mathbb{R})$  fiber bundle  $\pi : \mathcal{P}_n \rightarrow \mathcal{Q}_n$ , and  $\mathcal{Q}_n$  is identified with the space of friezes of width  $n - 3$ .

When  $n = 5$ , the space of friezes of width two is 2-dimensional, and a general frieze is of the form

$$\begin{array}{cccccc} \cdots & 1 & & 1 & & 1 & & 1 & & \cdots \\ & x & & \frac{y+1}{x} & & \frac{x+1}{y} & & y & & \frac{x+y+1}{xy} \\ \cdots & & y & & \frac{x+y+1}{xy} & & x & & \frac{y+1}{x} & \cdots \\ & 1 & & 1 & & 1 & & 1 & & 1 \end{array}$$

The area form is

$$\frac{dx \wedge dy}{xy},$$

and the Hamiltonian of the field  $\bar{\xi}$  is

$$H(x, y) = x + y + \frac{x+1}{y} + \frac{y+1}{x} + \frac{x+y+1}{xy}.$$

Up to a constant, this is the function  $J^2 - 4IK$  from the previous section.

The respective centroaffine decagons are constructed as follows. Start with two vectors,  $P_0$  and  $P_1$  with  $[P_0, P_1] = 1$ , and construct the next three using the discrete Hill's equation:

$$P_2 = xP_1 - P_0, \quad P_3 = \frac{y+1}{x}P_2 - P_1, \quad P_4 = \frac{x+1}{y}P_3 - P_2,$$

and the rest by central symmetry  $P_{i+5} = -P_i$ .

The affine-regular decagon corresponds to the frieze with the constant entries that are all equal to the golden ratio:

$$x = y = \frac{1 + \sqrt{5}}{2}, \quad \text{and} \quad H = \frac{5(1 + \sqrt{5})}{2}.$$

For centroaffine  $2n$ -gons, the following inequality holds:

$$\sum_i [P_{i-1}, P_{i+1}] \geq 2n \cos\left(\frac{\pi}{n}\right),$$

with the equality only for regular polygons, see [43]. In fact, the affine-regular polygons are the only critical points of the function  $\sum [P_{i-1}, P_{i+1}]$ .

In particular, the regular decagon minimizes the function  $H$  on the moduli space of  $\mathrm{SL}_2(\mathbb{R})$ -equivalence classes of centroaffine decagons, and the closed level curves  $H(x, y) = c$  foliate the first quadrant  $x, y > 0$ . It follows that the vector field  $\bar{\xi}$  is periodic on each such level curve.

Let  $M^2$  be a generic common level surface of the functions  $I, J, K$ , and let  $C$  be the respective level curve of the function  $H$ . Recall that  $M$  carries commuting vector fields  $\xi$  and  $\nu$ , and  $\nu$  is vertical with respect to the projection  $\pi : M \rightarrow C$ .

**Lemma 3.12.** *The surface  $M$  is compact, and its connected components are tori.*

*Proof.* First we show that  $M$  is compact.

Let  $(\alpha_i, r_i)$  be the polar coordinates of the vertex  $P_i$  of the decagon,  $i = 1, \dots, 10$ . It suffices to show that  $r_i$  are bounded away from zero and infinity, and that  $\alpha_{i+1} - \alpha_i$  are bounded away from zero.

The centroaffine condition is

$$r_i r_{i+1} \sin(\alpha_{i+1} - \alpha_i) = 1,$$

and

$$\begin{aligned} I &= \sum r_i r_{i+1} \cos \alpha_i \cos \alpha_{i+1} = \sum \frac{\cos \alpha_i \cos \alpha_{i+1}}{\sin(\alpha_{i+1} - \alpha_i)}, \\ J &= \sum r_i r_{i+1} (\cos \alpha_i \sin \alpha_{i+1} + \sin \alpha_i \cos \alpha_{i+1}) = \sum \frac{\sin(\alpha_i + \alpha_{i+1})}{\sin(\alpha_{i+1} - \alpha_i)}, \\ K &= \sum r_i r_{i+1} \sin \alpha_i \sin \alpha_{i+1} = \sum \frac{\sin \alpha_i \sin \alpha_{i+1}}{\sin(\alpha_{i+1} - \alpha_i)}. \end{aligned}$$

Since  $I, J, K$  are constant on  $M$  (in particular, are bounded above), it follows that  $(\alpha_{i+1} - \alpha_i)$  are bounded away from zero. Let  $\sin(\alpha_{i+1} - \alpha_i) \geq \delta > 0$  for all  $i$ , and set  $C = 1/\delta$ .

Then

$$1 \leq r_i r_{i+1} \leq C, \quad i = 1, \dots, 10. \quad (29)$$

Let  $R = C^{3/2}$ . We claim that  $r_i \leq R$  for all  $i$ .

Indeed, assume that  $r_0 > R$ . Then, due to equation (29),

$$r_1 < \frac{C}{R}, \quad r_2 > \frac{R}{C}, \quad r_3 < \frac{C^2}{R}, \quad r_4 > \frac{R}{C^2}, \quad r_5 < \frac{C^3}{R}.$$

But  $r_5 = r_0 > R$ , that is,  $C^3 > R^2$ , a contradiction.

It then follows from equation (29) that  $r_i \geq 1/R$ . Note that this, and the preceding arguments, apply, with obvious modifications, to every odd  $n$ .

Finally, since  $M$  carries two commuting vector fields, its connected components are tori.  $\square$

Let  $T$  be the cyclic permutation of the vertices of polygons:  $T(P_i) = P_{i+1}$ , generating the action of  $\mathbb{Z}_5$ . All the functions and vector fields involved are  $T$ -invariant; in particular, the level curves  $H = c$  are  $\mathbb{Z}_5$ -invariant.

Let  $\varphi_t$  be the  $t$ -flow of  $\bar{\xi}$ . There is a minimal value  $t_0$ , depending on the level  $c$ , such that the polygons  $T(\varphi_t(\mathbf{P}))$  and  $\varphi_{t+t_0}(\mathbf{P})$  are centroaffine equivalent:  $A(T(\varphi_t(\mathbf{P}))) = \varphi_{t+t_0}(\mathbf{P})$  for some  $A \in \text{SL}_2(\mathbb{R})$ .

In fact,  $A$  belongs to the 1-parameter subgroup whose infinitesimal generator is the vector field  $\nu$  (recall, that  $I, J$  and  $K$  are constant on  $M$ ) and, by compactness of  $M$ , this monodromy is an elliptic transformation (that is, it can be thought of as a rotation of the hyperbolic plane about some center through some angle).

The conjugacy class of this monodromy  $A$  depends only on  $c$ . If one finds  $\mathbf{P}$  such that  $A = Id$ , then its  $\xi$ -orbit will provide, after rescaling, a self-Bäcklund centroaffine curve with period 5.

One such example is presented in Figure 11. For more examples and animations, see [13].

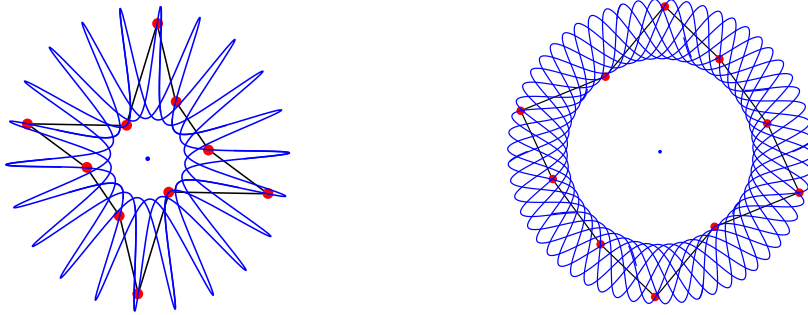


Figure 11: Examples of self-Bäcklund centroaffine curves constructed as described in Section 3.5.

**Remark 3.13.** The material of this section is analogous to that of [14, 15], where *carrousel*s and *Zindler carrousel*s were studied. A *carrousel* is a closed curve in the space of equilateral polygons such that each vertex moves with unit speed and the velocity of the midpoints of the sides are aligned with these sides. A *Zindler carrousel* has the additional property that the trajectories of all the vertices coincide.

In the situation at hand, the condition that every side is unit is replaced by the condition that  $[P_i, P_{i+1}] = 1$ , and the condition of the unit speed by the condition  $[P_i, P'_i] = 1$ .

## 4 Self-Bäcklund curves and Lamé equation

### 4.1 Traveling wave solutions of KdV and Wegner's ansatz

The first two in the hierarchy of integrals of the Korteweg-de Vries equation are the functionals

$$\int p(t) dt, \int p^2(t) dt \quad (30)$$

on centroaffine curves. In particular, KdV is the Hamiltonian flow of the former functional with respect to the symplectic form  $\int [V_f, V_g] dt$ , where we use formula (1) for tangent vector fields [40].

Consider a centroaffine curve that is a relative extremum of the second functional (30) subject to the constraint given by the first one. The next lemma is well known and we do not present its proof, see [22].

**Lemma 4.1.** *These relative extrema are characterized by the differential equation on the centroaffine curvature*

$$p''' = 6pp' + ap', \quad (31)$$

where  $a$  is a Lagrange multiplier.

Equation (31) describes traveling wave solutions of KdV, see [22]. For the centroaffine curves satisfying equation (31), the KdV evolution is described by the equation  $\dot{p} = ap'$ , that is, by a parameter shift of the curvature  $p(t)$ . Two centroaffine curves with the same curvature function differ by an element of  $\text{SL}_2(\mathbb{R})$ . Therefore these curves evolve in time by special linear transformations.

Equation (31) can be integrated to

$$(p')^2 = 2p^3 + ap^2 + 2bp + c, \quad (32)$$

where  $a, b, c$  are constants.

**Lemma 4.2.** *The curves described in Section 2.2 satisfy equation (31).*

*Proof.* Let  $q(t)$  be the centroaffine curvature of the curve  $f\gamma + c\gamma'$  where  $\gamma$  is a unit circle and  $f$  satisfies equation (4). Then

$$q = \frac{2}{c^2}(f^2 - 1) - 1,$$

see Lemma 3.1 in [45] for this calculation. Hence

$$q' = \frac{4ff'}{c^2} = \frac{4f}{c^2} \left( \frac{f^2}{c} + c - \frac{1}{c} \right).$$

One needs to check that  $(q')^2 = 2q^3 + aq^2 + 2bq + c$  for some constants  $a, b, c$ . One has

$$(q')^2 = \frac{16f^2}{c^4} \left( \frac{f^2}{c} + c - \frac{1}{c} \right)^2$$

a cubic polynomial in  $f^2$  with the leading coefficient  $16/c^6$ . The same holds for  $2q^3 + aq^2 + 2bq + c$ , so one can choose the coefficients  $a, b, c$  as needed.  $\square$

Now we develop a centroaffine analog of F. Wegner's approach to 2-dimensional bodies that float in equilibrium in all positions (or bicycle curves) [49, 50, 51].

Consider a centroaffine curve  $\gamma(t) = (r(t) \cos \alpha(t), r(t) \sin \alpha(t))$ . The centroaffine condition  $[\gamma, \gamma'] = 1$  is satisfied if  $\alpha' = r^{-2}$ . We use prime to denote the derivative with respect to  $t$ ; the derivative with respect to  $\alpha$  is denoted as  $r_\alpha$ .

Emulating Wegner's approach and using material of Section 2.1, fix a small  $\varepsilon$  and consider the curves  $\Gamma_\pm = \gamma \pm \varepsilon\gamma'$ . These curves are  $2\varepsilon$ -related. We want them to be obtained from the same curve,  $\Gamma$ , by rotating it through small angles  $\pm\delta$ . The assumption is that  $\delta$  is of order  $\varepsilon^3$ ; all the calculations below are mod  $\varepsilon^4$ . We use the notations in Figure 12.



One has

$$\begin{aligned}\varphi &= \varepsilon r^{-2} - \varepsilon^2 r^{-3} r' + \varepsilon^3 \left( r^{-4} r'^2 - \frac{1}{3} r^{-6} \right), \\ \varphi^2 &= \varepsilon^2 r^{-4} - 2\varepsilon^3 r^{-5} r', \varphi^3 = \varepsilon^3 r^{-6}, \\ \sqrt{r^2 + 2\varepsilon r r' + \varepsilon^2 (r^{-2} + r'^2)} &= r + \varepsilon r' + \frac{\varepsilon^2}{2} r^{-3} - \frac{\varepsilon^3}{2} r^{-4} r'.\end{aligned}$$

To expand the left hand side of equation (33), we calculate  $\rho_\alpha, \rho_{\alpha\alpha}$  and  $\rho_{\alpha\alpha\alpha}$ , using  $\alpha' = r^{-2}$ :

$$\rho_\alpha = r^2 \rho', \rho_{\alpha\alpha} = 2r^3 r' \rho' + r^4 \rho'', \rho_{\alpha\alpha\alpha} = 6r^4 r'^2 \rho' + 2r^5 r'' \rho' + 6r^5 r' \rho'' + r^6 \rho''''.$$

Now we have for the left hand side of equation (33)

$$\begin{aligned}\rho(\alpha + \varphi - \delta) &= \rho + \varphi \rho_\alpha + \frac{1}{2} \varphi^2 \rho_{\alpha\alpha} + \frac{1}{6} \varphi^3 \rho_{\alpha\alpha\alpha} - \delta \rho_\alpha = \\ &= \rho + \varepsilon \rho' + \frac{1}{2} \varepsilon^2 \rho'' + \frac{1}{6} \varepsilon^3 (r^{-2} \rho'''' + 2r^{-1} r'' \rho' - 2r^{-4} \rho') - c \varepsilon^3 r^2 \rho'.\end{aligned}$$

Thus

$$\begin{aligned}\rho + \frac{1}{2} \varepsilon^2 \rho'' &= r + \frac{1}{2} \varepsilon^2 r^{-3}, \\ \rho' + \frac{1}{6} \varepsilon^2 (\rho'''' + 2r^{-1} r'' \rho' - 2r^{-4} \rho' - 6c r^2 \rho') &= r' - \frac{1}{2} \varepsilon^2 r^{-4} r'.\end{aligned}$$

Differentiate the first equation and subtract from the second one, setting, following Wegner,  $\rho = r$  (since  $\varepsilon$  is infinitesimal), to obtain

$$r'''' - r^{-1} r' r'' + 4r^{-4} r' + 3c r^2 r' = 0.$$

Multiply this by  $r^{-1}$  and write it as

$$\left( r^{-1} r'' - r^{-4} - \frac{3}{2} c r^2 \right)' = 0,$$

or

$$r'' - r^{-3} + \frac{3}{2} c r^3 - b r = 0,$$

where  $b$  is a constant. Multiply this by  $2r'$  and write it as

$$\left( r'^2 + r^{-2} + \frac{3}{4} c r^4 - b r^2 \right)' = 0.$$

Hence

$$r'^2 = -r^{-2} - \frac{3}{4} c r^4 + b r^2 + a,$$

where  $a$  is another constant. Multiply by  $4r^2$  to obtain

$$4r^2 r'^2 = -4 - 3c r^6 + 4b r^4 + a r^2.$$

Finally, setting  $R = r^2$  and renaming the constants, we obtain the differential equation

$$R'^2 = aR^3 + bR^2 + cR - 4. \quad (34)$$

Thus  $R(t)$  is an elliptic function. The curve is given by a parametric equation

$$\Gamma(t) = (R(t)^{1/2} \cos \alpha(t), R(t)^{1/2} \sin \alpha(t)) \quad (35)$$

with  $R$  as in equation (34) and  $\alpha' = R^{-1}$ .

