## 1. 1. Outline.

### 1.1. **What's a Riemann surface?** Recall manifolds.

**Definition 1.1.** *A Riemann surface is a 2-dimensional real manifold whose transition functions $\psi_{\alpha,\beta}$ are all restrictions of holomorphic maps $\mathbb{C} \to \mathbb{C}$.*

### 1.2. **Examples.**

1. $\mathbb{C}$
2. $\mathbb{C}P^1$. Also called $\hat{\mathbb{C}}, \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}, S^2$. Charts thereon. Covered by two open sets. Overlap map $z \mapsto 1/z$.
3. $\{(z,w) \subset \mathbb{C}^2 : w^2 = f(z)\}$ provided ... ! Uses holomorphic version of IFT. State it!

Viewing the RS of $w = \sqrt{z^2 - 1}$ by making branch cuts.

Of $z = \sqrt{(z-r_1)(z-r_2)\dots(z-r_n)}$

Generalizing in two directions. a) $F(z,w) = 0$ in $\mathbb{C}^2$ provided...

b) the Riemann surface of any analytic function germ $z \mapsto h(z)$

4. Gauss: any analytic oriented surface embedded in $\mathbb{R}^3$ [Riemannian surfaces]

Uses a) Existence of "isothermal coordinates" [ hard work! - Gauss]

b) Any conformal diffeo is holomorphic or anti-holomorphic [ easy - local coord computation]

5. Quotients of known Riemann surface by discrete groups "acting properly discontinuously": Eg flat torus as $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$.
6. Algebraic curves. These are curves in projective spaces. To get such a curve in $\mathbb{CP}^2$ choose a homogeneous polynomial in 3 variables. For example, the Fermat surfaces are $x^n + y^n + z^n = 0$. Relate to Fermat's last theorem. Multiply $z$ by $-z$ when $n$ is odd. Bring to other side. Look for rational solutions ...

IFTs and condition for curve to be smooth here ...

### 1.3. **Maps between.** A map between Riemann surfaces is a map whose expression in any local coordinates is holomorphic.

A biholomorphism is a holomorphic map which is invertible. This inverse is necessarily holomorphic.

Examples.

1. Look at $z \mapsto exp(z)$. Since $exp(z) = exp(z + 2\pi i k)$ we have that $exp(z)$ is a well defined map on the quotient $\mathbb{C}/(2\pi i\mathbb{Z})$. This quotient space is a cylinder. A fundamental domain consists of a horizontal strip bounded by two parallel lines separated by $2\pi$. Thus $exp$ maps the cylinder biholomorphically onto the punctured plane $\mathbb{C} \setminus \{0\}$.
2. Take the hyperelliptic surface above. The projection $(z,w) \mapsto z$ is a holomorphic map which is 2:1 at all points except over the branch points, these being the points where $p(z) = 0$. Near the branch points we can choose local coordinates on the surface so that the map looks lie $\xi \mapsto \xi^2$.
3. [Hard !] The Riemann mapping theorem. Any bounded simply connected domain in $\mathbb{C}$ is biholomorphic to the open unit disc $|z| < 1$.
4. [Moderately Hard]. The open disc is NOT biholomorphic to the entire complex plane $\mathbb{C}$.
5. [Elliptic curves]. More complicated example. The two-torus, realized as $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, forms a Riemann surface. Going up to the universal cover, we try

to make functions on it as holomorphic functios invariant under $z \to z+1$ and $z \to z+i$. There are none except constants. But there are meromorphic functions. These are the famous elliptic functions. They necessarily have poles within, or on the boundary of each unit square. The classic one (see Ahlfors) is the Weierstrauss P-function, written with curly Gothic letters: $\mathcal{P}(z)$. The mapping $(x,y) = \mathcal{P}, \mathcal{P}'$ take the torus onto an "elliptic curve" $y^2 = x^2 + g_2 x + g_3$ which completes into $\mathbb{C}P^2$ to a cubic curve. The coefficients $g_2, g_3$ are given by infinite sums over the Gaussian integers.

6. To map an algebraic curve $X \subset \mathbb{CP}^1$ onto $\mathbb{CP}^1$ choose any point $P \notin X$, $P \in \mathbb{CP}^1$. The set of all lines passing through $P$ forms a $\mathbb{CP}^1$. For each $x \in X$ map $x$ to the line $Px$.

