1. Overview

We look at complex line bundles from the topological viewpoint, and then from the holomorphic viewpoint. Tensor product defines a group structure on the space of line bundles over a space X, modulo equivalence of line bundles. For topological line bundles we show this group is isomorphic to the 2nd cohomology group $H^2(X, \mathbb{Z})$ of the base space X. The isomorphism is the first Chern class of the line bundle. For holomorphic line bundles , we will show that this group is isomorphic to the Picard group of divisors modulo principal divisors. Forgetting the holomorphic structure defines a homorphism from the Picard variety to $H^2(X, \mathbb{Z})$. For Riemann surface this map corresponds to the degree of the divisor. The map is onto and its kernel (or fiber) is the Jacobian. The Jacobian can be viewed as the space of flat holomorphic line bundles over X which in turn is isomorphic to the character group of $H_2(X, \mathbb{Z})$.

Topological Theory. The main result in the topological theory is that

$$\text{(topological line bundles)/topological isomorphism} \cong H^2(X, \mathbb{Z}). \tag{1.1}$$

The isomorphism takes a line bundle L to its (1st) Chern class $c_1(L) \in H^2(X, \mathbb{Z})$. One can find the statement and the proof in Chern's book "Complex Manifolds without Potential Theory", p. 33-34. There are at least four ways to understand the Chern isomorphism: by way of sheaf theory, by way of obstruction theory, by way of homotopy theory and by way of differential geometry and curvature. The sheaf perspective flows naturally right from the definitions. The obstruction theory perspective is the most pictorial and visceral perspective on the Chern class. The homotopy theory approach is often the most direct for computations and global understanding. The differential geometric approach connects to analysis, and physics and is the way I first learned about Chern classes. The differential geometry approach is also the road by which Chern discovered these classes. In this approach the Chern class is the deRham class represented by the curvature of a connection on the line bundle.

Holomorphic Theory. In algebraic geometry holomorphic line bundles and divisors are almost the same thing. Recall

$$Pic(X) = \text{(divisors)/(principal divisors)}$$

We will show that also

$$Pic(X) \cong \text{(holomorphic line bundles)/(holomorphic equivalence)} . \tag{1.2}$$

The structure of this equivalence is easy to see from one direction. Given a holomorphic line bundle, take any meromorphic section. This section may have zeros and poles. Take the associated divisor: the sum of the section's zeros and poles.

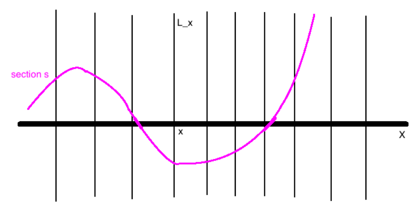
A holomorphic line bundle is de facto a topological line bundle, so we have a map

$$Pic(X) \to H^2(X, \mathbb{Z})$$

For Riemann surfaces this map is onto and represents the degree of the divisor D.

Flat line bundles and the Jacobian.

$Jac(X) = Pic_0(X)$ can be thought of as either the space of degree 0 divisors modulo principal divisors, or as (holomorphic line bundles which are topologically trivial)/(equivalence). Important here is that topological triviality does not imply



(a) A line bundle with a section

holomorphic triviality. We can also think of the Jacobian in terms of flat line bundles. (Atiyah on Topological Quantum Field Theory.)

We present the notion of a principal G-bundle, and of a flat G-bundle. We see how circle bundles correspond to line bundles. Flat G-bundles over X modulo equivalence correspond bijectively to representations of $\pi_1(X)$ into G modulo G-conjugation. For $G = S^1$ we have that the flat line bundles are $Hom(H_1(X, \mathbb{Z}), S^1) = H^1(X, S^1) = H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$ which is a $2g$ dimensional torus.... the Jacobian.

2. Line bundles, Informally, with examples.

Here are the informal definitions. A complex line bundle L over a space X is a collection $L_x, x \in X$ of complex lines which vary continuously with x. A holomorphic line bundle is a complex line bundle over a complex manifold X for which the lines L_x vary holomorphically with x. A section of a line bundle is a continuous map $s : X \to L$ which assigns to each $x \in X$ a vector $s(x) \in L_x$. The section s is holomorphic if as a map s is holomorphic.