**Remark 4.4.** If the curve is a centroaffine ellipse, one has  $a = 0$  in equation (34).

Concerning the centroaffine curvature of this curve, it is also an elliptic function.

**Lemma 4.5.** *One has*

$$p(t) = \frac{1}{2}aR(t) + \frac{1}{4}b.$$

*Proof.* Differentiating equation (35) twice, we find that

$$p = -\frac{1}{4}R^{-2}(R'^2 + 4) + \frac{1}{2}R^{-1}R''.$$

Differentiating equation (34), we obtain

$$R'' = \frac{3}{2}aR^2 + bR + \frac{1}{2}c.$$

Substitute this and equation (34) in the above formula for  $p$  to obtain the result.  $\square$

Renaming the constants again, we obtain from equation (34)

$$p'^2 = 2p^3 + ap^2 + bp + c,$$

which coincides with equation (32).

Let us also calculate the (Euclidean) curvature  $k$  of a curve satisfying equation (34).

**Lemma 4.6.** *One has*

$$k = -\frac{4aR + 2b}{(aR^2 + bR + c)^{\frac{3}{2}}}.$$

*Proof.* Since  $t$  is the centroaffine parameter, we have for the curvature

$$k = \frac{[\gamma', \gamma'']}{|\gamma'|^3} = \frac{-p(t)}{|\gamma'|^3}.$$

We have

$$|\gamma'| = \sqrt{r'^2 + r^2\alpha'^2} = \sqrt{\frac{R'^2}{4R} + \frac{1}{R}} = \sqrt{\frac{R'^2 + 4}{4R}} = \frac{\sqrt{aR^2 + bR + c}}{2}.$$



Hence

$$k = \frac{-8p(t)}{\sqrt{aR^2 + bR + c}^3} = -\frac{4aR + 2b}{(aR^2 + bR + c)^{\frac{3}{2}}}.$$

□

Thus the curvature is a function of the distance from the origin. This is a special class of curves, studied in [17, 41]. One can think of these curves as the trajectories of a charge in a rotationally symmetric magnetic field whose strength is a function of the distance from the origin. Note that Wegner's curves also have this property: their curvature satisfies  $k = ar^2 + b$ , where  $a, b$  are constants.

Likewise one can interpret equation  $\gamma'' = p\gamma$  as Newton's Second Law, that is,  $\gamma(t)$  is the trajectory of a point-mass in a central force field whose potential  $V$  is rotationally symmetric. By Lemma 4.5, and renaming the constants, one has  $V(r) = ar^4 + br^2 + c$ . Using conservation of energy and momentum, one can solve the equation of motion in quadratures.

**Remark 4.7.** Consider a particular case when  $V$  is a pure 4th power of the distance, that is, the force is proportional to  $r^3$ . According to a corollary of the Bohlin theorem, see Theorem 5, Appendix 1 in [5], some trajectories in this field are the images of straight lines under the conformal transformation  $w = z^{1/3}$ . These are cubic curves, see Figure 13.

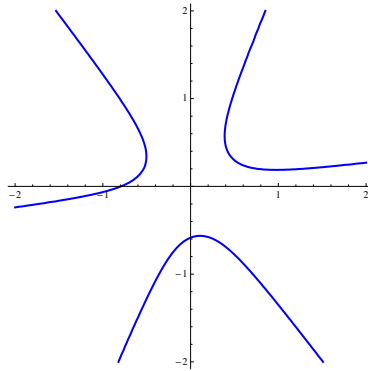


Figure 13: The curve  $2(x^3 - 3xy^2) - 5(3x^2y - y^3) + 1 = 0$ , the image of the line  $2a - 5b + 1 = 0$  under the conformal transformation  $w = z^{1/3}$ .

## 4.2 Self-Bäcklund curves as solutions of the Lamé equation

In this section we give an explicit construction of a large family of self-Bäcklund curves, given by the Wegner ansatz of Section 4.1. We shall make frequent use of standard facts about the Weierstrass elliptic functions  $\wp, \zeta, \sigma$ , such as: the

addition formulas [3, pages 40-41], quasi-periodicity properties [3, pages 35-37], reality conditions [39, pages 29-32], degenerate cases of Weierstrass functions [3, pages 201]. We shall also use applications of elliptic functions to the Lamé equation which can be found in [39, pages 48-54]. We collected most of the formulas and results that we are using in Appendix B.

#### 4.2.1 Constructing the curves

Our starting point is equation (32),

$$(p')^2 = 2p^3 + ap^2 + 2bp + c,$$

for the curvature  $p(t)$  of the self-Bäcklund curves suggested by the Wegner's ansatz. Comparing this equation to the equation satisfied by the Weierstrass  $\wp$  function,

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \quad (36)$$

we conclude that  $p(t)$  is given in terms of  $\wp$  by

$$p(t) = 2\wp(t + \omega') + C. \quad (37)$$

Here  $\wp$  is the Weierstrass function with half periods  $\omega, \omega'$ , where the first one is real and the second one is pure imaginary, see Figure 14. Since  $p(t)$  needs to be periodic, we are in the case of three real roots  $e_1 > e_2 > e_3$  of the right hand side of equation (36). In formula (37) the shift of the argument by  $\omega'$  is performed in order to get a real, smooth,  $2\omega$ -periodic potential  $p(t)$ .

The constant  $C$  can be written as  $C = \wp(a)$  for some  $a \in \mathbb{C}$ . Thus

$$p(t) = 2\wp(t + \omega') + \wp(a). \quad (38)$$

We write our curve in complex form  $X(t) = x(t) + iy(t)$ , satisfying

$$X'' + (-\wp(a) - 2\wp(t + \omega'))X = 0, \quad (39)$$

which is precisely the Lamé equation (equation (6) of [3, page 186]).

In order to construct a centroaffine  $\pi$ -anti-symmetric curve, we shall require the following:

1. The Wronskian  $[X, X'] = 1$ . This can be achieved by rescaling of any solution of equation (39) satisfying  $[X, X'] = \text{const} > 0$  (see item 4 of Proposition 4.8 below).
2.  $\omega = \pi/2k$  for some integer  $k \geq 2$ , so that  $p$  is  $\pi/k$ -periodic.
3. The solution  $X$  is rotated over the period  $2\omega$  by  $\pi n/k$ , where  $0 < n < k$  is odd and co-prime to  $k$ , so that after  $k$  periods we have  $X(t + \pi) = -X(t)$ . In other words, we require  $X(t)$  to be a complex  $2\omega$ -quasiperiodic solution of equation (39), with Floquet multiplier  $\mu = e^{i\pi n/k}$ :

$$X(t + 2\omega) = X(t)e^{i\pi n/k}.$$

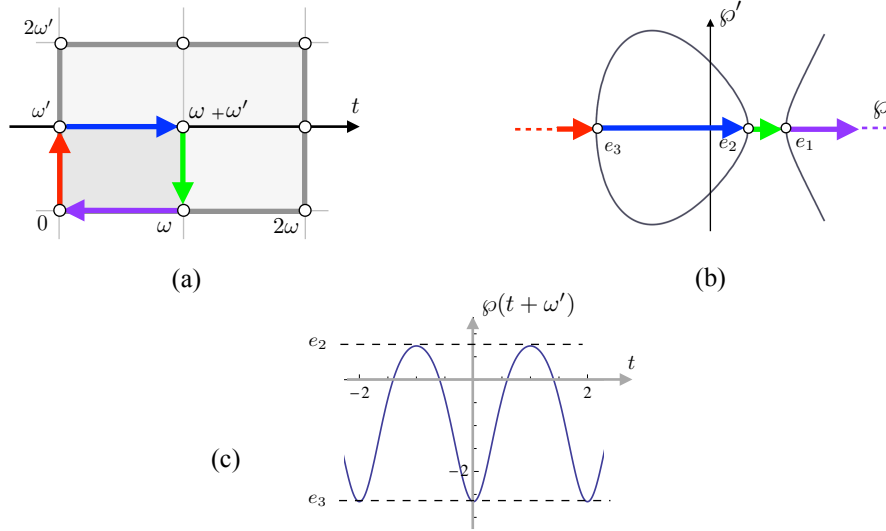


Figure 14: The Weierstrass function  $\wp(z)$  with real invariants and fundamental half periods  $\omega \in \mathbb{R}, \omega' \in i\mathbb{R}$ . (a) The fundamental rectangle in the  $z$  plane. The boundary of the rectangle  $(0, \omega', \omega + \omega', \omega)$  is mapped by  $\wp$  onto the extended real axis  $\mathbb{R} \cup \{\infty\}$ . (b) The phase plane of  $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ . (c) The line  $\{t + \omega' | t \in \mathbb{R}\}$  is mapped,  $2\omega$ -periodically, onto the segment  $[e_3, e_2]$ .

A basis  $X_+, X_-$  for the solutions of the Lamé equation (39) can be written in the following form (see [3, page 37]):

$$X_{\pm}(t) = e^{-t\zeta(\pm a)} \frac{\sigma(\pm a + t + \omega')\sigma(\omega')}{\sigma(\pm a + \omega')\sigma(t + \omega')}, \quad (40)$$

where  $\zeta, \sigma$  are the Weierstrass zeta and sigma functions, respectively.

The construction of the self-Bäcklund curves in this section boils down to a careful choice of the parameter  $a$  in equation (39).

**Proposition 4.8.** *For every  $a \in (0, \omega') \cup (\omega, \omega + \omega')$ ,*

1.  $\wp(a)$  is real, hence the potential  $2\wp(t + \omega') + \wp(a)$  in the Lamé equation (39) is real as well.
2.  $X_+(t)$  is a regular curve, that is,  $X'_+(t) \neq 0$  for all  $t$ .
3.  $X_+(0) = 1$  and  $X'_+(0) = ib$  for some  $b \in \mathbb{R}$ ,  $b > 0$ .
4.  $X_+(t)$  is locally star-shaped and positively oriented:

$$[X_+(t), X'_+(t)] = \text{const} > 0.$$

5.  $X_+(t + 2\omega) = X_+(t)e^{2f(a)}$ , where

$$f(a) := a\zeta(\omega) - \omega\zeta(a). \quad (41)$$

That is,  $X_+(t)$  is a  $2\omega$ -quasiperiodic solution of equation (39) with a Floquet multiplier  $\mu = e^{2f(a)}$ .

6. The function  $f$  of the previous item satisfies the identities

$$f(-a) = -f(a), \quad f(a + 2\omega) = f(a), \quad f(a + 2\omega') = f(a) + i\pi.$$

*Proof.* 1. See pages 31-32 of [39].

2. Differentiating equation (40), and using  $\zeta = \sigma'/\sigma$  and the addition formula for  $\zeta$ , we compute:

$$X'_+(t) = X_+(t) [\zeta(a + t + \omega') - \zeta(a) - \zeta(t + \omega')] = X_+(t) \frac{\wp'(a) - \wp'(t + \omega')}{2[\wp(a) - \wp(t + \omega')]}.$$

Notice that the numerator in the last fraction cannot vanish, since  $\wp'(t + \omega')$  is real and  $\wp'(a)$  is purely imaginary, both non-vanishing ( $\wp'$  vanishes in the fundamental rectangle only at  $0, \omega, \omega', \omega + \omega'$ ). It follows that  $X'_+(t)$  does not vanish.

3. Substituting  $t = 0$  into equation (40) gives  $X_+(0) = 1$ . From the previous item we have

$$X'_+(0) = \frac{\wp'(a)}{2(\wp(a) - e_3)}.$$

For  $a \in (0, \omega') \cup (\omega, \omega + \omega')$  the numerator  $\wp'(a)$  is purely imaginary and the denominator is real, both non-vanishing. Hence we can write  $X'_+(0) = ib$ ,  $b \in \mathbb{R}$ ,  $b \neq 0$ . Moreover,  $\wp(a) < e_3$  and  $\text{Im}[\wp'(a)] < 0$  for  $a \in (0, \omega')$ . When  $a \in (\omega, \omega + \omega')$  we have that  $\wp(a) > e_3$  is positive and  $\text{Im}[\wp'(a)] > 0$ . (All this is evident in Figure 14.) Hence, in both cases,  $b > 0$ .

4. Since  $X_+$  is a solution of Lamé equation (39), which has no  $X'$  term, one has

$$\text{Wronskian} = [X_+(t), X'_+(t)] = \text{const.}$$

The constant must be positive, due to item 2.

5. See [39] page 52.

6. See [39] page 86. □

Thus, due to requirement 3 and Proposition 4.8 (item 5), we need to solve  $2f(a) \equiv i\pi n/k \pmod{2\pi i}$ , or

$$f(a) = \frac{i\pi n}{2k} + i\pi m, \quad (42)$$

for some integers  $m, n \in \mathbb{Z}$ , where  $n$  is odd, relatively prime to  $k$ , and  $0 < n < k$ .

**Remark 4.9.** 1. To solve equation (42), it is enough to restrict  $a$  to the fundamental rectangle. Indeed, if  $a_1$  and  $a_2$  are two congruent solutions of equation (42), then the corresponding potentials (38) of Lamé equation are equal, and the curves constructed by formula (40) are equivalent under the action of  $\text{SL}_2(\mathbb{R})$ .

2. One may restrict to solutions of equation (42) where  $a$  belongs to one of the segments  $(0, \omega')$  or  $(\omega, \omega + \omega')$ , and  $m \geq 0$ . This follows from the properties of  $f$  listed in Proposition 4.8 and the monotonicity property of  $f$  on the segments  $[0, 2\omega']$  and  $[\omega, \omega + 2\omega']$ . On the segment  $[0, 2\omega']$ , the function  $f$  varies monotonically from  $+i\infty$  to  $-i\infty$ . On the segment  $[\omega, \omega + 2\omega']$  it varies from 0 to  $i\pi$ .