Bezout's theorem asserts this map is $d:1$ where $d$ is the degree of the curve. The singular *values* of the map are lines tangent to the curve passing through $P$.

## 2. History ?

Complex variables beginning. Poncelet - projective space. Gauss. Riemann. Before them: Abel. Jacobi . Elliptic functions and elliptic curves. ..

## 3. Classification of compact Riemann surfaces.

I. Prequel. The Classification of compact oriented topological surfaces. Puncture a hole in the surface. The result collapses (homotopes) to the wedge of $2g$ circles. This number $g$ is the genus.

One can construct such a surface by adding $g$ handles to a sphere. Any two such surfaces having the same genus are homeomorphic. They are also diffeomorphic.

We thus have a complete classification of these surfaces. There is exactly one for each $g = 0, 1, 2, \ldots$.

To get the classification, Morse theory is a big help. See Donaldson's book.

There is a standard polygonal model of the genus $g$ surface which shows us its fundamental group. Mention combinatorial topology. Stillwell's book.

Draw this polygon for $g = 2$, $g = 1$, general $g$. ; write the standard presentation for the fundamental group. The fundamental group has $2g$ generators, written $A_i, B_i, i = 1, \ldots g$ with a single relation

$$C_1 C_2 \ldots C_g = 1 \text{ where } C_i = A_i B_i A_i^{-1} B_i^{-1}$$

To reiterate: Any two compact oriented surfaces of genus $g$ are diffeomorphic. However, they are not equivalent as RS's, assuming they admit RS structures.

2. To show that every genus $g$ surface *admits* a RS structure we can use the hyperelliptic model. Choose $N = 2g+2$ distinct points. For simplicity, place them on the real line in order: $x_1 < x_2 < x_3 < \ldots x_{2g}$. Form $p(z) = (z-x_1)(z-x_2)\ldots(z-x_N)$ and let $M$ be the RS of $\sqrt{p(z)}$ - compactifying at infinity as per the embedding into $\mathbb{C}P^2$. We compute by using the view of $M$ as a 2-sheeted cover over $\mathbb{S}^2 = \mathbb{CP}^1$ that $M$ is obtained by adding $N/2$ tubes corresponding to the $N/2$ cuts $[x_1, x_2], [x_3, x_4], \ldots, [x_{N-1}, x_N]$. Inflate each cut to a tube. We have $N/2$ tubes joining two spheres -the two sheets. One tube joining two $\mathbb{S}^2$ is diffeomorphic to $\mathbb{S}^2$. Each additional tube corresponds to adding a handle. So we have added $N/2 - 1 = g$ handles.

PICTURES !

2. **Uniformization theorem.**

The universal cover $\tilde{\Sigma}$ of ANY connected compact Riemann surface $\Sigma$ is itself a Riemann surface [easy]. The following is hard and deep.

**Theorem 3.1** (Uniformization). *Every simply connected RS is biholomorphic to one, and only one of the the following three models* $\mathbb{M}$

- *a)* $\mathbb{C}P^1$ *(genus 0)*
- *b)* $\mathbb{C}$ *(genus 1)*
- *c) the open unit disc* $\mathbb{D}$ *(genus 2 and greater)*

Another model for (c) is the Poincaré upper half plane $\mathcal{H}$: the open disc is biholomorphic to the upper half plane $\mathcal{H}$.

The biholomorphisms (invertible holomorphic maps) in all cases are Lie groups. They are

$PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ by fractional linear transformations. 6 dimensional

Addition and multiplication - so the “ax + b ” group of affine transformations. 4 dimensional

$PSL(2,\mathbb{R}) \subset PGL(2,\mathbb{C})$ acts on $\mathcal{H}$. 3 dimensional

The fundamental group $\Gamma = \pi_1(\Sigma)$ embeds into the group of holomorphic automorphisms of the model universal cover, acting in a nice fixed point free manner (“properly discontinously”) so that the quotient space $\tilde{/}\Gamma$ is a nice Riemann surface, biholomorphic to the original $\Sigma$.

3. [**Teichmuller space and moduli space.** ] How many RS’s of a given genus are there?

As far as real diffeomorphisms are concerned, there is only one. Call two RS’s equivalent (over $\mathbb{C}$) is there is holomorphic diffeomorphism between them. We can then talk about the ‘space’ of all RSs of genus $g$.