We go right to the motivating examples.

Example 2.1 (Tautological line bundle)**.** *A point of $\mathbb{CP}^1$ is a one-dimensional complex linear subspace of $\mathbb{C}^2$. Attach to that point, the line it stands for, viewed as a one-dimensional complex linear space. In this way we get a family of complex lines depending smoothly on the points of $\mathbb{CP}^1$. This line bundle is called the "tautological line bundle" over $\mathbb{CP}^1$. Its points consist of pairs $(\ell, v) \in \mathbb{CP}^1 \times \mathbb{C}^2$ such that $v \in \ell$. Thus, if $v \neq 0$, then $\ell = [v] = span(v)$.*

Exercise 2.2. *Describe the tautological line bundle as a subvariety of $\mathbb{CP}^1 \times \mathbb{C}^2$ by finding a polynomial equation in 4 variables which defines it here.*

Example 2.3. *The dual of the tautological line bundle, denoted γ^* is the bundle whose fiber γ^*_p consists of complex linear functions $\gamma_x \to \mathbb{C}$.*

Exercise 2.4. *Describe the tautological line bundle as a subquotient of $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^2$. Subquotient means: submanifold (constraint) and quotient (by some group operation).*

Exercise 2.5. *Define the tautological line bundle over $\mathbb{CP}^n$.*

Terminology. The tautological line bundle is sometimes also called the canonical line bundle or the universal line bundle.

Example 2.6. *The tangent bundle TX of a Riemann surface X is a complex line bundle.*

Exercise 2.7. *On any complex line bundle L, and for any $\lambda \in \mathbb{C}$ we have the operation of scalar multiplication by λ, acting on each L_x by scalar multiplication by λ. How do you multiply by i on the tangent bundle of a Riemann surface ? By $e^{i\pi/4}$?*

Hint: Write out $v = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}$, $v \in T_x X$. Suppose that $z = x + iy$ is a holomorphic coordinate.

Example 2.8. *The dual of the tangent bundle is the cotangent bundle.*

Example 2.9. *The bundle whose sections are $(1,0)$ forms is another complex line bundle, isomorphic to T^*X.*

Example 2.10. *The trivial bundle is the bundle $X \times \mathbb{C} \to X$.*

A section of a line bundle is a continuous map $s : X \to L$ such that $s(x) \in L_x$ for each $x \in X$. If L is a holomorphic line bundle, then it makes sense to talk about holomorphic sections. We say that s vanishes at x, or that x is a zero of s if $s(x) = 0_x \in L_x$.

Proposition 2.11. *If $L \to X$ admits a nowhere vanishing section then L is isomorphic to the trivial bundle.*

Proof. Let s be the non-vanishing section. Then $s(x)$ is a basis for each L_x. The map $(x, \lambda) \mapsto \lambda s(x)$ defines an isomorphism $X \times \mathbb{C} \to L$; which is linear on each fiber and covers the identity on X.

Exercise 2.12. *We have used the notion of an isomorphism between line bundles. Give an informal definition of an isomorphism between line bundles. Give an informal definition of a map between line bundles.*

Exercise 2.13. *Show that the tautological line bundle is not trivial.*

3. Group structure on line bundles.

The tensor product over $\mathbb{C}$ of one-dimensional complex vector spaces is again a complex one-dimensional vector space. Thus if $L, E \to X$ are complex line bundles over X, so is $L \otimes E \to X$ where $(L \otimes E)_x = L_x \otimes E_x$. Tensor product gives the space of continuous isomorphism classes of line bundles the structure of an Abelian group. The identity is the trivial bundle

$$\text{Identity:} \qquad \epsilon = X \times \mathbb{C},$$

What is L^{-1}?