**Theorem 5.** *Consider equation (42) for fixed integers  $k, n$ , where  $k \geq 2$  and  $n$  is odd, relative prime to  $k$ , and  $0 < n < k$ . Then*

1. For each integer  $m \geq 0$  there is a unique solution  $a_m \in (0, \omega') \cup (\omega, \omega + \omega')$ .
2. For  $m > 0$ ,  $a_m \in (0, \omega')$ .
3. For  $m = 0$ ,  $a_0 \in (\omega, \omega + \omega')$ .
4. The sequence  $\lambda_m(\mu) := -\wp(a_m)$  is strictly monotone increasing and, in particular, the value  $\lambda_0(\mu) = -\wp(a_0)$  is the smallest one.

*Proof.* The proof of items 1–3 uses the behavior of the function  $f$ . Since  $\frac{\pi n}{2k} < \frac{\pi}{2}$ , for  $m = 0$  there is a unique solution  $a_0$  in the segment  $[\omega, \omega + \omega']$ , because  $f$  is pure imaginary on  $[\omega, \omega + \omega']$  and varies monotonically from 0 at  $\omega$  to  $i\pi/2$  at  $\omega + \omega'$ .

For  $m > 0$ , one can find a unique  $a_m$  in the segment  $[0, \omega']$  since there  $f$  is pure imaginary, varying monotonically from  $+i\infty$  at 0 to  $i\pi/2$  at  $\omega'$ . Moreover, the sequence  $a_m$  is monotone decreasing on  $[0, \omega']$ .

In order to prove 4, notice that on the segment  $[0, \omega']$  the function  $\wp$  is real-valued and monotone increasing from  $-\infty$  to  $e_3$ . Hence  $-\wp(a_m)$  is monotone increasing for  $m \geq 1$ . Moreover,  $-\wp(a_m) > -e_3$  for every  $m \geq 1$ . As for  $m = 0$ ,

$$-\wp(a_0) \in (-e_1, -e_2),$$

because on the interval  $[\omega, \omega + \omega']$  the function  $\wp$  is monotonically decreasing and takes the values  $e_1, e_2$  at the end points, respectively. Since  $e_3 < e_2 < e_1$ , this proves item 4 (see Fig. 14).  $\square$

Moreover we have the following result.

**Theorem 6.** *For each  $k, m, n$  as in Theorem 5, consider the curve  $X_+$  determined by the value  $a_m$ .*

1.  $X_+$  is locally star-shaped  $\pi$ -antisymmetric curve, with winding number

$$w = 2k \left\lceil \frac{m}{2} \right\rceil + n.$$

2.  $X_+$  is embedded (simple) if and only if  $m = 0, n = 1$ .

*Proof.* It follows from Theorem 5 that the sequence  $\lambda_m(\mu) := -\wp(a_m)$  is the sequence of Floquet eigenvalues for the problem

$$X'' + (\lambda - 2\wp(t + \omega'))X = 0, \quad X(t + 2\omega) = \mu X(t), \quad \mu := e^{i\pi n/k},$$

and that  $\lambda_m(\mu)$  is monotone increasing.

It follows from Proposition 4.8 that the curve is locally star-shaped and positively oriented.

In order to compute the winding number of the curve, we need first to see what happens over one period  $[0, 2\omega]$ . Denote by  $y_m(t)$  the imaginary part of the solution  $X_+$  corresponding to  $a_m$ . We know by Proposition 4.8 (claim 2) that at the end points of the period one has

$$y_m(0) = 0, \quad y'_m(0) > 0, \quad y_m(2\omega) = \sin\left(\frac{\pi n}{k}\right) > 0.$$

This implies that the number of zeroes of  $y_m$  on  $(0, 2\omega]$  is even for every  $m$ .

In order to find the number of zeroes of  $y_m$  on the interval  $(0, 2\omega)$  we use Sturm theory, comparing  $y_m$  with the Dirichlet eigenfunctions of the Lamé equation, as follows.

Let us denote by  $\Lambda_m, \Psi_m, m \geq 0$ , the eigenvalues and eigenfunctions corresponding to Dirichlet boundary conditions of the equation

$$\Psi'' + (\lambda - 2\wp(t + \omega'))\Psi = 0. \quad (43)$$

Thus the eigenfunctions  $\Psi_m$  vanish at the end points of the interval  $[0, 2\omega]$  and have exactly  $m$  zeros in  $(0, 2\omega)$ .

We claim that the number of zeroes of  $y_m$  in  $(0, 2\omega)$  is given by the formula:

$$\#\{t \in (0, 2\omega) : y_m(t) = 0\} = 2 \left\lceil \frac{m}{2} \right\rceil. \quad (44)$$

To prove this, we shall consider two cases (see Figure 15):

1. If  $m = 2l$  then  $\Lambda_{2l-1} < \lambda_{2l}(\mu) < \Lambda_{2l}$ . In this case, the zeroes of  $\Psi_{2l-1}$  divide the interval into  $2l$  subsegments. In each of them,  $y_{2l}$  must vanish somewhere (by Sturm theory). Hence there are at least  $2l$  zeroes. In fact, this number must be exactly  $2l$ , because otherwise it would be at least  $2l + 2$  zeros ( $y_m$  has an even number of zeroes). But then  $\Psi_{2l}$  would have more than  $2l$  zeroes.

2. If  $m = 2l + 1$  then  $\Lambda_{2l} < \lambda_{2l+1}(\mu) < \Lambda_{2l+1}$ . The zeroes of  $\Psi_{2l}$  divide the interval into  $2l + 1$  subintervals, in each of which  $y_{2l+1}$  must vanish somewhere (by Sturm theory), implying that  $y_{2l+1}$  has at least  $2l + 1$  zeroes. But then this number is at least  $2l + 2$ , because it is even. Hence, the number of zeroes of  $y_{2l+1}$  is exactly  $2l + 2$ , because otherwise  $\Psi_{2l+1}$  would have more than  $2l + 1$  zeroes. This completes the proof of the claim.

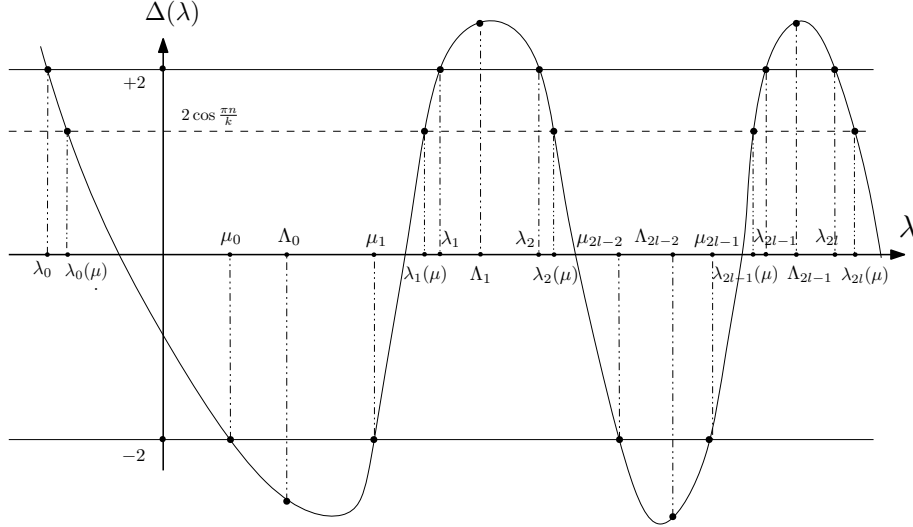


Figure 15: Graph of the function  $\Delta(\lambda) := y_1(\lambda, 2\omega) + y_2'(\lambda, 2\omega)$ , where  $y_1(\lambda, t), y_2(\lambda, t)$  are the basic solutions of equation (43) with  $y_1(\lambda, 0) = y_2'(\lambda, 0) = 1, y_1'(\lambda, 0) = y_2(\lambda, 0) = 0$ ; the positions of the periodic ( $\lambda_n$ ), anti-periodic ( $\mu_n$ ), Dirichlet ( $\Lambda_n$ ), and Floquet ( $\lambda_n(\mu)$ ) eigenvalues are indicated.

As a consequence of formula (44), we see that for  $a = a_m$  the solution  $X_+$  makes  $\lceil \frac{m}{2} \rceil$  full turns over the period  $[0, 2\omega]$ , plus an angle of  $\frac{\pi n}{k}$ , which is a  $\frac{n}{2k}$  fraction of a full turn. Altogether, after  $2k$  periods, the number of turns is

$$w = 2k \left( \left\lceil \frac{m}{2} \right\rceil + \frac{n}{2k} \right) = 2k \left\lceil \frac{m}{2} \right\rceil + n.$$

This proves the first claim of Theorem 6.

The last formula implies that the curve is simple, that is,  $w = 1$ , if and only if  $m = 0, n = 1$ , proving the second claim. This completes the proof.  $\square$

#### 4.2.2 Establishing the self-Bäcklund property

**Proposition 4.10.** *The curve  $X_+$  of equation (40) satisfies the self-Bäcklund property  $[X_+(t), X_+(t + \alpha)] = \text{const}$  for a value of the parameter  $\alpha \in (0, \pi)$  if and only if*

$$\sigma(a + \alpha) = e^{2\alpha\zeta(a)} \sigma(a - \alpha). \quad (45)$$

*Proof.* Set  $\beta = \alpha/2$ . Then equation (2) can be rewritten as

$$\text{Im} \left( X_+(t + \beta) \overline{X_+(t - \beta)} \right) = c,$$

where overline denotes the complex conjugation. We can rewrite this equation as

$$X_+(t + \beta)X_-(t - \beta) - X_-(t + \beta)X_+(t - \beta) = 2c.$$

Next we substitute in the last equation the expressions for  $X_{\pm}$  from equation (40):

$$2c = e^{-(t+\beta)\zeta(a)} \frac{\sigma(a+t+\beta+\omega')\sigma(\omega')}{\sigma(a+\omega')\sigma(t+\beta+\omega')} e^{(t-\beta)\zeta(a)} \frac{\sigma(-a+t-\beta+\omega')\sigma(\omega')}{\sigma(-a+\omega')\sigma(t-\beta+\omega')} - \\ - e^{(t+\beta)\zeta(a)} \frac{\sigma(-a+t+\beta+\omega')\sigma(\omega')}{\sigma(-a+\omega')\sigma(t+\beta+\omega')} e^{-(t-\beta)\zeta(a)} \frac{\sigma(a+t-\beta+\omega')\sigma(\omega')}{\sigma(a+\omega')\sigma(t-\beta+\omega')}.$$

This can be simplified, using the identity

$$\wp(z) - \wp(w) = -\frac{\sigma(z-w)\sigma(z+w)}{\sigma^2(z)\sigma^2(w)} \quad (46)$$

(see [39, page 25]). We get

$$2c = e^{-2\beta\zeta(a)} \frac{[\wp(t+\omega') - \wp(a+\beta)]\sigma^2(a+\beta)\sigma^2(\omega')}{[\wp(t+\omega') - \wp(\beta)]\sigma^2(\beta)\sigma(a+\omega')\sigma(-a+\omega')} - \\ - e^{2\beta\zeta(a)} \frac{(\wp(t+\omega') - \wp(a-\beta))\sigma^2(a-\beta)\sigma^2(\omega')}{[\wp(t+\omega') - \wp(\beta)]\sigma^2(\beta)\sigma(a+\omega')\sigma(-a+\omega')}.$$

Multiplying by the common denominator and renaming the constant,

$$\tilde{c} := 2c\sigma^2(\beta)\sigma(a+\omega')\sigma(-a+\omega')/\sigma^2(\omega'),$$

we get

$$\tilde{c} [\wp(t+\omega') - \wp(\beta)] = e^{-2\beta\zeta(a)} [\wp(t+\omega') - \wp(a+\beta)]\sigma^2(a+\beta) - \\ - e^{2\beta\zeta(a)} [\wp(t+\omega') - \wp(a-\beta)]\sigma^2(a-\beta).$$

Thus we must have

$$\tilde{c} = e^{-2\beta\zeta(a)}\sigma^2(a+\beta) - e^{2\beta\zeta(a)}\sigma^2(a-\beta) \\ \wp(\beta)\tilde{c} = e^{-2\beta\zeta(a)}\wp(a+\beta)\sigma^2(a+\beta) - e^{2\beta\zeta(a)}\wp(a-\beta)\sigma^2(a-\beta).$$

Substituting  $\tilde{c}$  from the first identity into the second and simplifying, we get

$$\sigma^2(a+\beta) [\wp(a+\beta) - \wp(\beta)] = e^{4\beta\zeta(a)}\sigma^2(a-\beta) [\wp(a-\beta) - \wp(\beta)].$$

Now, using equation (46) again, we obtain  $\sigma(a+\alpha) = e^{2\alpha\zeta(a)}\sigma(a-\alpha)$ , as needed.  $\square$

The next theorem states the self-Bäcklund property of the curves  $X_+$ .

**Theorem 7.** *For each  $k, m, n$  as in Theorem 5, the associated curve  $X_+$  satisfies the self-Bäcklund property  $[X_+(t), X_+(t+\alpha)] = \text{const}$  for  $k-2$  values of  $\alpha \in (0, \pi)$ .*



**Example 4.11.** Let us look for solutions of equation (45) of the form  $\alpha = l\omega$ , where  $l$  is an integer. Using the quasi-periodicity property of  $\sigma$  (see [3, page 37], [39, page 20]), we write

$$\begin{aligned}\sigma(a + \alpha) &= \sigma(a + l\omega) = \sigma(a - \alpha + 2l\omega) = (-1)^l e^{2l\zeta(\omega)(a - \alpha + l\omega)} \sigma(a - \alpha) = \\ &= (-1)^l e^{2la\zeta(\omega)} \sigma(a - \alpha).\end{aligned}$$

Comparing with equation (45), we require  $(-1)^l e^{2la\zeta(\omega)} = e^{2\alpha\zeta(a)}$ . We choose  $l$  to be odd and require

$$2\alpha\zeta(a) = 2l\omega\zeta(a) = 2la\zeta(\omega) - i\pi.$$

Hence  $f(a) = a\zeta(\omega) - \omega\zeta(a) = i\pi/2l$ . But, according to equation (42),  $f(a) = i\pi n/2k + i\pi m$ . Therefore, choosing  $m = 0, n = 1$ , implies  $l = k$ , and so  $\alpha = l\omega = k\pi/2k = \pi/2$ . In this way, we construct an infinite family of self-Bäcklund simple closed curves with rotation number  $\alpha = \pi/2$ , as discussed in Section 3.3, but now we have an analytical example. See Figure 16.