**Theorem 3.2** (A big deal). *The space of RSs of a given genus forms a connected complex algebraic variety of complex dimension* $3(g-1)$ *when* $g > 1$*. In particular, the dimension of htis space grows linearly with* $g$*.*

This space is called “moduli space” . It is the quotient of space diffeomorphic to $\mathbb{C}^{3g-3}$ by the action of a ‘big’ discrete group called the mapping class group.

The hyperelliptic ones form a subvariety. One can count dimension by using the locations of the N $= 2g+2$ roots as variables and then modding out by the action of the group of Mobius transformations. This count yields $N-3 = 2g-1$ hyperelliptic RSs. Since $3g$ grows faster than $g$ , for large $g$ we see that most RSs are NOT hyperelliptic. We get $3g-3 = 2g-1$ when $g = 2$. Beyond $g = 2$ we expect ‘most’ RSs are not hyperelliptic.

## 4. Genus and fundamental group.

Various characterizations of : $\chi(\Sigma) = 2-2g$ where $\chi = b_0 - b_1 + b_2 = V - E + G$. where $b_i = dim(H_i(M,\mathbb{R}))$.

$2g$ = # of circles to which $M$ deformation retracts onto upon having a disc deleted from $M$.

$b_1 = 2g$

$rank(H_1(\Sigma,\mathbb{Z}) = rank(H^1(\Sigma,\mathbb{Z})) = dim(H^1(\Sigma,\mathbb{R})) = 2g$.

$g$ = # handles added to a sphere to build surface.

$2g$ = # of simple curves you must remove in order to make the result connected and simply connected.

Two analytic characterizations of genus.

1. The genus $g$ is the (complex) dimension of the space of holomorphic differentials.

2. If I place any $g+1$ distinct points on $M$ there is a meromorphic function having these points as simple poles and no other poles.

The assertion that this number $g+1$ works is Donaldson, prop 26, p 113, combined with the duality between $H^{0,1}$ and $H^{1,0}$. Indeed, he shows any number $N > dim(H^{0,1})$ of points can be chosen as simple poles.

To make an invariant out of this 'pole placement number" we have to also say "we can't do better": Namely: it is impossible to build such a function having $g$ or fewer simple pole points, provided these points are chosen "generically" Indeed this is a theorem **provided $M$ is not hyperelliptic** ! See Miranda, Cor. 2.9, p 210. It is a beautiful theorem, using the canonical embedding $M \to \mathbb{CP}^{g-1}$ defined by the holomorphic differentials to define what it means for $g$ points to be 'generic': their image under the canonical embedding is generic in the linear algebra sense: these $g$ image points spans all of $\mathbb{CP}^{g-1}$.

For special positions of $g$ points it may be possible to do better. Indeed, the version of RR that Miranda calls the Geometric Form of RR tells precisely that you can, and by how much in terms of the dimension of the span.

NB. If $M$ is hyperelliptic just take a function like $z$ or $1/z$ with a simple pole on $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Pull it back by the map $M \to \mathbb{CP}^1$, assuming the pole is not a branch point, to get a meromorphic function on genus $g$ $M$ having exactly TWO simple poles. They are related by symmetry, since they project to the same point, so they are not 'generic". Indeed, a characterization of 'hyperelliptic' is that it admist a meromorphic function with exactly two simple poles.

More .. on 'doing better" or not. That usually you cannot do it with fewer must be the assertion that typically, $L(K-D) = 0$ if $D$ consists of $g$ generic points. I don't know how to do this, but it must work out. Also, it must be a theorem that a) $L(K-D) = 0$ if $D$ consists of $g+1$ distinct points . Here we are using RR : $L(D) = d + (1-g) + L(K-D) = 1 + L(K-D)$. and we want $L(D) = 2$, the constant functions and one other. See Cor in Miranda to work out details ...!

3. A metric view of uniformization. On the metric level, a la Gauss's isothermal coordinates, the uniformization theorem is as follows :

Every **Riemannian** surface (as opposed to Riemann surface) admits a metric in its same conformal class which has constant curvature $K$.

The simply connected models for the Riemannian surfaces of constant curvature COMPLETE riemannian manifolds are the SAME as above,

a) $K > 0$. $\Sigma = S^2(\sqrt{K}) = \mathbb{C}P^1 = \tilde{\Sigma}$ b) $K = 0$. $\tilde{\Sigma} = \mathbb{R}^2 = \mathbb{C}$ c) $K < 0$, $\tilde{\Sigma} = \mathbb{D}^2 = \mathcal{H}$, Poincare metric scaled.