Proposition 3.1. *$L \otimes L^* \cong \epsilon$ as line bundles, so that under tensor product as multiplication we have $L^{-1} = L^*$.*

Proof. $L \otimes L^* \cong \epsilon = Hom(L, L)$ can be identifies with the line bundle whose fiber over x is linear homorphisms $L_x \to L_x$. The identity is a global section. Use prop 2.11.

We will give several proofs of the following fundamental theorems, which we will refer to as

Theorem 3.2. *[The Test Case] The group of continuous isomorphism classes of line bundles over $\mathbb{CP}^1$ is isomorphic to the group $\mathbb{Z}$ of integers. As generator '1 we can take either the tautological line bundle γ or its inverse γ^*.*

Exercise 3.3. *Show that $\gamma^{-2} = \gamma^* \otimes \gamma^* \cong T\mathbb{CP}^1$.*

Theorem 3.4. *The isomorphism classes of (topological) complex line bundles over a manifold X is isomorphic to the Abelian group $H^2(X, \mathbb{Z})$. (See eq 1.1 above).*

3.1. **Formal theory. Line bundles a cocycles.** In this subsection take X to be a topological space.

Definition 3.1. A line bundle L over X is a topological space L endowed with a continuous surjection $\pi : L \to X$ such that

- (i)for each $x \in X, \pi^{-1}(x) := L_x$ is homeomorphic to the line $\mathbb{C}$
- (ii) [local triviality], X admits a cover by open sets U_α such that $\pi^{-1}(U_\alpha)$ is homeomorphic to $U_\alpha \times \mathbb{C}$ by a homeomorphism $h_\alpha : U_\alpha \times \mathbb{C} \to \pi^{-1}(U_\alpha)$:

$$\begin{array}{ccc} U_\alpha \times \mathbb{C} & \xrightarrow{h_\alpha} & \pi^{-1}(U_\alpha) \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & U_\alpha & \end{array}$$

  which commutes with the projections to U_α.

Over the non-trivial overlaps $U_\alpha \cap U_\beta$ the local trivializations h_α, h_β conspire to yield a map $h_\beta^{-1} \circ h_\alpha : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}$. Item (ii) of the defintion of a line bundle implies that this map must have the form $h_\beta^{-1} \circ h_\alpha(x, z) = (x, \psi_{\alpha\beta}(x, z)$ where

$$\psi_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to \mathbb{C}.$$

Definition 3.2 (Part 2)**.** . The functions $\psi_{\alpha\beta}(x, z)$ just described have the particular form

$$\psi_{\alpha\beta}(x, z) = g_{\alpha\beta}(x) z$$

for some continuous map

$$g_{\alpha\beta} : (U_\beta \cap U_\alpha) \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}.$$

The collection of $g_{\alpha\beta}$ are called the "transition functions" or "clutching functions" associated to the line bundle and the particular collection $\{U_\alpha, h_\alpha\}$ of trivializations.

Remark: This part of the definition seems like the most technical. But it is exactly the part that makes all of the L_x have the structure of a vector space. Please compare it to the coordinate overlap condition in the definition of a manifold.

3.1.1. *Equivalence of sections and trivializations.* A local section of the line bundle $\pi : L \to X$ is a continuous map $s_\alpha : U_\alpha \to L$ such that $\pi \circ s_\alpha$ is the identity on U_α. In other words, $s_\alpha(x) \in L_x$ for each $x \in U_\alpha$.

PICTURE.

A local section which vanishes nowhere defines a local trivialization by the rule

$$h_\alpha(x, z) = z s_\alpha(x).$$

This equality is saying is that the s_α give us a continuously varying choice of basis element, or "1" for the $L_x \cong \mathbb{C}$, $x \in U_\alpha$.

Conversely, a local trivalization defines a local section $s_\alpha(x) = h_\alpha(x, 1)$.

In terms of the associated local sections the transition overlap maps are defined by

$$s_\alpha(x) = g_{\alpha\beta}(x)s_\beta(x).$$

Exercise 3.5. *Verify that the above relation between non-vanishing local sections is equivalent to the original definition of the transition functions.*

Example 3.6 (Tautological line bundle, redux)**.** *Take for cover* $U_0 = \{[z_0, z_1] : z_1 \neq 0\}$ *and* $U_\infty = \{[z_0, z_1] : z_0 \neq 0\}$.