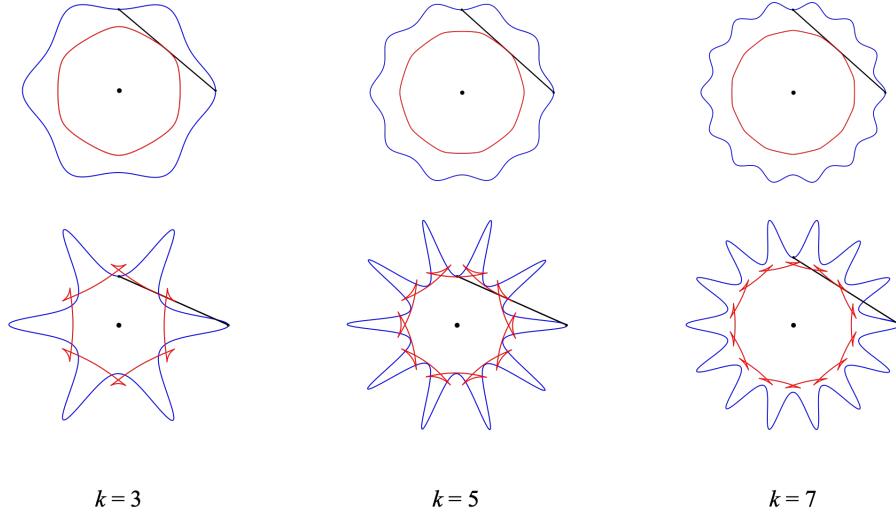


Figure 16: Example 4.11. Self-Bäcklund centroaffine simple curves  $X_+(t)$  of Wegner type (blue) with  $2k$ -fold symmetry,  $k = 3, 5, 7$ , with rotation number  $\alpha = \pi/2$  (one quarter of a turn). The red curve is traced by the midpoint of the line segment  $X_+(t)X_+(t + \pi/2)$  (black) and is tangent to it. For large enough  $\omega'$ , the midpoint curve is smooth and convex (top); as  $\omega'$  becomes smaller, cusps appear (bottom).

### 4.2.3 Proof of the self-Bäcklund property (Theorem 7)

We shall distinguish between two cases. In both cases we shall rewrite equation (45) in a more tractable form.

**Case 1.** Let us start with the most important case  $m = 0$  (the curve is simple if and only if  $n = 1$ ). For  $m = 0$  we have from equation (42) that  $f(a) = \frac{i\pi n}{2k}$ , where

$$a = \omega + ib \in [\omega, \omega + \omega'], \quad b \in \mathbb{R}.$$

We have from equation (45) that

$$-\frac{\sigma(\alpha + \omega + ib)}{\sigma(\alpha - \omega - ib)} = e^{2\alpha\zeta(\omega+ib)}. \quad (47)$$

Using the quasi-periodicity of  $\sigma$ , one has

$$-\sigma(\alpha + \omega + ib) = \sigma(\alpha - \omega + ib)e^{2\zeta(\omega)(\alpha+ib)}.$$

Substituting into equation (47), we get

$$\frac{\sigma(\alpha - \omega + ib)}{\sigma(\alpha - \omega - ib)} = e^{2\alpha\zeta(\omega+ib) - 2\zeta(\omega)(\alpha+ib)} = e^{2\alpha[\zeta(\omega+ib) - \zeta(\omega)] - 2i\zeta(\omega)b},$$

or, equivalently,

$$-\frac{\sigma(\alpha - \omega + ib)}{\sigma(-\alpha + \omega + ib)} = e^{2\alpha[\zeta(\omega+ib) - \zeta(\omega)] - 2i\zeta(\omega)b}.$$

Taking log, we obtain

$$i2\pi l + \int_{-\alpha+\omega}^{\alpha-\omega} \zeta(ib+t)dt = i\pi + 2\alpha[\zeta(\omega+ib) - \zeta(\omega)] - 2i\zeta(\omega)b.$$

Hence

$$\pi l + \operatorname{Im} \left( \int_0^{\alpha-\omega} \zeta(ib+t)dt \right) = \frac{\pi}{2} + \frac{\alpha}{i} [\zeta(\omega+ib) - \zeta(\omega)] - \zeta(\omega)b. \quad (48)$$

Let us denote

$$g(\alpha) := \operatorname{Im} \left( \int_0^{\alpha-\omega} \zeta(ib+t)dt \right).$$

**Lemma 4.12.** *For any  $r \in \mathbb{N} \cup \{0\}$ , we have*

$$\operatorname{Im} \left( \int_0^{2\omega r - \omega} \zeta(ib+t)dt \right) = (2r-1)b\zeta(\omega) - \pi r + \frac{\pi}{2}.$$

*Proof.* Apply the Cauchy residue formula to the rectangular path

$$-\omega(2r-1)+ib \rightarrow \omega(2r-1)+ib \rightarrow \omega(2r-1)-ib \rightarrow -\omega(2r-1)-ib \rightarrow -\omega(2r-1)+ib$$

to obtain the result.  $\square$

Using the quasi-periodicity of  $\zeta$  and Lemma 4.12, we have

$$\begin{aligned}
g(\alpha + 2\omega) &= \operatorname{Im} \left( \int_0^{2\omega r + \omega} \zeta(ib + t) dt \right) \\
&= \operatorname{Im} \left( \int_0^{2\omega r - \omega} \zeta(ib + t) dt \right) + \operatorname{Im} \left( \int_{-\omega}^{\omega} \zeta(ib + t) dt \right) \\
&= \operatorname{Im} \left( \int_0^{2\omega r - \omega} \zeta(ib + t) dt \right) + 2\operatorname{Im} \left( \int_0^{\omega} \zeta(ib + t) dt \right) \\
&= \operatorname{Im} \left( \int_0^{2\omega r - \omega} \zeta(ib + t) dt \right) + 2b\zeta(\omega) - \pi = g(\alpha) + 2b\zeta(\omega) - \pi.
\end{aligned}$$

Therefore we can write  $g$  in the form

$$g(\alpha) = \left( \frac{2b\zeta(\omega) - \pi}{2\omega} \right) \alpha + h(\alpha), \quad (49)$$

where  $h$  is a  $2\omega$ -periodic function. Moreover, by Lemma 4.12 (with  $r = 0$ ),

$$h(0) = g(0) = -b\zeta(\omega) + \frac{\pi}{2}.$$

It is convenient to use  $h_0$  instead of  $h$ :

$$h_0(\alpha) := h(\alpha) - h(0) = h(\alpha) + b\zeta(\omega) - \frac{\pi}{2},$$

so that  $h_0$  is  $2\omega$ -periodic with  $h_0(0) = 0$ . Thus

$$g(\alpha) = \left( \frac{2b\zeta(\omega) - \pi}{2\omega} \right) \alpha + h_0(\alpha) - b\zeta(\omega) + \frac{\pi}{2}. \quad (50)$$

Substituting equation (50) into equation (48), we obtain the equation:

$$\begin{aligned}
\pi l + \left( \frac{2b\zeta(\omega) - \pi}{2\omega} \right) \alpha + h_0(\alpha) - b\zeta(\omega) + \frac{\pi}{2} \\
= \frac{\pi}{2} + \frac{\alpha}{i} [\zeta(\omega + ib) - \zeta(\omega)] - \zeta(\omega)b.
\end{aligned}$$

This is the same as

$$\begin{aligned}
\pi l + h_0(\alpha) &= \alpha \left( \frac{-2b\zeta(\omega) + \pi}{2\omega} + \frac{(\zeta(\omega + ib) - \zeta(\omega))}{i} \right) \\
&= \alpha \left( \frac{\pi}{2\omega} + \frac{2\omega\zeta(\omega + ib) - 2\omega\zeta(\omega) - 2ib\zeta(\omega)}{2i\omega} \right) \\
&= \alpha \left( \frac{\pi}{2\omega} - \frac{2f(\omega + ib)}{2i\omega} \right) = \alpha \left( \frac{\pi}{2\omega} - \frac{2f(a)}{2i\omega} \right).
\end{aligned} \quad (51)$$

Taking into account that  $f(a) = \frac{i\pi n}{2k}$  and  $2\omega k = \pi$ , we come to the final form of the equation:

$$\pi l + h_0(\alpha) = \alpha(k - n). \quad (52)$$

We claim that equation (52) has at least  $k - n - 1$  solutions for  $\alpha$  in the open interval  $(0, \pi)$ .

Indeed, since  $h_0(0) = h_0(\pi) = 0$ , the end points  $\alpha = 0, \alpha = \pi$  of the open interval are solutions of equation (52) for  $l = 0$  and  $l = k - n$ , respectively. (These two solutions are geometrically trivial, corresponding to  $\alpha = 2\beta = 0$  and  $\alpha = 2\beta = \pi$  for the initial geometric problem.) Therefore, for all intermediate levels of  $l$ , that is, for  $l \in [1, k - n - 1]$ , there exists a solution of equation (52). This proves the claim.

We shall prove now that the number of solutions of equation (52) in the interval  $(0, \pi)$  is exactly equal to  $(k - n - 1)$ . For equation (48), it suffices to show that the function

$$\operatorname{Im} \left( \int_0^{\alpha - \omega} \zeta(ib + t) dt \right) - \frac{\alpha}{i} [\zeta(\omega + ib) - \zeta(\omega)]$$

has non-vanishing derivative with respect to  $\alpha$ . Arguing by contradiction, suppose that

$$\operatorname{Im} (\zeta(ib + \alpha - \omega) - [\zeta(\omega + ib) - \zeta(\omega)]) = 0.$$

Notice that  $\zeta(\omega)$  is real, and  $\zeta(\omega + \alpha + ib)$  and  $\zeta(-\omega + \alpha + ib)$  have the same imaginary part. Hence

$$\operatorname{Im} (\zeta(ib + \alpha + \omega) - \zeta(\omega + ib)) = 0. \quad (53)$$

Using the addition formula, we have

$$\zeta(ib + \omega + \alpha) = \zeta(ib + \omega) + \zeta(\alpha) + \frac{\wp'(ib + \omega) - \wp'(\alpha)}{2(\wp(ib + \omega) - \wp(\alpha))}.$$

It then follows from equation (53) that

$$\zeta(\alpha) + \frac{\wp'(ib + \omega) - \wp'(\alpha)}{2(\wp(ib + \omega) - \wp(\alpha))} \in \mathbb{R}.$$

Moreover, the values  $\zeta(\alpha)$ ,  $\wp(ib + \omega)$ ,  $\wp(\alpha)$ ,  $\wp'(\alpha)$  are all real. We conclude that  $\wp'(ib + \omega) \in \mathbb{R}$ .

On the other hand,

$$ib + \omega \in (\omega, \omega') \Rightarrow e_2 < \wp(ib + \omega) < e_1.$$

Thus the equation  $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$  implies that  $\wp'(ib + \omega) \in i\mathbb{R}$ , a contradiction. This completes the proof of Theorem 7 in Case 1.

**Case 2.** In this case  $m > 0$ ,  $a = ib \in [0, \omega']$ ,  $b \in \mathbb{R}$ . Using  $\frac{\sigma'}{\sigma} = \zeta$ , we write

$$\sigma(z) = \sigma(z_0) \exp \left( \int_{z_0}^z \zeta(t) dt \right).$$

Taking log, we rewrite equation (45) in the form

$$\int_{-\alpha}^{\alpha} \zeta(ib + t) dt + 2\pi il = 2\alpha \zeta(ib), \quad l \in \mathbb{Z}.$$

Using that  $\zeta$  is odd, rewrite this as

$$2\pi il + \int_0^\alpha [\zeta(ib + t) - \zeta(-ib + t)]dt = 2\alpha\zeta(ib).$$

Notice that both sides of this equation are purely imaginary, and hence

$$\pi l + \operatorname{Im} \left( \int_0^\alpha \zeta(ib + t)dt \right) = \frac{1}{i}\alpha\zeta(ib). \quad (54)$$

On the right hand side we have a linear function of  $\alpha$ . Let us denote the integral on the left hand side of equation (54) by

$$g(\alpha) := \operatorname{Im} \left( \int_0^\alpha \zeta(ib + t)dt \right).$$

**Lemma 4.13.** *For any  $r \in \mathbb{N}$ , we have*

$$\operatorname{Im} \left( \int_0^{2\omega r} [\zeta(ib + t)dt] \right) = -\pi r + 2r\zeta(\omega)b.$$

*Proof.* This follows from the residue formula for the rectangular path

$$ib \rightarrow 2\omega r + ib \rightarrow 2\omega r - ib \rightarrow -ib \rightarrow ib,$$

avoiding the singular points of  $\zeta$  at 0 and  $2\omega r$  by small half circles.  $\square$

In particular, using this lemma for  $r = 1$  and the quasi-periodicity of  $\zeta$ , we compute

$$g(\alpha + 2\omega) = g(\alpha) + \frac{1}{i} \int_0^{2\omega} \zeta(ib + t)dt = g(\alpha) - \pi + 2\zeta(\omega)b.$$

Using this, one can express  $g$  as the sum of a linear and a  $2\omega$ -periodic function as follows:

$$g(\alpha) = \left( \frac{-\pi + 2\zeta(\omega)b}{2\omega} \right) \alpha + h(\alpha), \quad g(0) = h(0) = 0,$$

where  $h$  is  $2\omega$ -periodic. Therefore, equation (54) takes the form

$$\pi l + h(\alpha) = - \left( \frac{-\pi + 2\zeta(\omega)b}{2\omega} \right) \alpha + \frac{1}{i}\alpha\zeta(ib),$$

hence

$$\pi l + h(\alpha) = \alpha \left( \frac{1}{i}\zeta(ib) - \frac{-\pi + 2\zeta(\omega)b}{2\omega} \right).$$

Thus we arrive at the following equation

$$\pi l + h(\alpha) = \alpha \left( \frac{\pi}{2\omega} + \frac{2\omega\zeta(ib) - 2\zeta(\omega)ib}{2\omega i} \right) = \alpha \left( \frac{\pi}{2\omega} - \frac{2f(ib)}{2\omega i} \right).$$

Next, taking into account that  $f(ib) = f(a) = \frac{i\pi n}{2k}$  and  $2\omega k = \pi$ , we obtain the simplest possible form:

$$\pi l + h(\alpha) = \alpha(k - n). \quad (55)$$

Also in this case we claim that equation (55) has at least  $k - n - 1$  solutions for  $\alpha$  in the open interval  $(0, \pi)$ .

Indeed, since  $h(0) = h(\pi) = 0$ , the end points  $\alpha = 0, \alpha = \pi$  of the open interval are solutions of equation (55) for  $l = 0$  and  $l = k - n$ , respectively. Therefore, for all intermediate levels of  $l$ , that is, for  $l \in [1, k - n - 1]$ , there exists a solution of equation (55). This proves the claim.

We shall prove now that the number of solutions of equations (55) in the interval  $(0, \pi)$  equals exactly  $k - n - 1$ . Consider equation (54). We shall check that the function

$$\operatorname{Im} \left( \int_0^\alpha \zeta(ib + t) dt - \alpha \zeta(ib) \right)$$

has everywhere non-vanishing derivative with respect to  $\alpha$  when  $ib \in (0, \omega')$ .