The group of rigid motions (orientation preserving isometries) in all cases is a subgroup of the group of holomorphic automorphisms. Only in the last case are the two groups equal!.

The fundamental group of the compact Riemann surface embeds as a discrete subgroup of this group of rigid motions.

Depending on the genus,

a) $K > 0$. $\Sigma = S^2 = \mathbb{C}P^1 = \tilde{\Sigma}$ b) $K = 0$. $\Sigma = \mathbb{T}^2$, a flat torus, which is $\mathbb{R}^2/\mathbb{Z}^2$ c) $K < 0$. $\tilde{\Sigma} = D^2 = \mathcal{H}$.

The action of the fundamental group $\Gamma$ on $\tilde{\Sigma}$ is by isometries.

We can realize the quotient $\tilde{X}/\Gamma$ by using Dirichlet (or "Voronoi cells) to find a 'fundamental region ' for the action of $\Gamma$. So: $-\Gamma$ acts on $\tilde{\Sigma}$. Pick any point $x_0 \in \tilde{\Sigma}$. Look at its orbit $\Gamma x_0$ – a "lattice" in the model space $\tilde{\Sigma}$. Set $R(x_0) = \{y \in M : d(x_0, y) \le d(x_i, y) \forall x_i \in \Gamma x_0 \setminus \{x_0\}\}$. Then $R(x_0)$ is the interior of a finite-sided geodesic polygon and we can make $\Sigma$ by using $\Gamma$ to identify edges.

The uniformization theorem plus an understanding of the relation between the universal cover of a surface (manifold) and the original implies

**Theorem 4.1.** *Every compact RS is the quotient of one of the three models* $\mathbb{M}$ *by a "co-compact" lattice: an embedded image of* $\pi_1(\Sigma)$ *into* $Aut(\mathbb{M})$*, where the image acts properly discontinuously on* $\mathbb{M}$

4.1. **How many Riemann surfaces are there?** For each $g$, topologically there is only one.

Teichmuller theory: the space of inequivalent Riemann surfaces forms $6g-6$-dimensional manifold with singularities known as "moduli space":

$$\mathcal{M}_g = \text{ compact Riem surfaces of genus g / biholomorpisms}$$

This moduli space has a complex structure so forms a $3g-3$ dimensional complex manifold which is in fact algebraic. Its singularities correspond to the Riemann surfaces with "additional symmetries".

Using the Poincaré upper half-plane representation, we are looking for parameterizing a particular class of representations, i.e injective homomomorphisms

$$\rho : \Gamma \to SL(2, \mathbb{R})$$

The homomorphism can be specified by $g$ matrices $\rho(A_i), \rho(B_i)$ subject to the single relation ... Counting: 3 parameters for each of $\rho(A_i), \rho(B_i)$, for $3 * 2g = 6g$. Constraints 3 for the relation. Then quotient by inner automorphisms of $SL(2, \mathbb{R})$ : subtract another 3. $6g - 3 - 3 = 6g - 6$ real parameters.

We cannot use just any such homomorphisms. We require the image of $\rho$ acts properly discontinuously on the upper half plane. This requires that each $\rho(A_i), \rho(B_i)$ be hyperbolic, among other things...

The complex structure on moduli space is another more complicated story ...

For $g > 1$ "most' Riemann surfaces have no symmetries: no non-identity biholomorphisms.

4.2. **Function theory on Riemann surfaces.** Again, $\Sigma$ a compact Riemann surface.

**Proposition 4.1.** $\Sigma$ *admits* **no** *(global) holomorphic functions.*

Proof. Homework. Use the maximum modulus theorem from complex variables and the fact that on a compact manifold any function (eg $|f(z)|$ ) has a minimum.

Lacking any holomorphic functions, but desiring of some functions, we allow for meromorphic functions.

Definition of. Verification.

Example $\mathbb{C}P^1$. $f(z) = z$. (What's it look like, near $\infty$?) $f(z)$. At infinity we use the chart $w$ with $w = 1/z$. Then the function $z$ becomes $1/w$: infinity is a simple pole.