Then

$$s_0([z, 1]) = ([z, 1], (z, 1))$$

and

$$s_1([1, w]) = ([1, w], (1, w)).$$

define corresponding local sections. We see that the transition map is

$$g_{01}([z, 1]) = \frac{1}{z}$$

on the overlap $U_0 \cap U_\infty$.

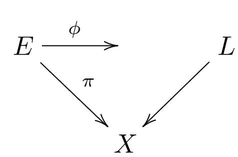
Example 3.7 (Cotangent bundle of a Riemann surface, redux)**.** *Let* z_α *be coordinates on the Riemann surface with corresponding neighborhoods* U_α. *Then* $s_\alpha = dz_\alpha$ *is a non-vanishing local section of* $T^{(1,0)}X$ *over* U_α. *Comparing two such, we get that the transition functions are just*

$$g_{\alpha\beta}(p) = \frac{dz_\alpha}{dz_\beta}(p),$$

the derivative of the coordinate overlap map, evaluated at points of $U_\alpha \cap U_\beta$.

Holomorphic line bundles; Smooth line bundles Insist that the transition functions $g_{\alpha\beta}$ be holomorphic, or be smooth, instead of merely continuous. Then we are speaking of the category of holomorphic line bundles, or of smooth line bundles. All the other maps, the sections s_α, the local trivializations h_α would then holomorphic, or be smooth. All the above examples were holomorphic line bundles.

3.2. **Maps between line bundles.** Suppose that $\pi_E : E \to X, \pi_L : L \to X$ are two complex line bundles, over the same space X. A bundle map $\phi : E \to L$ is a continuous map $E \to L$ which covers the identity on X :



and is linear on each fiber: $\phi_x : \phi|_{E_x} E_x \to L_x$ is a complex linear map.

Suppose we have locally trivialized E, L over the same collection of open sets $U_\alpha \subset X$. Let $s_\alpha, \tilde{s}_\alpha$ be the corresponding local non-vanishing sections. Then, in terms of local trivializations, ϕ is given by a collection $\phi_\alpha : U_\alpha \to \mathbb{C}$ of local trivializations via:

$$\phi(s_\alpha(x)) = \phi_\alpha(x)\tilde{s}_\alpha(x).$$

And one has

$$\phi_\beta = \tilde{g}_{\beta\alpha}\phi_\alpha g_{\alpha\beta} \tag{3.1}$$

where $g_{\alpha\beta}, \tilde{g}_{\alpha\beta}$ are the corresponding transition functions.

3.3. **Triviality.**

Definition 3.3. A line bundle L is trivial if there is an invertible bundle map between L and the trivial bundle.

A line bundle is holomorphically trivial if there is an invertible holomorphic bundle map $L \to \epsilon$.

Exercise. A bundle is trivial if and only if it admits a global nonvanishing section.

A bundle is holomorphically trivial if and only if it admits a global non-vanishing *holomorphic section*.

Exercise. A bundle is trivial if and only if it admits a family of local trivializations such that the corresponding transition functions $g_{\alpha\beta}$ satisfy:

$$g_{\alpha\beta}(x) = \phi_\alpha(x)\phi_\beta^{-1}(x) \tag{3.2}$$

for some collection of continuos maps

$$\phi_\alpha : U_\alpha \to \mathbb{C} \setminus \{0\}.$$

Exercise. The tautological line bundle is not trivial.

3.4. **Equivalance.**

As a slight variant of equations (3.2), (3.1) we observe that two collections of transition functions define isomorphic line bundles if

$$\tilde{g}_{\alpha\beta} = \phi_\alpha g_{\alpha\beta}\phi_\beta^{-1}$$

for some collection of smooth maps $\phi_\alpha : U_\alpha \to \mathbb{C}^*$.

3.5. **Group structure in terms of transition functions.**

In our informal discussion we described how the space of line bundles modulo equivalence forms a group under tensor product. The inverse of a line bundle was its dual.