Suppose, on the contrary, that the derivative vanishes for some  $\alpha$ :

$$\operatorname{Im} (\zeta(ib + \alpha) - \zeta(ib)) = 0. \quad (56)$$

Using the addition formula for  $\zeta$ , we have

$$\zeta(ib + \alpha) = \zeta(ib) + \zeta(\alpha) + \frac{\wp'(ib) - \wp'(\alpha)}{2(\wp(ib) - \wp(\alpha))}.$$

Taking the imaginary part and using equation (56), we obtain

$$\zeta(\alpha) + \frac{\wp'(ib) - \wp'(\alpha)}{2(\wp(ib) - \wp(\alpha))} \in \mathbb{R}.$$

Also we know that  $\zeta(\alpha)$ ,  $\wp(ib)$ ,  $\wp(\alpha)$ ,  $\wp'(\alpha)$  are all real. Therefore we conclude that  $\wp'(ib) \in \mathbb{R}$ . But, on the other hand,  $\wp$  satisfies the equation  $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ . Moreover,

$$ib \in (0, \omega') \Rightarrow \wp(ib) < e_3 \Rightarrow \wp'(ib) \in i\mathbb{R}.$$

This contradiction completes the proof in Case 2.  $\square$

The preceding theorem has the next corollary.

**Corollary 4.14.** *All the solutions of equation (45) are transversal and hence change smoothly as one varies the parameter  $\omega'$  of the elliptic functions involved.*

### 4.3 Self-Bäcklund curves as deformations of conics

In this section we construct a genuine deformation of a Bäcklund curve to a central conic. This material is related to that in Section 3.1.

We restrict consideration here to simple curves only. Thus we shall assume everywhere in this section that  $n = 1, m = 0$ , in accordance with Theorem 6. Hence, while constructing the curve  $X_+$ , we solve equation (42) for  $m = 0$  and get the unique solution  $a \in (\omega, \omega + \omega')$  (we simplify the notations and drop the index 0)

As before, we shall fix a positive integer  $k$  and set  $2\omega = \frac{\pi}{k}$ . We shall consider the deformation of the Weierstrass functions as  $\omega' \rightarrow \infty$ :

$$\wp_s(\cdot) = \wp(\cdot; \omega'_s), \quad \omega'_s = \frac{\omega'}{s}, \quad s \in [1, 0].$$

The functions  $\zeta_s, \sigma_s$  are determined accordingly. Such a deformation can be realized by collapsing two roots  $e_3, e_2 \rightarrow -c, e_1 \rightarrow 2c$ .

**Remark 4.15.** It is convenient to think about this deformation in terms of the elliptic invariants  $(g_2(s), g_3(s))$ , which can be extended smoothly to the closed interval  $[0, 1]$ . This can be deduced from the series expressing the  $\wp$ -function via the invariants  $g_2, g_3$ .

With these remarks it is clear that  $\wp_s, \zeta_s, \sigma_s, s \in [0, 1]$  become smooth (analytic) families. It turns out that ([3], page 201)

$$\begin{aligned} c &= \frac{1}{3} \left( \frac{\pi}{2\omega} \right)^2, \quad \wp_0(z) = -c + 3c \frac{1}{\sin^2(\sqrt{3cz})}, \\ \zeta_0(z) &= cz + \sqrt{3c} \cot(\sqrt{3cz}), \quad \sigma_0(z) = \frac{1}{\sqrt{3c}} e^{cz^2/2} \sin(\sqrt{3cz}). \end{aligned} \quad (57)$$

In what follows we start with the objects considered at  $s = 0$  and then extend them to positive  $s$  using smoothness of the families and a transversality argument. Next we extend the objects smoothly to the whole interval  $[0, 1]$ .

For example, consider equation (42) on  $a$  for  $s = 0$ . Write  $a$  in the form  $a = \omega + i\omega'b$ . We have:

$$f_0(a) = a\zeta_0(\omega) - \omega\zeta_0(a) = \frac{i\pi}{2k} = i\omega. \quad (58)$$

Using the explicit formula (57) for  $\zeta_0$ , we compute that equation (58) is equivalent to

$$a = \omega + i\omega'b, \quad \tanh\left(\frac{\pi b}{2}\right) = \frac{2\omega}{\pi} = \frac{1}{k}. \quad (59)$$

Notice that there is a unique solution of this equation and, moreover, it is non-degenerate. Therefore there exists a unique solution  $a_s$  of the equation

$$f_s(a) = a_s\zeta_s(\omega) - \omega\zeta_s(a_s) = \frac{i\pi}{2k} = i\omega \quad (60)$$

for  $s \geq 0$ , smoothly depending on  $s$ . (By the implicit function theorem we get this fact for small positive  $s$ , and then for the whole segment  $s \in [0, 1]$ , since we know a priori the existence, uniqueness, and smooth dependence of  $a_s$  on  $s$  for positive  $s$ .)

Also we have that the solution of equation (58) satisfies

$$\zeta_0(a) = -i + \frac{1}{3} \left( \frac{\pi}{2\omega} \right)^2 a.$$

Now we are in position to state the existence of the deformations of the circle. The idea is to write the functions  $X_s, s \in [0, 1]$ , as Floquet eigenfunctions of a Hill operator on  $[0, 2\omega]$  and to use a general argument of smooth dependence of the eigenfunction on  $s$ .

Following this idea, we consider the family of  $2\omega$ -periodic functions for  $s \in [0, 1]$ ,  $2\omega = \frac{\pi}{k}$ :

$$\begin{cases} q_s(t) = \wp_s(t + \omega'_s), \omega'_s = \frac{\omega'_s}{s}, s \neq 0, \\ q_0(t) = -c = -\frac{1}{3} \left( \frac{\pi}{2\omega} \right)^2, s = 0. \end{cases}$$

**Theorem 8.** 1. The functions  $q_s$  smoothly depend on  $s \in [0, 1]$ .

2. The functions

$$X_s(t) = \begin{cases} e^{-t\zeta_s(a_s)} \frac{\sigma_s(a_s + t + \omega'_s) \sigma_s(\omega'_s)}{\sigma_s(a_s + \omega'_s) \sigma_s(t + \omega'_s)} & \text{for } s \neq 0, \\ e^{it} & \text{for } s = 0 \end{cases}$$

are eigenfunctions corresponding to the smallest eigenvalue  $\lambda_0^{(s)}$  for the Floquet problem

$$X'' + (\lambda - 2q_s(t))X = 0, \quad X(t + 2\omega) = \mu X(t), \quad \mu = e^{i\frac{\pi}{k}}, \quad 2\omega = \pi/k. \quad (61)$$

3. The family of functions  $X_s, s \in [0, 1]$ , depend smoothly on  $s$ . The functions  $X_s$  determine a deformation of the unit circle through centrally symmetric star-shaped curves. See Figure 17.

*Proof.* 1. We use the series defining the  $\wp_s$  function:

$$\begin{aligned} q_s(t) &= \wp_s(t + \omega'_s) = \\ &= \frac{1}{(t + i\frac{\omega'_s}{s})^2} + \sum_{(m,n) \neq 0} \left( \frac{1}{(t + 2n\omega + i\frac{\omega'_s}{s} + 2im\frac{\omega'_s}{s})^2} - \frac{1}{(2n\omega + 2im\frac{\omega'_s}{s})^2} \right) \\ &= \frac{s^2}{(st + i\omega')^2} + \sum_{(m,n) \neq 0} \left( \frac{s^2}{(st + s2n\omega + i\omega' + 2im\omega')^2} - \frac{s^2}{(s2n\omega + 2im\omega')^2} \right). \end{aligned}$$

From this series one sees that, for  $s = 0$ ,

$$q_0(t) = \sum_{n \neq 0} - \left( \frac{1}{2n\omega} \right)^2 = -\frac{1}{(2\omega)^2} \frac{\pi}{3} = -c.$$

Also it is clear that the series can be differentiated.



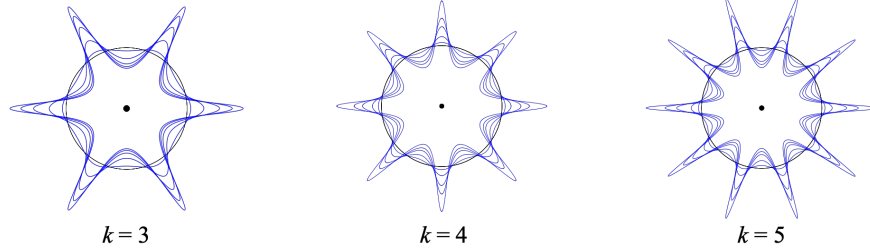


Figure 17: Theorem 8. Three families of deformations of the circle (black) through a 1-parameter family of self-Bäcklund curves  $X_s$  (blue) with  $2k$ -fold symmetry,  $k = 3, 4, 5$ , realizing the respective infinitesimal deformations of Theorem 3.

2. For  $s > 0$ , the functions  $X_s$  were constructed exactly as eigenfunctions of the Floquet problem for the smallest eigenvalue. Let us consider the case  $s = 0$ . In this case,

$$X_0'' + (\lambda + 2c)X_0 = 0.$$

Hence

$$X_0(t) = e^{i\sqrt{2c+\lambda}t}.$$

Since the Floquet multiplier is  $e^{i2\omega}$ , we obtain

$$2c + \lambda = \left(1 + \frac{\pi m}{\omega}\right)^2 = (1 + 2km)^2 \geq 1.$$

Hence the smallest  $\lambda$  is  $1 - 2c$ , and we get  $X_0 = e^{it}$ , as needed.

3. Notice that, for a given periodic potential  $q(t)$ , the problem (61) of Floquet eigenvalues has the following properties (see [23], page 32):

1. The eigenvalues  $\lambda_m(\mu)$  are solutions of the equation

$$\Delta(\lambda) = 2 \cos\left(\frac{\pi}{k}\right). \quad (62)$$

(Recall that  $\Delta(\lambda)$  is the trace of the period map of equation (61).)

2. The graph of the function  $\Delta(\lambda)$  (see Figure 15) is such that all the solutions of equation (62) are transversal. Hence all  $\lambda_{m,s}(\mu)$ , and, in particular,  $\lambda_{0,s}(\mu)$ , depend smoothly on the parameter  $s$ .
3. All Floquet eigenvalues  $\lambda_{m,s}(\mu)$  have multiplicity 1, because if  $X$  is an eigenfunction for some non-real Floquet exponent, then  $\overline{X}$  is not.

One concludes from these properties that the eigenfunctions depend smoothly on the parameter  $s$ . Indeed, let us fix a basis of solutions of the second order differential equation

$$X'' + (\lambda - 2q_s(t))X = 0$$

$$y_1(\lambda, s, t), y_2(\lambda, s, t) : \quad y_1(\lambda, s, 0) = y_2'(\lambda, s, 0) = 1, \quad y_1'(\lambda, s, 0) = y_2(\lambda, s, 0) = 0.$$

Hence we can write the eigenfunction corresponding to  $\lambda_{m,s}(\mu)$  in the form

$$X = Ay_1 + By_2,$$

and then the Floquet boundary conditions can be written, in terms of  $A, B$ , in the form

$$M(\lambda, s) \cdot \begin{pmatrix} A \\ B \end{pmatrix} = 0 \quad (63)$$

for a 2 by 2 matrix  $M(s, \lambda)$ , smoothly depending on  $\lambda, s$ .

Moreover, it follows from 2) that, for  $\lambda = \lambda_{m,s}$ , the matrix  $M$  has rank 1 and that  $M(\lambda_{m,s}, s)$  smoothly depends on  $s$ . But then the solution  $\begin{pmatrix} A \\ B \end{pmatrix}$  of equation (63) can be also chosen smoothly depending on  $s$ . Therefore the eigenfunction corresponding to  $\lambda_{m,s}$  smoothly depends on  $s$  as well.  $\square$

The next step is to evaluate  $\alpha_s$ . This is covered by the following theorem.

**Theorem 9.** *For every  $s \in [0, 1]$ , the curves determined by  $X_s$  are self-Bäcklund for  $k - 2$  values of  $\alpha_s \in (0, \pi)$ . These  $\alpha_s$  satisfy the equation*

$$\frac{\sigma_s(a_s + \alpha_s)}{\sigma_s(a_s - \alpha_s)} = e^{2\alpha_s \zeta_s(a_s)}. \quad (64)$$

For  $s = 0$ , this equation reduces to equation (18),

$$k \tan(\alpha) = \tan(k\alpha).$$

Moreover, all  $k - 2$  solutions  $\alpha_s$  (see Theorem 3, part 2) depend smoothly on  $s \in [0, 1]$ .

*Proof.* Consider equation (64) on  $\alpha$  for  $s = 0$  :

$$\frac{\sigma_0(a + \alpha)}{\sigma_0(a - \alpha)} = e^{2\alpha \zeta_0(a)},$$

where  $a$  is the solution of equation (58). Set

$$F(\alpha) := \frac{\sigma_0(a + \alpha)}{\sigma_0(a - \alpha)} e^{-2\alpha \zeta_0(a)}.$$

Using the explicit formulas (57)-(59) and  $\pi = 2k\omega$ , we have:

$$F(\alpha) = \frac{\sin\left(\frac{\pi}{2\omega}(a + \alpha)\right)}{\sin\left(\frac{\pi}{2\omega}(a - \alpha)\right)} e^{i2\alpha} = \frac{1 - i\frac{1}{k}\tan(k\alpha)}{1 + i\frac{1}{k}\tan(k\alpha)} e^{i2\alpha}.$$

This immediately implies that the equation  $F = 1$  is equivalent to the familiar equation (18):

$$k \tan(\alpha) = \tan(k\alpha).$$

This means that, for  $s = 0$ , equation (64) has precisely  $k - 2$  solutions for  $\alpha \in (0, \pi)$ .

Moreover, differentiating  $F$  at a point  $\alpha$  where  $F(\alpha) = 1$  we have:

$$\begin{aligned} F'(\alpha) &= 2iF + e^{2i\alpha} \frac{\frac{-2i}{\cos^2(k\alpha)}}{\left(1 + \frac{i}{k} \tan k\alpha\right)^2} \\ &= 2i - \frac{\frac{2i}{\cos^2(k\alpha)}}{1 - \left(\frac{i}{k} \tan k\alpha\right)^2} = 2i \frac{(1 - k^2) \tan^2 k\alpha}{k^2 + \tan^2 k\alpha} \neq 0. \end{aligned}$$

Applying the implicit function theorem, we conclude that all  $k - 2$  solutions of equation (64) can be smoothly extended from  $s = 0$  to  $s > 0$ . This, together with Theorem 7 and Corollary 4.14, implies the existence of  $k - 2$  solutions for every  $s \in [0, 1]$ , smoothly depending on  $s$ .  $\square$

## 5 Self-Bäcklund polygons

### 5.1 Centroaffine butterflies, Bianchi permutability

The central projection  $\mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{RP}^1$  takes a centroaffine curve to a curve in the projective line. Conversely, a projective curve admits a unique lift to a centroaffine curve. Bianchi permutability for  $c$ -relation was established for projective curves, in [45]. Here we do it for centroaffine curves.