**Theorem 4.2.** *Any meromorphic function on* $\mathbb{C}P^1$ *is the quotient of two polynomials*

Example. Elliptic curves. The Weierstrass $\mathcal{P}$ function. Any meromorphic function on an elliptic curve is a rational function of $\mathcal{P}$ and its derivative $\mathcal{P}'$.

In terms of the model, an elliptic function is a meromorphic function on $\mathbb{C}$ invariant under the period.

A topological perspective. A meromorphic map is precisely the same thing as a holomorphic map to $\mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$. Poles get sent to infinity.

Degree. Degree of a map from $\Sigma$ to $\Sigma'$. Degree and local sum formula for, using notion of multiplicity.

A surprising theorem. See Donaldson, Cor 1 , p. 45.

This requires the notion of degree and the summation formula for degree in terms of mulitplicities, valid at ALL points $p$ and for all holomorphic maps between cpt RSs :

$$deg(f) = \Sigma_{x:f(x)=p} mult_p(f)$$

The multiplicity is always a positive number. Locally we can choose coord $z, w$ in the domain and range, centered at $p$ and $x_0$ with $f(x_0) = p$ so that $w = f(z)$ has the form $w = z^k$. Then $k$ is the multiplicity of $f$ at $x_0$.

**Theorem 4.3.** *If $\Sigma$ admits a meromorphic function having a* **single** *simple pole then $\Sigma = \mathbb{CP}^1$.*

Proof. To say $f : \Sigma \dashrightarrow \mathbb{C}$ has a simple pole at $p$ means that $f(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + \ldots$. In terms of the chart at infinity on $\mathbb{C}P^1$ we get that $w = 1/f(z) = 1/(\frac{a_{-1}}{z} + a_0 + a_1 z + \ldots) = z/(a_{-1} + a_0 z + .$ It follows that $f$ has degree 1. By the Riemann- Hurwitz theorem $\Sigma$ must have the genus of $\mathbb{C}P^1$

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Higher genus . Fuchsian functions. Holomorphic functions on the upper half plane invariant under the realization of the fundamental group defining the surface:

$$f(\gamma z) = f(z), \gamma \in \Gamma \subset SL(2, \mathbb{R}).$$

2. Holomorphic characterization of $g$. The genus is the (complex) dimension of the space of globally defined holomorphic one-forms. Locally such an $\omega$ has the form $f(z)dz$.

Eg. $g = 0$. The only holomorphic one-form on $\mathbb{C}P^1$ is $0 - \omega = 0dz/$ Dually, $\mathbb{C}P^1$ admits many holomorphic vector fields – indeed the space of such is complex three-dimensional, corresponding to the dimension of $PGL(2, \mathbb{C})$.

$g = 1$. Any elliptic curve is holomorphic to $\mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}$-span of $1, \tau = a + ib$ with $b > 0$. The form $dz$ on $\mathbb{C}$ is well defined on the torus and any other holomorphic one-form is a complex multiple of it. The space of holomorphic forms is complex 1-dimensional.

Dually, the space of holomorphic vector fields is also complex 1 dimensional. They are all of the form $a\frac{\partial}{\partial z}$, $a \in \mathbb{C}$.

$g > 1$. Lets take the hyperelliptic case, $w^2 = p(z)$ where all the roots of $p$ are simple.

HW. Use the defining relation to verify that $\frac{dz}{w}$ is nonzero and has no pole at the branch points, these being the zeros of $p$. Also look at what happens at infinity.

The forms $\frac{dz}{w}, \frac{zdz}{w}, \frac{z^k dz}{w}$ form a basis for the holomorphic differentials, where $2k + 1$ is the degree of

## 5. The great synthesis

Mumford's "great sythesis" taken from the beginning of his lecture notes (a book) 'Curves and Their Jacobians".

The following objects are naturally equivalent:

a) [Algebra] Finitely generated field extensions of $\mathbb{C}$ having transcendence degree 1.

b) [Geometry] Algebraic curves in $\mathbb{C}P^n$

c) [Analysis ] Compact Riemann surfaces

We have been talking about (c). Let's talk a bit about (a) and (b) for awhile.

### 5.1. **Algebra and field extensions.** Transcendence degree.

Let $\mathbb{C}(x)$ = field of rational functions in a single complex variable $x$, which is the same as the ring of fractions of the polynomial ring $\mathbb{C}[x]$. It is a "field extension of $\mathbb{C}$" of transcendence degree 1.