Exercise 3.8. *Suppose that $g_{\alpha\beta}$ are the transition functions for L. Verify that the transition functions for L^* are $h_{\alpha\beta} = g_{\alpha\beta}^{-1}$.*

Suppose that $\tilde{g}_{\alpha\beta}$ are the transition functions for another line bundle E. Verify that $h_{\alpha\beta} = \tilde{g}_{\alpha\beta}/g_{\alpha\beta}$ are the transition functions for the line bundle $E \otimes L^* = Hom(L, E)$.

3.6. **Cocycle conditions.**

Observe that from the definitions we have that

$$g_{\beta\alpha} = 1/g_{\alpha\beta}$$

and that:

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$$

is the constant function 1 on any triple overlap $U_\alpha \cap U_\beta \cap U_\gamma$. These two conditions are called "cocycle conditions".

Conversely,suppose we are given a cover $U\alpha$ of X and collection of continuous functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, on the overlaps which satisfying this cocycle condition. We can define a complex line bundle over X by forming the disjoint union of the $U_\alpha \times \mathbb{C}$ and gluing by using the equivalence relation that declares that $(x, z_\alpha) \sim (x, z_\beta)$ if and only if $g_{\alpha\beta}(x)z_\beta = z_\alpha$.

The trivial cocycle is one defined by $g_{\alpha\beta} = h_\alpha h_\beta^{-1}$ for some collection of functions $h_\alpha : U_\alpha \to \mathbb{C}^*$. We have seen that two cocycles represent equivalent line bundles if and only if their 'difference' $g_{\alpha\beta}\tilde{g}_{\alpha\beta}^{-1}$ is a trivial cocycle.

We formalize and codify this with the language of sheaves and sheaf cohomology. Write $\mathcal{C}^*$ for the sheaf of continuous functions on X with values in $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Thus $\mathcal{C}^*$ assigns to each open set $U \subset X$ the group of all continuous functions $g : U \to \mathbb{C}^*$. The group multiplication of two such functions is pointwise multiplication.

Proposition 3.9. *The group of line bundles modulo continuous equivalence is isomorphic, to the 1st sheaf cohomology group $H^1(\mathcal{C}^*)$ where $\mathcal{C}^*$ is the sheaf of continuous functions on X with values in the multiplicative group $\mathbb{C}^*$.*

and we have the holomorphic version of this theorem:

Proposition 3.10. *The group of holomorphic line bundles modulo holomorphic equivalence (over a complex manifold X) is isomorphic, to the 1st sheaf cohomology group $H^1(\mathcal{O}^*)$ where $\mathcal{O}^*$ is the sheaf of nowhere vanishing holomorphic functions on X – that is to say: holomorphic functions taking values in the multiplicative group $\mathbb{C}^*$.*

3.7. **Sheaves, cocycles, sheaf cohomology.** Def. Presheaf. Sheaf.

Def. Cocycle for an index set...; The δ map.

Def. Sheaf cohomology.

Short exact sequence of sheaves leads to long exact sequence of sheaf cohomologies.

The sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C} \xrightarrow{e^{2\pi i()}} \mathcal{C}^* \longrightarrow 0$$

(Cf. p. 139 Griffiths Harris.)

Will show: $H^1(\mathcal{C}) = H^2(\mathcal{C}) = 0$.

Cor: $H^1(\mathcal{C}^*) = H^2(X; \mathbb{Z})$.

This is theorem 3.4. The connecting homomorphism δ here has for its image the Chern class of the corresponding line bundle. x

Difference between sheaves of analytic functions vs. smooth functions.

Fine sheaves: partitions of unity. Big theorem for fine sheaves $H^1 = 0 = H^2 = \ldots$.

Computations: good cover.

Test case: Computation of $H^2(\mathbb{CP}^1, \mathbb{Z})$ using a good (tetrahedron-induced) cover of $S^2 = \mathbb{CP}^1$.