Let us say that a quadrilateral  $P_1P_2P_3P_4$  forms a *centroaffine butterfly* if

$$[P_1, P_2] = [P_4, P_3] \quad \text{and} \quad [P_2, P_3] = [P_1, P_4]. \quad (65)$$

Note that a centroaffine butterfly is not necessarily a centroaffine polygon.

**Lemma 5.1.** *A generic quadrilateral  $P_1P_2P_3P_4$  is a centroaffine butterfly if and only if any of the following equivalent conditions are satisfied:*

1. *There is a linear involution  $I \in \text{GL}_2(\mathbb{R})$  interchanging  $P_1P_2$  and  $P_3P_4$ . That is,  $I(P_1) = P_3$ ,  $I(P_2) = P_4$ ,  $I(P_3) = P_1$ ,  $I(P_4) = P_2$ .*
2. *The line segments  $P_1P_3, P_2P_4$  are parallel and their midpoints are collinear. See Figure 18.*
3.  *$P_aP_bP_cP_d$  is a centroaffine butterfly, where  $abcd$  is any of the 8 permutations of 1234 generated by (1234), (24), (12)(34).*

*Proof.* 1. By applying a linear transformation, we can assume the  $P_1 = (1, 0)$ ,  $P_3 = (0, 1)$ . Let  $P_2 = (c, d)$ . Then equations 65 imply  $P_4 = (d, c)$ . Thus  $I : (x, y) \mapsto (y, x)$  is the required symmetry.

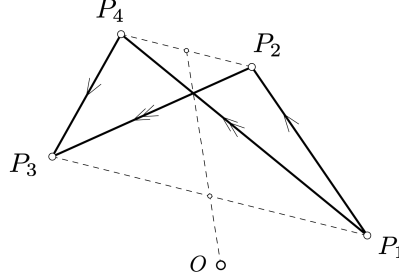


Figure 18: A centroaffine butterfly

2. Note that the said segments are parallel and their midpoints are collinear if and only if  $[P_1 \pm P_3, P_2 \pm P_4] = 0$  ('-' for the 1st statement, '+' for the 2nd). By expanding these expressions we see that they are equivalent to  $[P_1, P_2] = [P_4, P_3]$ ,  $[P_2, P_3] = [P_1, P_4]$ .

3. This is a simple verification (omitted).  $\square$

It follows from this lemma that, given a generic triple of points  $P_1, P_2, P_3$ , there is a unique fourth point  $P_4$  such that  $P_1P_2P_3P_4$  form a centroaffine butterfly. Namely, by property 1,  $P_4 = IP_2$  where I is defined by  $IP_1 = P_3$ ,  $IP_3 = P_1$ . More geometrically, by property 2, one constructs the line  $\ell$  through  $P_2$  and parallel to  $P_1P_3$ , intersect  $\ell$  with the line through the origin  $O$  and the midpoint of  $P_1P_3$ , then finds the unique point  $P_4$  on  $\ell$  such that this intersection point is the midpoint of  $P_2P_4$ .

**Theorem 10** (Bianchi permutability). *Consider three centroaffine curves  $\gamma, \delta$ , and  $\Gamma$  such that  $\Gamma$  and  $\delta$  are  $b$ - and  $c$ -related to  $\gamma$  (respectively). Then there exists a fourth centroaffine curve  $\Delta$  that is  $b$ -related to  $\delta$  and  $c$ -related to  $\Gamma$ . In fact,  $\Delta(t)$  is the unique point such that  $\delta(t)\gamma(t)\Gamma(t)\Delta(t)$  form a centroaffine butterfly.*

*Proof.* We have

$$[\gamma, \delta] = [\Gamma, \Delta] = c, \quad [\gamma, \Gamma] = [\delta, \Delta] = b,$$

and need to check that  $\Delta(t)$  is a centroaffine curve, that is,  $[\Delta, \Delta'] = 1$ .

Using the above relations, one can write  $\Delta$  as a linear combination of  $\delta$  and  $\Gamma$ ,

$$\Delta = \frac{[\gamma, \delta]}{[\Gamma, \delta]} \delta - \frac{[\gamma, \Gamma]}{[\Gamma, \delta]} \Gamma = \frac{c\delta - b\Gamma}{[\Gamma, \delta]}.$$

Then

$$[\Delta, \Delta'] = \frac{[c\delta - b\Gamma, c\delta' - b\Gamma']}{[\Gamma, \delta]^2} = \frac{b^2 + c^2 - bc([\delta, \Gamma'] + [\Gamma, \delta'])}{[\Gamma, \delta]^2}.$$

Thus we want to show that

$$b^2 + c^2 - bc([\delta, \Gamma'] + [\Gamma, \delta']) = [\Gamma, \delta]^2. \quad (66)$$

We have

$$\delta = f\gamma + c\gamma', \quad \Gamma = g\gamma + b\gamma',$$

hence

$$\delta' = (f' + cp)\gamma + f\gamma', \quad \Gamma' = (g' + bp)\gamma + g\gamma'.$$

It follows that

$$[\Gamma, \delta] = cg - bf, \quad [\delta, \Gamma'] = fg - cg' - bcp, \quad [\Gamma, \delta'] = fg - bf' - bcp.$$

In addition, one has equations (3):

$$cf' = f^2 - c^2p - 1, \quad bg' = g^2 - b^2p - 1.$$

Substitute these formulas into equation (66) to obtain a true identity.  $\square$

## 5.2 Discrete Bäcklund transformation

In this section, we describe the centroaffine version of the transformation thoroughly studied in [7]. That paper concerned polygons in  $\mathbb{RP}^1$ , which is close to, but not exactly the same as, centroaffine polygons in the plane (the difference occurs for projective even-gons, see Section 8.4 of [36] for details).

Let  $\mathbf{P}$  and  $\mathbf{Q}$  be two origin-symmetric  $2n$ -gons in  $\mathbb{R}^2$  with vertices  $P_i$  and  $Q_i, i = 1, \dots, 2n$ , such that  $[P_i, P_{i+1}] = [Q_i, Q_{i+1}] = 1$  for all  $i$ . The polygons are Bäcklund transformations of each other if there exist a constant  $c$  such that  $[P_i, Q_i] = c$  for all  $i$ .

The above described relation between centroaffine polygons is an analog of the discrete bicycle transformation studied in [46]. In particular, it admits the following geometric construction.

Let  $\mathbf{P} = (P_1, P_2, \dots)$  be given, and let the first vertex,  $Q_1$ , be given as well. Then one can construct the next vertex  $Q_2$  by requiring  $[Q_1, Q_2] = 1$  and  $[P_1, Q_1] = [P_2, Q_2]$ . That is,  $P_1P_2Q_2Q_1$  is a centroaffine butterfly.

Continuing in this way, one constructs the points  $Q_3, Q_4, \dots, Q_{2n}, Q_{2n+1}$ . In general,  $Q_{2n+1} \neq Q_1$ , and one has a non-trivial monodromy associated with  $\mathbf{P}$ . However if  $Q_{2n+1} = Q_1$ , one obtains a centroaffine polygon  $\mathbf{Q}$  that is Bäcklund transformation of  $\mathbf{P}$ .

One may extend this construction to more general types of polygons: given  $\mathbf{P}$ , construct  $\mathbf{Q}$  such that  $[P_i, P_{i+1}] = [Q_i, Q_{i+1}]$  for all  $i$ , and  $[P_i, Q_i]$  having the same value for all  $i$ . This is in fact the case in the next result, since, as we noted before, centroaffine butterflies, by definition, are not centroaffine polygons. The next result has an analog for the discrete bicycle transformation, see [46].

**Lemma 5.2.** *Centroaffine butterflies have trivial monodromy: if  $\mathbf{P}$  is a butterfly then any of its Bäcklund transforms is a closed quadrilateral  $\mathbf{Q}$  that is also a centroaffine butterfly. See Figure 19.*

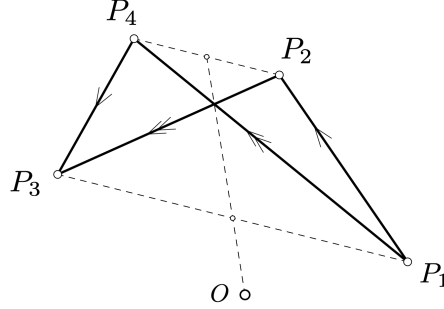


Figure 19: Two centroaffine butterflies are Bäcklund transformations of each other.

*Proof.* Since  $P_i P_{i+1} Q_{i+1} Q_i$  is a centroaffine butterfly we have, by Lemma 5.1, that  $Q_{i+1} = I_{Q_i P_{i+1}}(P_i)$ ,  $i \geq 1$ , where  $I_{AB}$  is the linear map interchanging  $A$  and  $B$ . Explicitly,

$$I_{AB}(X) = \frac{[A, X]B + [X, B]A}{[A, B]}. \quad (67)$$

Applying a linear transformation, we may assume that

$$P_1 = (1, 0), P_2 = (a, b), P_3 = (0, 1), P_4 = (b, a), Q_1 = (x, y),$$

where  $a, b$  are parameters that characterize the butterfly  $\mathbf{P}$ . Applying equation (67) repeatedly to  $Q_{i+1} = I_{Q_i P_{i+1}}(P_i)$ , we find

$$Q_2 = \frac{1}{bx - ay}(ab - xy, b^2 - y^2), \quad Q_3 = \left( -y, \frac{-y(a^2 + b^2) + abx + y^3}{ab - xy} \right),$$

$$Q_4 = \frac{1}{ax - by}(ab - xy, a^2 - y^2), \quad Q_5 = (x, y).$$

This completes the proof.  $\square$

We can define centroaffine polygon recutting in analogy with polygon recutting introduced by V. Adler [1, 2]. Let  $\mathbf{P}$  be a centroaffine  $n$ -gon. For every  $i \bmod n$ , define the transformation  $T_i$ : all vertices stay put, except  $P_i$ , which is replaced by  $P'_i$  so that  $P_{i-1}P_iP_{i+1}P'_i$  be a centroaffine butterfly, see Figure 20 (left).

Note that if  $[P_{i-1}, P_i] = [P_i, P_{i+1}]$ , then  $P'_i = P_i$ , and the recutting is a trivial operation.

The centroaffine polygon recutting transformation of  $\mathbf{P}$  is the composition  $T_n T_{n-1} \dots T_1$ . It appears that this transformation is completely integrable, and that it commutes with and shares the integrals of the centroaffine Bäcklund transformation, similarly to what holds for Adler's polygon recutting, see [46]. See Figure 20 (right). We do not dwell on this matter here.

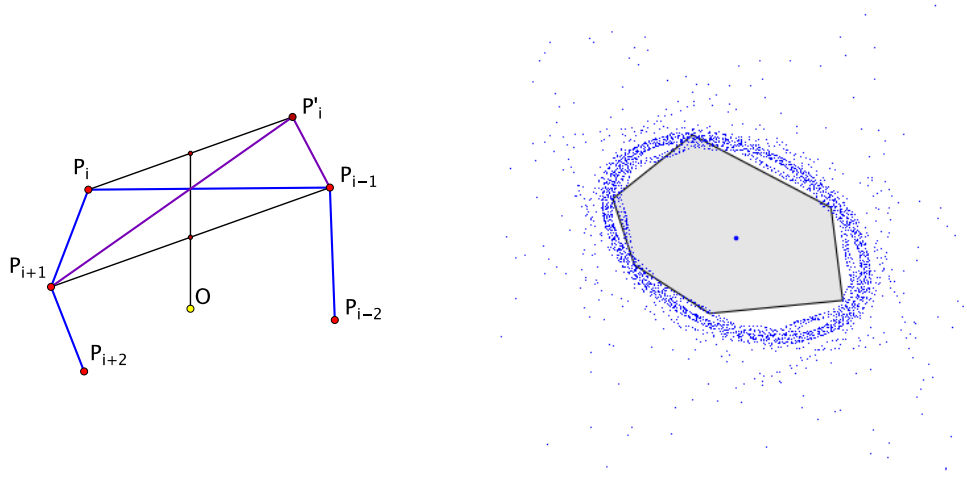


Figure 20: Centoraffine polygon recutting. Left: the definition. Right: an examples of hexagon evolution under recutting. The vertices of several thousand iteration are drawn, showing evidence of complete integrability.

### 5.3 Rigidity results and flexible examples

Recall that an origin-symmetric  $2n$ -gon  $\mathbf{P}$  in  $\mathbb{R}^2$  with vertices  $P_i, i = 1, \dots, 2n$ , is called a *self-Bäcklund*  $(n, k)$ -gon if

$$[P_i, P_{i+1}] = 1, [P_i, P_{i+k}] = c$$

for all  $i$  and  $2 \leq k \leq n - 2$ . Such polygons are acted upon by  $\mathrm{SL}_2(\mathbb{R})$ . Since  $P_{i+n} = -P_i$ , we can assume, without loss of generality, that  $k \leq n/2$ .

A regular  $2n$ -gon is a self-Bäcklund  $(n, k)$ -gon for all  $2 \leq k \leq n/2$ . We call these self-Bäcklund  $(n, k)$ -gons and their  $\mathrm{SL}_2(\mathbb{R})$  images trivial. The problem is to find non-trivial self-Bäcklund  $(n, k)$ -gons.

The next result is analogous to Theorem 9 of [42].

**Theorem 11.** *In the following cases every self-Bäcklund  $(n, k)$ -gon is trivial:*

1.  $n$  is arbitrary,  $k = 2$ ;
2.  $n$  is odd,  $k = 3$ ;
3.  $k$  is arbitrary,  $n = 2k + 1$ .
4.  $n = 3k$ .

*On the other hand, there exist non-trivial self-Bäcklund  $(n, k)$ -gons in the following cases:*

1.  $n$  is even and  $k$  is odd;
2.  $n = 2k$ .

*Proof.* Each next vertex is a linear combination of the preceding two:  $P_{i+2} = a_i P_{i+1} - P_i$ .

Let  $k = 2$ . Then  $[P_i, P_{i+2}] = c$ , hence  $a_i = c$  for all  $i$ . Let  $A$  be the linear map defined by

$$A(P_1) = P_2, \quad A(P_2) = P_3.$$

We claim that  $A$  is area preserving and  $A(P_i) = P_{i+1}$  for all  $i$ . This would imply that the polygon  $\mathbf{P}$  is centroaffine regular, that is, trivial.