Eg: consider $e \in \mathbb{R}$. It satisfies no polynomial equation with rational coefficients. so $\mathbb{Q}[e]$ is a field extension of $\mathbb{Q}$, finitely generated (by $e$) and having transcendence degree 1. On the other hand $\mathbb{Q}(\sqrt{2})$ has transcendence degree 0.

Consider two fields $\mathbb{K} \subset \tilde{\mathbb{K}}$. The transcendence degree of $\tilde{\mathbb{K}}$ over $\mathbb{K}$ is positive if the vector space $\tilde{\mathbb{K}}/\mathbb{K}$ is infinite dimensional. If this holds, that means there is at least one element $x \in \tilde{\mathbb{K}}$ which satisfies NO algebraic relation over $\mathbb{K}$. Thus all polynomials in $x$ are linearly independent *over* $\mathbb{K}$. To say that the transcendence degree is **exactly** one means that if we select any other $y \in \mathbb{K}$, then $y$ is NOT transcendental over $\mathbb{K}[x]$. (Note: $\mathbb{K} \subset \mathbb{K}[x] \subset \tilde{\mathbb{K}}$.) In particular, there must be some polynomal $P$ over $\mathbb{K}[x]$ such that $P(y) = 0$, otherwise the elements $y, y^2, \ldots, y^n, \ldots$ are all linearly independent (over $\mathbb{K}[x]$.) But such a polynomial simply a polynomial in $x$ and $y$. Thus, to say that $\tilde{\mathbb{K}}$ has transcendence degree 1 means that any two $x, y \in \tilde{\mathbb{K}} \setminus \mathbb{K}$ satisfy some polynomial relation $P(x, y) = 0$.

Example. Elliptic curves. The field of meromorphic functions on an elliptic curve is generated by $\mathcal{P}, \mathcal{P}'$. The relation is cubic ...

### 5.2. **Algebraic curves.** A bit on (b) of the Great Synthesis.

Simplest view of algebraic curves.

$$P(x, y) = 0$$

a curve in $\mathbb{C}^2$. Eg. Elliptic curve:

$$y^2 = x^3 + ax^2 + bx + c$$

Does not look compact!

Compactify by allowing $x \to \infty$. How many points to add at infinity? In what manner?

Think of it as lying in $\mathbb{C}P^2$ by adding another variable $w$ and rewriting the eqn as a homogeneous cubic:

do naive Riem surface , function element method EARLIER to understand compactification: $y = \sqrt{P(x)}$ .. -RM

$$y^2 w = x^3 + ax^2 w + bxw^2 + cw^3$$

which reduced to the original upon setting $w = 1$.

Thus, the old 'x,y' are reinterpreted as affine coordinates $x/w, y/w$ on $\mathbb{C}P^2$. 'Infinity' becomes the line $w = 0$. How many intersections of our curve with infinity? $w = 0 \implies 0 = x^3$. So we intersect at infinity at one point $[0, 1, 0]$. To see local form of equation, focus on affine chart $y = 1$, so variables $X = x/y, W = w/y$. Dividing the equation by $y^3$ we get

$$W = X^3 + aX^2W + bXW^2 + cW^3.$$

In the new variables we have a non-singular cubic near $W = X = 0$. Check.

One point added at infinity. It is an inflection point - look how it touches its tangent line $W = 0$ (the old line at infinity $w = 0$). This is important later in the addition law which turns the elliptic curve into an abelian group. We can take the point at infinity as the 'identity' precisely because it is an inflection point.

More generally, take ANY polynomial $P(x, y)$ in two variables and look at the curve $P(x, y) = 0$ in $\mathbb{C}^2$. Homogenize it to get a single homogeneous polynomial $Q(x, y, w)$ with $P(x, y) = Q(x, y, 1)$.

Problem: descibe the degree of $Q$ in terms of the degrees of the monomials occuring in $P(x, y)$.

Answer: $g = (d-2)(d-1)/2$. (??) seems to work for $d = 2, 3$

**Yikes!** Problem. Not every integer is a triangular number. There are genuses which cannot be embedded smoothly as algebraic curves in $\mathbb{C}P^2$. Eg, $g = 4$. For these, we need more dimensions:

$$\Sigma \subset \mathbb{C}P^n$$

for some $n$. ...