4. Divisors as holomorphic line bundles.

If L is a holomorphic line bundle over a Riemann surface X and $s : X \to L$ is a meromorphic section of L, we have the corresponding divisor $(s) = (s)_0 - (s)_\infty$. Any other meromorphic section s_1 can be written as $s_1 = fs$ for some meromorphic function f. Thus $(s_1) = (s) + (f)$ differs from (s) by a principal divisor. It follows that this construction leads to a well defined map from the line bundle to the Picard variety of divisors modulo principal divisors.

Theorem 4.1. *The map associating a holomorphic line bundle to its divisor is an isomorphism between holomorphic equivalence classes of line bundles and the Picard divisor.*

Cf. theorem **??** above.

To construct the inverse map, from the Picard variety to the space of holomorphic isomorphism classes of line bundles, we need to see how to build a line bundle out of a divisor.

References: Donaldson, p. Miranda, p. 347. Griffiths-Harris p. 133-134.

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4.1. **The line bundle of a point.** .

Start with the simplest nontrivial case of $D = p$, the divisor of a single point. Cover X with two charts, $U_0 = X \setminus \{p\}$ and U_1 a coordinate neighborhood centered at p. Let $f = z$ be any coordinate vanishing at p. Set

$$g_{01} = \frac{1}{z}, \qquad g_{10} = z$$

thus defining a line bundle L_p. Over U_0 we have the local section s_0 represented by 1. Since

$$s_1 = g_{10} s_0$$

we see that this same local section is given by z in U_1. Being defined everywhere, it is a global section vanishing exactly at p, with the order of vanishing being 1.

Example: The dual of the tautological bundle corresponds to the line bundle associated to the point at infinity.

4.2. **The line bundle of a general divisor.** . Now, let $D = \Sigma n_i p_i$ be a general divisor. Cover X by charts U_0, U_i so that $p_i \notin U_0$ and the U_i are coordinate charts with coordinates z_i centered at p_i. Set

$$g_{i0} = z_i^{n_i}, g_{ij} = z_i^{n_i}/z_j^{n_j}, g_{0i} = \frac{1}{z_i^{n_i}}$$

Verify that the section corresponding to the local trivialization over U_0 – the section represented by 1 over U_0 – extends uniquely to a global meromorphic section with zeros at the p_i of order n_i if $n_i > 0$, and poles at the p_i of order $|n_i|$ when $n_i < 0$.

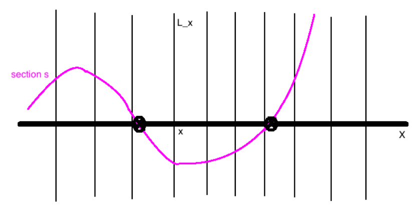
These bundles just constructed are holomorphic: The transition overlaps are all holomorphic. When we piece them together to build our global complex 2-manifold L, that manifold is complex holomorphic. The projection map $L \to X$ is holomorphic. The section built is meromorphic. We are in the category of holomorphic line bundles!

5. Obstruction theory.

Above, we took the inverse image of the zero section, minus the inverse image of infinty and made them into a divisor. We can perform these same operations in the continuous category. When viewed homologically we get the obstruction theoretic point of view on the Chern class. Perhaps the original theorem from this perspective is the Poincare Hopf theorem.

Theorem 5.1. *Let X be a compact oriented surface and $v : X \to TX$ a C^1 vector field on X with isolated zeros. Assign a degree to each zero p, as per vector fields in the plane. Then*

$$\chi(X) = \Sigma_{\{p \in X; v(p)=0\}} deg_p(v).$$



(a) The associated divisor is marked with Xs.