That  $A$  is area preserving follows from  $[P_1, P_2] = [P_2, P_3]$ . Next,

$$P_3 = -P_1 + cP_2, \quad \text{hence} \quad A(P_3) = -P_2 + cP_3 = P_4.$$

Repeating this argument, we obtain  $A(P_i) = P_{i+1}$  for all  $i$ .

Now let  $n$  be odd and  $k = 3$ . Consider four consecutive vertices of  $\mathbf{P}$ ; they satisfy the Ptolemy-Plücker relation

$$[P_i, P_{i+1}][P_{i+2}, P_{i+3}] + [P_{i+1}, P_{i+2}][P_i, P_{i+3}] = [P_i, P_{i+2}][P_{i+1}, P_{i+3}].$$

Therefore

$$1 + c = [P_i, P_{i+2}][P_{i+1}, P_{i+3}].$$

It follows that  $[P_i, P_{i+2}] = [P_{i+2}, P_{i+4}]$  for all  $i$ .

Now recall that  $n$  is odd and that  $P_{i+n} = -P_i$  for all  $i$ . This implies that

$$[P_i, P_{i+2}] = [P_{n+i}, P_{n+i+2}] = [P_{i+1}, P_{i+3}],$$

and hence  $[P_i, P_{i+2}]$  has the same value for all  $i$ . Thus  $\mathbf{P}$  is a self-Bäcklund  $(n, 2)$ -gon, the already considered case.

Next, let  $n = 2k + 1$ . First we notice that  $[P_i, P_{i+k+1}] = c$ . Indeed,

$$[P_i, P_{i+k}] = [P_{i+k+1}, P_{i+n}] = [P_i, P_{i+k+1}].$$

Now consider the quadruple of vertices  $P_i, P_{i+1}, P_{i+k}, P_{i+k+1}$ . The Ptolemy-Plücker relation implies that

$$[P_{i+1}, P_{i+k}] = \frac{c^2 - 1}{c}$$

for all  $i$ . That is,  $[P_i, P_{i+k-1}]$  is independent of  $i$ .

Continuing in the same way, we reduce  $k$  until we get to the case  $k = 2$ , considered above, and we conclude that  $\mathbf{P}$  is centroaffine regular.

Now let  $n = 3k$ . Let us scale the polygon so that  $[P_i, P_{i+k}] = \sqrt{3}/2$  for all  $i$  (as for a regular  $6k$ -gon inscribed in a unit circle). Then  $[P_i, P_{i+1}] = t$ , a constant.

Each hexagon  $\mathbf{P}_i := (P_i, P_{i+k}, P_{i+2k}, P_{i+3k}, P_{i+4k}, P_{i+5k})$  is affine-regular, and they are all equivalent under  $\text{SL}_2(\mathbb{R})$ . Hence we assume, without loss of generality, that the vertices of  $\mathbf{P}_0$  are the sixth roots of unity. Let  $A \in \text{SL}_2(\mathbb{R})$  take  $\mathbf{P}_0$  to  $\mathbf{P}_1$ . A quick calculation, using the equations

$$[P_0, P_1] = [P_k, P_{k+1}] = [P_{2k}, P_{2k+1}] = [P_{3k}, P_{3k+1}] = [P_{4k}, P_{4k+1}] = [P_{5k}, P_{5k+1}] = t,$$



reveals that  $A$  is a rotation

$$A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}, \quad t = \sin \alpha.$$

The same argument, applied to the linear map that takes  $\mathbf{P}_1$  to  $\mathbf{P}_2$ , shows that this map is the same rotation,  $A$ . And so on, showing that the polygon is regular.

Let us construct non-trivial self-Bäcklund  $(n, k)$ -gons for even  $n$  and odd  $k$ . Start with a regular  $2n$ -gon, and consider the midpoints of its sides. These points are the vertices of another regular  $2n$ -gon. Dilate the latter  $2n$ -gon with the center of dilation at its center. We obtain a centrally symmetric  $4n$ -gon having a dihedral symmetry, and this symmetry implies  $[P_i, P_{i+k}] = [P_{i+1}, P_{i+k+1}]$ . See Figure 21 on the left. (The projection of this polygon to  $\mathbb{R}\mathbb{P}^1$  is a regular  $n$ -gon therein).

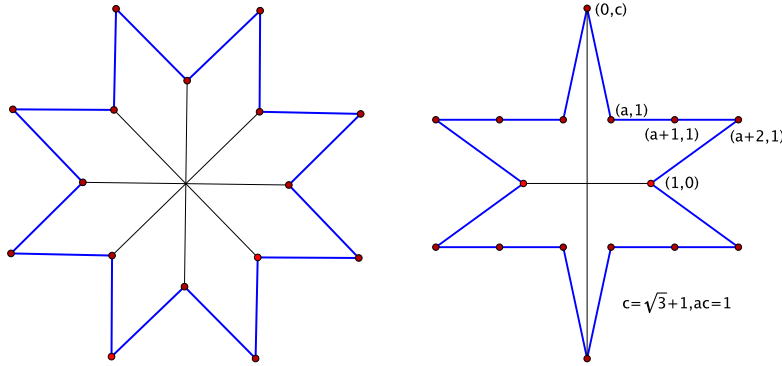


Figure 21: Left: a self-Bäcklund  $(8, 3)$ -gon. Right: a self-Bäcklund  $(8, 4)$ -gon.

The construction of a non-trivial self-Bäcklund  $(2k + 4, k + 2)$ -gon is presented in Figure 21 on the right (where  $k = 2$ ).<sup>1</sup> This polygon has two axes of symmetry. In the general case, one has points  $(a, 1), (a + 1, 1), \dots, (a + k, 1)$  on a horizontal line with

$$a = \frac{\sqrt{k^2 + 8} - k}{4}, \quad c = \frac{\sqrt{k^2 + 8} + k}{2}.$$

One checks that  $[P_i, P_{i+1}] = 1$  and  $[P_i, P_{i+k+2}] = c$  for all  $i$ .  $\square$

#### 5.4 Infinitesimal deformations of regular polygons

Here we consider the linearized problem, that is, infinitesimal deformations of regular polygons as self-Bäcklund  $(n, k)$ -gons; this is a discrete analog of the material in Section 3.1.

<sup>1</sup>We are grateful to Michael Cuntz for suggesting this construction.

Call a regular polygon *infinitesimally rigid* as a self-Bäcklund  $(n, k)$ -gon if each of its infinitesimal deformations in the class of self-Bäcklund  $(n, k)$ -gons is induced by the action of  $\mathfrak{sl}(2, \mathbb{R})$ .

**Theorem 12.** *A regular  $2n$ -gon is infinitesimally rigid as a self-Bäcklund  $(n, k)$ -gon unless one of the following holds:*

1.  $n$  is even and  $k$  is odd;
2.  $n = 2k$  with even  $k > 2$ ;
3. there exists an integer  $j$  with  $2 \leq j \leq n - 2$  such that  $n = 2(k + j)$  and  $n$  divides  $(k - 1)(j - 1)$ .

**Corollary 5.3.** *A regular  $2n$ -gon is infinitesimally rigid as a self-Bäcklund  $(n, k)$ -gon if  $n$  is odd, or if both  $n$  and  $k$  are even,  $k < n/2$ , and  $\gcd(n, k) > 2$ .*

*Proof.* The first statement of the corollary follows immediately from the theorem.

For the second statement, assume that a non-trivial infinitesimal deformation exists. We claim that  $k$  and  $j$  are coprime. Indeed, if  $(j, k) = p$ , then  $n = 2(j + k) \equiv 0 \pmod{p}$ , but  $(j - 1)(k - 1) \equiv 1 \pmod{p}$ . This contradicts to the fact that  $n$  divides  $(j - 1)(k - 1)$ . It follows that

$$(n, k) = (2(j + k), k) = 2(j, k) = 2,$$

proving the second statement. □

Now we prove Theorem 12.

*Proof.* Let

$$P_j = \left( \cos\left(\frac{\pi j}{n}\right), \sin\left(\frac{\pi j}{n}\right) \right), \quad j = 1, \dots, 2n,$$

be the vertices of a regular  $2n$ -gon. We have

$$[P_j, P_{j+1}] = \sin\left(\frac{\pi}{n}\right) = a, \quad [P_j, P_{j+k}] = \sin\left(\frac{\pi k}{n}\right) = b.$$

(One can rescale to have  $a = 1$ , but it is not really needed for the argument.)

We also have the respective second-order linear recurrence

$$P_{j+1} = 2 \cos\left(\frac{\pi}{n}\right) P_j - P_{j-1}. \quad (68)$$

Consider an infinitesimal deformation  $P_j + \varepsilon V_j$ , where  $V_j$  is an  $n$ -anti-periodic sequence of vectors, that is,  $V_{j+n} = -V_j$  for all  $j$ , and assume that the resulting polygon is a self-Bäcklund  $(n, k)$ -gon. By applying a dilation, we may assume that the constant  $a$  does not change. Then, calculating modulo  $\varepsilon^2$ , we obtain two systems of equations

$$[P_j, V_{j+1}] + [V_j, P_{j+1}] = 0, \quad j = 1, \dots, n, \quad (69)$$

and

$$[P_j, V_{j+k}] + [V_j, P_{j+k}] = C, \quad j = 1, \dots, n, \quad (70)$$

where  $C$  is a constant.

Consider the system (69). Let

$$V_j = a_j P_j + b_j P_{j+1} = c_j P_j + d_j P_{j-1}.$$

Then the recurrence (68) implies that

$$\frac{c_j - a_j}{b_j} = 2 \cos\left(\frac{\pi}{n}\right), \quad \frac{d_j}{b_j} = -1.$$

Substitute vectors the  $V_j$  into equation (69) to obtain

$$a_j = -c_{j+1}, \quad b_j = \frac{c_j + c_{j+1}}{2 \cos(\pi/n)}, \quad d_j = -\frac{c_j + c_{j+1}}{2 \cos(\pi/n)}, \quad (71)$$

where  $c_j$  is an  $n$ -periodic sequence to be determined.

Now consider the system (70). Substituting vectors  $V_j$ , using equation (71), and collecting terms yields the linear system

$$\mu_{k-1} c_j - \mu_{k+1} c_{j+1} + \mu_{k+1} c_{j+k} - \mu_{k-1} c_{j+k+1} = C, \quad j = 1, \dots, n, \quad (72)$$

where  $\mu_k = \sin(\pi k/n)$ .

First, we note that  $C$  must be zero. Indeed, add equations (72): the left hand side vanishes, and so must the right hand side.

Second, system (72) has a 3-dimensional space of trivial solutions that correspond to the action of the Lie algebra  $\mathfrak{sl}_2(\mathbb{R})$ . These solutions are given by the formulas

$$c_j = 1; \quad c_j = \cos\left(\frac{\pi(2j-1)}{n}\right); \quad c_j = \sin\left(\frac{\pi(2j-1)}{n}\right).$$

We need to find out when there are no other solutions.

To this end, consider the eigenvalues of the matrix defining the system (72). This is a circulant matrix, and its eigenvalues are given by the formula

$$\lambda_j = \mu_{k-1} - \mu_{k+1} \omega_j + \mu_{k+1} \omega_j^k - \mu_{k-1} \omega_j^{k+1}, \quad j = 0, \dots, n-1,$$

where  $\omega_j = e^{i \frac{2\pi j}{n}}$ , see [20].

We are interested in zero eigenvalues. One has  $\lambda_j = 0$  if and only if

$$\omega_j^{k+1} = \frac{\mu_{k-1} - \mu_{k+1} \omega_j}{\mu_{k-1} - \mu_{k+1} \bar{\omega}_j}.$$

Let  $2\alpha$  be the argument of the unit complex number on the right. A direct calculation yields

$$\tan \alpha = -\frac{\sin\left(\frac{\pi(k+1)}{n}\right) \sin\left(\frac{2\pi j}{n}\right)}{\sin\left(\frac{\pi(k-1)}{n}\right) - \sin\left(\frac{\pi(k+1)}{n}\right) \cos\left(\frac{2\pi j}{n}\right)}.$$

The argument of  $\omega_j^{k+1}$  is  $2\pi j(k+1)/n$ , hence (after cleaning up the formulas)

$$\sin\left(\frac{\pi j(k+1)}{n}\right) \sin\left(\frac{\pi(k-1)}{n}\right) = \sin\left(\frac{\pi j(k-1)}{n}\right) \sin\left(\frac{\pi(k+1)}{n}\right),$$

or, equivalently,

$$\tan\left(\frac{\pi j}{n}\right) \tan\left(\frac{\pi k}{n}\right) = \tan\left(\frac{\pi j k}{n}\right) \tan\left(\frac{\pi}{n}\right). \quad (73)$$

Note the trivial solutions  $j = 0, 1, n - 1$ , corresponding to the action of  $\mathfrak{sl}_2(\mathbb{R})$ . Let us assume that  $2 \leq j \leq n - 2$ .

One also has other trivial solutions, when both sides of equation (73) are infinite:  $n = 2j$  and  $k$  odd, and  $n = 2k$  and  $j$  odd. Note that, in the latter case,  $k > 2$ . Indeed, if  $k = 2$ , then  $n = 4$ , and since  $2 \leq j \leq n - 2$ , we have  $j = 2$ , contradicting that  $j$  is odd.

Equation (73) appeared in [42] and in [4], and it was solved in [18]. This equation has non-trivial solutions if and only if  $n = 2(j + k)$  and  $n$  divides  $(j - 1)(k - 1)$ . This completes the proof.  $\square$

**Remark 5.4.** As we know from Theorem 11, if  $n$  is even and  $k$  is odd, or if  $n = 2k$ , non-trivial self-Bäcklund  $(n, k)$ -gons indeed exist. The smallest values in case 3) of Theorem 12 are  $k = 4, n = 30$ . Does there exist a non-trivial self-Bäcklund  $(30, 4)$ -gon?

**Remark 5.5.** One wonders whether the symmetry between  $k$  and  $j$  in the formulation of Theorem 12 corresponds to some kind of duality between self-Bäcklund  $(n, k)$ - and  $(n, j)$ -gons.

## A From the centroaffine plane to the hyperbolic plane

In this appendix we connect two geometries associated with the group  $\mathrm{SL}_2(\mathbb{R})$ , the centroaffine and the hyperbolic ones.

Consider the 3-dimensional space of quadratic forms  $ax^2 + 2bxy + cy^2$  with the pseudo-Euclidean metric given by quadratic form  $b^2 - ac$ , the negative of the determinant of the quadratic form. The projectivization of the subspace of the positive-definite forms is the hyperbolic plane  $H^2$ ; the degenerate forms comprise the circle at infinity. In the modern literature, this approach to hyperbolic geometry was developed in [6].