What if we 'try' to embed a genus 4 surface holomorphically into $\mathbb{C}P^2$? By the above count, we cannot. When we try, we get singularities. These can be resolved by various methods. The result can be embedded, as above, in a higher $\mathbb{C}P^n$.

Example of a singular curve and its resolution: Eg $P(x, y) = x^3 - y^2$. ...

For the simplest version of the problem, consider instead of a degree 3 polynomial a degree 4 one,

$$y^2 = x^4 + ax^3 + bx^2 + cx + d$$

by the same process we are led to

$$w^2y^2 = x^4 + awx^3 + bw^2x^2 + cw^3x + dw^4$$

setting $w = 0$ yields $0 = x^4$. Again, the curve intersects the line at infinity in a single point $w = x = 0$, so the point with homogeneous coordinates $[0, 1, 0]$. What does the curve look like near this point? Dividing by $y^4$ and using $X = w/y$, $W = w/y$ yields

$$W^2 = X^4 + aWX^3 + bW^2X^2 + cW^3X + dW^4$$

which is a singular curve at $W = X = 0$. It splits into two branches , the local structure of which can be found by Puiseux (or Newton-Puiseux) expansions. ...

leads to singularity theory; CTC Wall etc..

How does the dictionary work, in the great synthesis, the dictionary relating (a), (b), (c).

5.3. **The dictionary.** Given (c), a compact Riemann surface, consider the space of all meromorphic functions on $\Sigma$. This is a field. It contains $\mathbb{C}$, the constant functions. Its transcendence degree is 1. It is generated by a finite set $f_1, \ldots, f_k$ of meromorphic functions [Hard theorem?]. This brings us to (a). Saying it has transcendence degree 1, means, in particular, that there are algebraic relations $R(f_1, \ldots, f_k) = 0$ .. or maybe .. $R_j(f_1, f_j) = 0$??

The meromorphic functions $f_1, \ldots f_k$ can be chosen, by taking $k$ large enough, so that there is no point $p \in \Sigma$ such that all $f_i$ satisfy $f_i(p) = 0$. Also, there is no point which is a common pole of all the $f_i$. Then $p \mapsto [f_1(p), \ldots, f_k(p)]$ defines an embedding into $\mathbb{C}P^n$ where $n = k - 1$. Due to the relations from (a), it is algebraic.

Conversely, given an algebraic curve $\Sigma \subset \mathbb{C}P^n$, take any affine function $X_i = X_i/X_1$ or more generally $P(X)/Q(X)$ where $P, Q$ are homogeneous polynomials of the same degree on $\mathbb{C}^{n+1}$ and thus a rational function on $\mathbb{C}P^n$. Testrict it to the curve to get a meromorphic function. (Hard?) theorem: all meromorphic functions on $\Sigma$ arise in this way.

Eg. The theorem for $\mathbb{C}P^1$. Any meromorphic function on $\mathbb{C}P^1$ is the quotient of homogeneous polynomials on $\mathbb{C}^2$ having the same degree. In an affine chart this looks like the quotient of two polynomials $p(x)/q(x)$ (of any two degrees).

$\mathbb{C}P^n$. Defs. Affine charts. As projective Hilbert space for an n+1 level system. Algebraic varieties. Curves. ...

## 6. A topological -analytic theorem

Recall degree between maps of compact surfaces.

**Theorem 6.1.** *If $f$ is a meromorphic function on $\Sigma$ with exactly one simple pole, then $\Sigma$ is biholomorphic to $\mathbb{C}P^1$.*

Proof: Look at the degree near the pole. It is one, since , viewed as a map to $\mathbb{C}P^1$, $f^{-1}(\infty)$ consists of a single point and in coordinates near infinity on $\mathbb{C}P^1$ the map is of the form $\xi \mapsto \xi$. Now use that degree, counted using multiplicities, is CONSTANT on any RS.

This proof culled from Donaldson, Cor. 1, p 45.

for degree see Farkas-Kra or Donaldson Prop 7, p44.

## 7. Integration formulae. The historical heart of the theory

Riemann surface theory grew in part out of trying to compute integrals such as $\int \frac{(1-k^2x^2)dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

See Griffiths, Variations on a Theme of Abel, p. 327 for a guided exercise on how arclength along an ellipse converts to this integral.