This theorem relates the Euler characteristic of X to a 'divisor' on X formed out of the zeros of the section v. It is a special case of

Theorem 5.2. *Let $L \to X$ be a complex line bundle over the compact oriented surface X. Let $s : X \to L$ be a section with isolated zeros. Then*

$$\langle c_1(L), [X]\rangle = \Sigma_{p\in X:s(p)=0} deg_p s.$$

How to define $deg_p s$? Choose any local trivialization over an open set U containing p, Relative to this trivialization s becomes a function $\tilde{s} : U \to \mathbb{C}$ having a zero at $p \in U$. Choose a norm on $\mathbb{C}$ so that $\tilde{s}/\|\tilde{s}\| : S^1_\epsilon \to S^1$ represents a map from the circle to itself. Here S^1_ϵ is a small circle surrounding p, Since X is oriented we may suppose that $U \cong V \subset \mathbb{R}^2$ is oriented and hence S^1_ϵ is oriented. The degree of s at p is the degree of this map from the circle to itself. One verifes that if we write s relative to a different trivialization, the degree does not change. (The transition map $g_{\alpha\beta}$ extends as a non-vanishing function across the entire disc centered at p bounding S^1_ϵ.

In case p is a transverse zero then $deg_p s$ is the sign of the determinant of $d\tilde{s}_p$. Verify that this is independent of the choice of local trivialization.

Exercise 5.3. *Formulate the obstruction theoretic formula for c_1 of a general line bundle L over a smooth n-dimensional manifold X.*

Hint: Take a section s. Perturb it to be transverse to the zero section. Represent $s^{-1}(0) = ker(s)$ as a cycle of codimension 2 on X. The Chern class paired with a 2-dimensional homology cycle , must yield an integer. How does this 2 dimensional cycle intersect with $ker(s)$?

Exercise 5.4. *Suppose that s is a meromorphic section of a holomorphic line bundle with a pole of order n at p. By choosing a norm near p and cutting off s to the neighborhood $s : \|s\| \le \frac{1}{\epsilon}$ show that we can find a section w which agrees with s off of a small neighborhood of p is continuous in this neighborhood of p and has a zero of degree $-n$ at the pole. (Use $1/z^n = \bar{z}^n/r^{2n}$.)*

Test case: the bundles over $\mathbb{CP}^1$....

6. Homotopy Theory.

6.1. **Pull-back bundles.** Suppose that $f : X \to Y$ is a continuous map and that $\pi_Y : L \to Y$ is a complex line bundle. We can use f to pullback the line bundle L

for yield a bundle $f^*L \to X$:

$$(f^*L)_x = L_{f(x)}$$

The associated commutative diagram is

$$\begin{array}{ccc} f^*L & \longrightarrow & L \\ \downarrow & & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y \end{array}$$

Set-theoretically

$$f^*L = \{(x, v) \in X \times L : f(x) = \pi_Y(v)\} \subset X \times L.$$

Lemma 6.1. *Let X be compact and $L \to X \times [0, 1]$ be a line bundle over X times the interval. Let $i_t : X \to X \times [0, 1]$ be the inclusions $i_t(x) = (x, t)$ for $0 \leq t \leq 1$. Then all the line bundles $i_t^*L \to X$ are isomorphic to one another. In particular $i_0^*L \cong i_1^*L$.*

Proof. My favorite proof involves choosing an Ehreshman connection for $X \times [0, 1] \to X$. Maybe see next section?

Corollary 6.2. *Suppose that $f \sim g : X \to Y$ are homotopic maps and that $L \to Y$ is a line bundle. Then $f^*L \cong g^*L$ as line bundles over X.*

Corollary 6.3. *If X is a contractible space and $L \to X$ is a line bundle then L is the trivial bundle.*

6.2. **Universal bundles.** Suppose we could find a space U, endowed with a line bundle $L_U \to U$ such that for every compact space X and every line bundle $L \to X$ there exists a map $f : X \to U$ such that $L \cong f^*L_U$. Then, according to corollary **??**, the theory of line bundles over X would be essentially the same as the theory of the space $[X, U]$ of homotopy classes of maps $X \to U$.

We call such a space a "universal space" for line bundles.