In the coordinates  $(u, v, w)$ , such that

$$a = u + v, \quad b = w, \quad c = u - v,$$

one has the standard Minkowski metric  $v^2 + w^2 - u^2$ . The unit-determinant quadratic forms comprise the hyperboloid of two sheets, and the condition  $a + c > 0$  describes its upper half, the pseudo-sphere.

A “unit” central ellipse of area  $\pi$  is an  $\mathrm{SL}_2(\mathbb{R})$  image of the unit circle, given by an equation of the form  $ax^2 + 2bxy + cy^2 = 1$  with  $ac - b^2 = 1$  and  $a + c > 0$ . This defines a point of the hyperbolic plane  $H^2$  in the pseudo-sphere model.

Likewise, a central hyperbola, which is an  $\mathrm{SL}_2(\mathbb{R})$  image of the “unit” hyperbola  $xy = 1$ , is given by an equation of the form  $ax^2 + 2bxy + cy^2 = 1$  with  $ac - b^2 = -1$ . It defines a point of the hyperboloid of one sheet.

**Lemma A.1.** *Let a unit central ellipse  $ax^2 + 2bxy + cy^2 = 1$  and a unit central hyperbola  $a'x^2 + 2b'xy + c'y^2 = 1$  be tangent at point  $(x, y)$ . Then the vectors  $(a, b, c)$  and  $(a', b', c')$  are orthogonal.*

*Proof.* The group  $\mathrm{SL}_2(\mathbb{R})$  acts transitively on the space of contact elements of the punctured plane whose line does not pass through the origin. And it acts by isometries on the space of quadratic forms. Therefore it suffices to consider the point  $(1, 0)$  and the vertical direction. In this case the two conics are  $x^2 + y^2 = 1$  and  $x^2 - y^2 = 1$ , and the vectors  $(1, 0, 1)$  and  $(1, 0, -1)$  are indeed orthogonal.  $\square$

To a point  $(x, y)$  of the punctured plane there corresponds the affine plane  $ax^2 + 2bxy + cy^2 = 1$  in the 3-dimensional space of quadratic forms. The normal vector of this plane is isotropic, and this plane lies above the origin. Hence its intersection with the pseudo-sphere is a horocycle in  $H^2$ . The symmetric point  $(-x, -y)$  yields the same horocycle.

To summarize, a point of the centoraffine plane is a horocycle in  $H^2$ , and a unit central ellipse is a point of  $H^2$ .

Let  $\gamma(t)$  be a centoraffine curve. The *osculating ellipse* at a point  $(x, y) = \gamma(t)$  is a unit central ellipse tangent to  $\gamma$  at this point. As  $t$  varies, one obtains a curve  $\gamma^*(t) \subset H^2$ , the dual curve of  $\gamma$ . Due to the central symmetry of  $\gamma$ , this curve closes up after  $t$  is increased by  $\pi$ . Equivalently, the curve  $\gamma^*$  is the envelope of the horocycles corresponding to the points of the curve  $\gamma$ .

**Lemma A.2.** *If  $[\gamma(t), \gamma'(t)] = 1$ , then  $|\gamma^*(t)'| = |1 + p(t)|$ .*

*Proof.* Let  $\gamma(t) = (x(t), y(t))$ . Then  $xy' - x'y = 1$ .

The osculating ellipse at a point  $(x, y)$  satisfies the equations

$$ax^2 + 2bxy + cy^2 = 1, \quad (ax + by, bx + cy) \cdot (x', y') = 0.$$

Taking  $ac - b^2 = 1$  into account, one solves these equations to obtain

$$a = y^2 + y'^2, \quad b = -(xy + x'y'), \quad c = x^2 + x'^2.$$

This is the equation of  $\gamma^*$ .

Next,  $x'' = px, y'' = py$ . Then

$$(\gamma^*)' = (1 + p)(2yy', -(x'y + xy'), 2xx'),$$

and  $|(\gamma^*)'| = |1 + p|$ , as claimed.  $\square$

Let  $k$  be curvature of the curve  $\gamma^*$ .

**Lemma A.3.** *One has*

$$k = \frac{1-p}{1+p} \quad \text{or} \quad (1+p)(1+k) = 2.$$

For example, when  $\gamma$  is a unit central ellipse with  $p = -1$ , the dual curve is a point, and the formula accordingly gives  $k = \infty$ . If  $\gamma$  is a unit central hyperbola with  $p = 1$ , then the formula gives  $k = 0$ . Indeed, Lemma A.1 implies that  $\gamma^*$  is a straight line, the intersection of the pseudo-sphere with the 2-dimensional subspace orthogonal to the vector corresponding to this hyperbola.

*Proof.* Let  $\tau$  be the arc length parameter on  $\gamma^*$ . Then  $dt/d\tau = 1/(1+p)$ .

The curvature is the magnitude of the projection of the vector  $d^2\gamma^*/d\tau^2$  on the pseudosphere. If  $u$  is a position vector of a point of the pseudo-sphere and  $v$  is a vector with foot point  $u$ , then the projection of  $u$  is given by  $u + (u \cdot v)v$ .

From the previous lemma, we know that

$$\frac{d\gamma^*}{d\tau} = (2yy', -(x'y + xy'), 2xx'),$$

hence

$$\frac{d^2\gamma^*}{d\tau^2} = \frac{1}{1+p} (2yy', -(x'y + xy'), 2xx')' = \frac{2}{1+p} (py^2 + y'^2, -pxy - x'y', px^2 + x'^2).$$

Next,

$$\frac{d\gamma^*}{d\tau} \cdot \gamma^* = 0 \Rightarrow \frac{d^2\gamma^*}{d\tau^2} \cdot \gamma^* + \frac{d\gamma^*}{d\tau} \cdot \frac{d\gamma^*}{d\tau} = 0 \Rightarrow \frac{d^2\gamma^*}{d\tau^2} \cdot \gamma^* = -1,$$

therefore the projection of  $d^2\gamma^*/d\tau^2$  on the pseudosphere is

$$\begin{aligned} \frac{d^2\gamma^*}{d\tau^2} - \gamma^* &= \frac{2}{1+p} (py^2 + y'^2, -pxy - x'y', px^2 + x'^2) - \\ &(y^2 + y'^2, -(xy + x'y'), x^2 + x'^2) = \frac{1-p}{1+p} (y'^2 - y^2, xy - x'y', x'^2 - x^2), \end{aligned}$$

and it remains to notice that the vector in the parentheses is unit.  $\square$

**Remark A.4.** According to a theorem of E. Ghys, see [38], the potential  $p(t)$  of the curve  $\gamma$  assumes the value -1 at least four times on the period  $[0, \pi)$ . It follows that the curve  $\gamma^*$  has at least four cusps; in particular, it cannot be smooth.

## B Weierstrass elliptic functions

These are meromorphic functions  $\wp, \zeta, \sigma : \mathbb{C} \rightarrow \mathbb{CP}^1$ , defined for each rank 2 lattice  $\Lambda = \mathbb{Z}2\omega + \mathbb{Z}2\omega'$ , where  $\omega, \omega' \in \mathbb{C}^*$ ,  $\omega'/\omega \notin \mathbb{R}$ . Define also  $\Lambda' = \Lambda \setminus 0$ .

**Alternative (useful) notation :**  $\omega_1 := \omega$ ,  $\omega_2 := -(\omega + \omega')$ ,  $\omega_3 = \omega'$ , so  $\sum \omega_i = 0$ .

## B.1 The $\wp$ -function

**Definition:**

- Infinite sum

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda'} \left[ \frac{1}{(z + \lambda)^2} - \frac{1}{\lambda^2} \right].$$

- ODE:

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) = 4\wp^3 - g_2\wp - g_3,$$

$$\text{so } e_1 + e_2 + e_3 = 0.$$

- Integral formula. Let  $\Sigma \subset \mathbb{CP}^2$  be the Riemann surface given in affine coordinates  $(x : y : 1)$  by  $y^2 = 4(x - e_1)(x - e_2)(x - e_3)$ . Then  $z \mapsto (\wp(z), \wp'(z))$  defines a biholomorphism  $\mathbb{C}/\Lambda \simeq \Sigma$ . The inverse  $\Sigma \rightarrow \mathbb{C}/\Lambda$  is given by

$$(x, y) \mapsto \int_x^\infty \frac{dx}{y} \pmod{\Lambda}.$$

The integral does not depend, mod  $\Lambda$ , on the integration path.

**Properties:** meromorphic, even,  $\Lambda$ -periodic, defining a double cover  $\mathbb{C}/\Gamma \rightarrow \mathbb{CP}^1$ , branched over 4 pts,

$$\wp(0) = \infty, \quad \wp(\omega) = e_1, \quad \wp(\omega + \omega') = e_2, \quad \wp(\omega'') = e_3.$$

Alternatively,  $\wp(\omega_i) = e_i$ ,  $i = 1, 2, 3$ ,  $\sum e_i = 0$ .

At these 4 branch points,  $\wp' = 0$ . In particular, the poles of  $\wp$  occur at  $\Lambda$  and are of order 2.

## B.2 The $\zeta$ function

**Definition:**

- Infinite sum:

$$\zeta(z) := \frac{1}{z} + \sum_{\lambda \in \Lambda'} \left[ \frac{1}{z + \lambda} - \frac{1}{\lambda} + \frac{z}{\lambda^2} \right].$$

- ODE:

$$\zeta'(z) = -\wp(z), \quad \zeta = \frac{1}{z} + \text{holomorphic function, near } z = 0.$$

- Integral formula:

$$\zeta(z) = \frac{1}{z} - \int_0^z \left( \wp(u) - \frac{1}{u^2} \right) du.$$

**Properties:** odd, meromorphic, simple poles at  $\Lambda$ ,  $\Lambda$ -quasi-periodic ([3, p. 35]):

$$\zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i, \text{ where } \eta_i := \zeta(\omega_i), \quad i = 1, 2, 3.$$

**Important relation:**

$$\eta\omega' - \eta'\omega = \frac{i\pi}{2} \quad \text{if } \text{Im}(\omega'/\omega) > 0.$$

### B.3 The $\sigma$ function

**Definition:**

- Infinite product

$$\sigma(z) = z \prod_{\lambda \in \Lambda'} \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}\right).$$

- ODE

$$\frac{\sigma'}{\sigma} = \zeta.$$

**Properties:** entire, quasi periodic [3, p. 37]:

$$\sigma(z + 2\omega_i) = -e^{2\eta_i(z + \omega_i)}\sigma(z),$$

where  $\omega_1 = \omega$ ,  $\omega_2 = -(\omega + \omega')$ ,  $\omega_3 = \omega'$ ,  $\eta_i = \zeta(\omega_i)$ , so that  $\sum \omega_i = \sum \eta_i = 0$ .

### B.4 Addition formulas

Express the relations between  $\wp, \zeta, \sigma$  at  $u \pm v, u, v$  [3, p. 271]:

$$\begin{aligned} \wp(u) - \wp(v) &= -\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)}, \\ \wp(u+v) + \wp(u) + \wp(v) &= \frac{1}{4} \left[ \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2, \\ \zeta(u+v) - \zeta(u) - \zeta(v) &= \frac{1}{2} \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}. \end{aligned}$$

### B.5 Reality condition

See [3, page 104]. If  $g_2, g_3 \in \mathbb{R}$  then  $4x^3 - g_2x - g_3 = 0$  has at least 1 real root. We want  $\wp$  to be oscillating, that is, bounded, so we better have 3 real roots (in case of multiple roots  $\wp$  is not doubly periodic, that is, not elliptic). In this case  $e_1 > e_2 > e_3$ ,  $\omega \in \mathbb{R}$ ,  $\omega' \in i\mathbb{R}$ . This is proved by showing

$$\omega = \int_{e_1}^{\infty} \frac{dx}{y}, \quad \omega' = i \int_{-e_3}^{\infty} \frac{dx}{y}, \quad y^2 = 4x^3 - g_2x - g_3 = 4 \prod (x - e_i).$$



Also,  $\wp$  maps

$$(\infty, \omega') \mapsto (\infty, e_3), [\omega', \omega + \omega'] \mapsto [e_3, e_2], [\omega + \omega', \omega] \mapsto [e_2, e_1].$$

See Figure 22. So  $t \mapsto \wp(\omega' + t)$  describes a particle bouncing back and fourth along  $[e_3, e_2]$ .

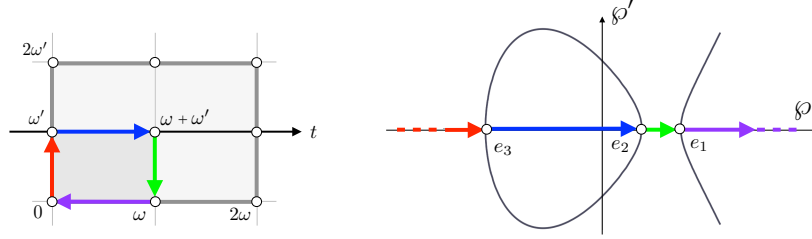


Figure 22: The Weierstrass  $\wp$  function with real invariants  $g_2, g_3$  and 3 real roots  $e_i$ : its fundamental rectangle (left) and the phase plane of  $(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$  (right). It maps  $(0, \omega', \omega + \omega', \omega) \mapsto (\infty, e_3, e_2, e_1)$ , and the horizontal axis  $\{\omega' + t \mid t \in \mathbb{R}\}$ ,  $2\omega$ -periodically, onto the segment  $[e_3, e_2]$ .

## B.6 The Lamé equation

This has the form  $X'' = (A\wp(z) + B)X$  for some constants  $A, B$ . When  $A = n(n + 1)$  all solutions are meromorphic [3, p.184]. By a theorem of Picard [3, Equation (6), p.182-3], there is then a basis of solutions which are  $\Lambda$ -quasiperiodic (classically, “doubly periodic of the 2nd kind”). That is,  $X(z + 2\omega) = \mu X(z)$ ,  $X(z + 2\omega') = \mu' X(z)$ . In our case,  $n = 1$ :

$$X'' = (2\wp(z) + B)X,$$

and such a basis is

$$X_{\pm}(z) = e^{-z\zeta(\pm a)} \frac{\sigma(z \pm a)}{\sigma(z)}, \quad \wp(a) = B.$$

These two solutions are linearly independent if  $B \neq e_i$ . The associated multipliers are

$$\mu_{\pm} = e^{\pm 2[a\eta - \omega\zeta(a)]}, \quad \mu'_{\pm} = e^{\pm 2[a\eta' - \omega'\zeta(a)]},$$

where  $\wp(a) = B$ ,  $\eta = \zeta(\omega)$ ,  $\eta' = \zeta(\omega')$ .

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