Proposition 6.4. $\mathbb{CP}^\infty$ *with its tautological line bundle is universal for line bundles over compact spaces.*

$\mathbb{CP}^1$ *with its tautological line bundle is universal for line bundles over compact surfaces.*

Sketch proof. Let X and L be given. Cover X by finitely many trivializations U_i, $i = 1, \ldots, N + 1$ for L with associated sections $s_i : U_i \to L$ which are non-vanishing over U_i. Let ρ_i be a partition of unity subordinate to the U_i. Then the sections $\rho_i s_i : X \to L$ are global sections. Put them together into one 'vector" of length $N + 1$: $\tilde{s}(x_=(\rho_1(x)s_1(x), \rho_2(x)s_2(x), \ldots \rho_{N+1}(x)s_{N+1}(x))$. To try to make sense of this as a vector in $\mathbb{C}^{N+1}$ take any local trivialization h_α whose base U_α contains the the point x. Expressing the $\rho_i(x)s_i(x)$ in this local trivialization we get a point of $\mathbb{C}^{N+1}$. The vector never vanishes since the ρ_i are a partition of unity and the s_i are non-vanishing. If we choose another trivialization h_β, then this this vector is multiplied the scalar function $g_{\alpha\beta}(x)$. Consequently $[\tilde{s}]$ is well defined as a point of $\mathbb{CP}^N$.

We leave it to the reader to show that $f^*\gamma \cong L$.

Now $\mathbb{CP}^\infty$ is the projective limit of the $\mathbb{CP}^N$ so we can view $[\tilde{s}]$ as a map into $\mathbb{CP}^\infty$ and we again get that $f^*\gamma = L$.

QED

with the property that for every x there is an i such that $s_i(x) \neq 0$ defines a map $X \to \mathbb{CP}^N$.

If X is a CW complex and Y is a compact k-dimensional manifold, then any map $X \to Y$ can be homotoped onto the k-skeleton of Y. Suppose that there are no cells in dimension $k+1$. Then two such maps are homotopic if and only if the corresponding retractions onto the k-skeleton are homotopic.

$\mathbb{CP}^\infty$ is a CW complex with only even dimensional cells. These cells fit together like $0 \subset \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \ldots$. It follows that if $dim(X) = 2$ then the homotopy classes of maps $[X, \mathbb{CP}^\infty]$ is isomorphic to the homotopy classes of maps $[X, \mathbb{CP}^1]$. Classical degree theory asserts that two maps $f, g : X \to \mathbb{CP}^1$ are homotopic if and only if they have the same degree $d \in \mathbb{Z}$.

From this we see that the degree of the inducing map $f \to \mathbb{CP}^1$ is a complete invariant of the topological type of the line bundles $L \to X$.

General case. $H^*(\mathbb{CP}^\infty)$ is the polynomial ring in a single generator u in degree 2. (This u is the class of the Kahler form which is also the Chern class of the dual of the universal line bundle over $\mathbb{CP}^\infty$.) The Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ from the previous section – the cocycle version – is related to u by

$$c_1(L) = f^*u \qquad ; L = f^*\gamma^*$$

7. Analytic theory

Connections.

Curvature. Curvature two-form.

Chern class as $\frac{1}{2\pi i}$ times deRham class of the curvature two-form

Independence of choices.

The 'Levi-Civita connection" of a holomorphic complex line bundles with a fiber Hermitian metric.

8. Higher dimensional divisors and their line bundles

Use meromorphic or rational functions f_i whose local vanishing on open sets (affine sets) U_i define divisors D_i of $D = \Sigma n_i D_i$. Thus: the support of D in U_i is defined by $f_i = 0$. If f_j defines the *same* divisor on U_j then f_i/f_j is nowhere vanishing on $U_i \cap U_j$. *Set*

$$g_{ij} = f_i \cap f_j : U_i \cap U_j \to \mathbb{C}^*$$

thus defining meromorphic transition functions for a line bundle.

If none of the D_a intersect U_0 we have the function 1 representing the divisor on U_0.

The same analysis as before now gives us a line bundle with a section (the section 1 on U_0 whose divisor is the given divisor D.

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Definition 8.1. A line bundle $L \to X$ over a Riemann surface is called a holomorphic line bundle if L is a complex manifold and if all the maps $\pi, h_\alpha, g_{\beta\alpha}$ in the definition of 'line bundle' are holomorphic functions