7.1. **Warm up.** To warm up, we will integrate $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$ not by messing about with trig substitutions a la Calc 1 or 2, but by viewing the Riemann surface of $y = \sqrt{1-x^2}$ as a branched cover over the $x$ plane, drawing branches and cuts, and doing a path integral by residues.

Picture here !

We get a surprise or two. The Riemann surface is a sphere. The integrand is $dx/y$ and has simple poles at the north and south pole, $\pm i\infty$.

7.2. **Indefinite integrals, multi-valued functions, and the universal cover.** What about the indefinite integral? i.e $\int_0^z \frac{dx}{\sqrt{1-x^2}}$ but with $z \in \mathbb{C}$ varying over the complex plane. It is a 'multi-valued function, $Arcsin(z)$..

$$Arcsin(z) = \int_0^z \frac{dx}{\sqrt{1-x^2}}$$

We are taught to avoid mult-valued functions. Put a hex symbol over them. But they come up in doing integrals all the time:

$$Log(z) = \int_1^z \frac{dz}{z}$$

We are taught to limit the domain of 'Log' or 'Arcsin' so that it becomes a well-defined function, to use cuts or restrictions on $z$. In a way, Riemann's approach is the reverse: we enlarge the domain, by replacing the Riemann surface (eg the punctured complex plane) by its universal cover.

**Slogan: Multi-valued functions are regular functions on the universal cover.**

Indefinite integrals are multi-valued functions.

Let us move to the world of manifolds. Suppose that $\alpha \in \Omega^1(M)$ is any closed real-valued one form on a manifold $M$. Fix a point $z_0 \in M$ and define the "function"

$$I(z) = \int_{z_0}^z \alpha$$

For $z$ close to $z_0$ , varying about in a small disc, the function is well-defined and smooth and $dI = \alpha$. We can make the domain larger and insure that $I$ remains single valued as long as $z$ varies in a simply connected region about $z_0$. But, if we move far way, uh-oh, we get a multi-valued function. However, if we integrate along two different paths, $c_1, c_2$ connecting $z_0$ to $z$ we generally get two different answers. Now consider the loop $c = c_2^{-1} c_1$ based at $z_0$. We have that the two different values of $I$ differ by the integral of $\alpha$ around the loop. This integral depends only on the homotopy class $[c]$ of the loop. [In fact, only on the homology class.] We could write this as

$$I_2(z) - I_1(z) = per([c]) \tag{1}$$

$$per : \Gamma = \pi_1(M) \to \mathbb{R}; per([c]) = \int_c \alpha \tag{2}$$

It follows that $\tilde{I} : \tilde{M} \to \mathbb{R}$ is well defined, where $\tilde{M}$ is the universal cover of $M$. Write $\Gamma = \pi_1(M, z_0)$ so that $\Gamma$ acts on $\tilde{M}$ smoothly and $M = \tilde{M}/\Gamma$. Here is how we can define $\tilde{I}$. Take the form $\alpha$ and pull it up to $\tilde{M}$ by the covering map $\pi : \tilde{M} \to M$. Set

$$\tilde{I}(\tilde{z}) = \int_{\tilde{z}_0}^{\tilde{z}} \tilde{\alpha}; \tilde{\alpha} = \pi^* \alpha.$$

Since a point $\tilde{z} \in \tilde{M}$ corresponds to a point $z \in M$ and a choice of homotopy classes of paths from $z_0, z$ we have that $\tilde{I}$ is our multi-valued function.

Moreover, the period mapping defines an additive homomorphism :

$$per : \Gamma \to \mathbb{R}$$

and we have

$$\tilde{I}(g\tilde{z}) = \tilde{I}(\tilde{z}) + per(g)$$

which is just another way to write the 'multi-valued" relation (1).

All this works perfectly well for complex valued closed one-forms. Meromorphic one-forms, also known as Abelian differentials, are such forms on Riemann surfaces. What gives the whole theory added power in the world of Riemann surfaces is the uniformization theorem. We basically **know** what the universal cover is going to be! And $\Gamma$, the fundamental group, must sit inside $Aut(\mathbb{M})$ as a discrete subgroup.

Let's return to the known examples as a reality check and then keep going. The meromorphic one-form associated to *Log* is $dz/z$, the integrand central to residues. Its domain is the punctured plane $\mathbb{C} \setminus \{0\}$ which we have seen is a cylinder. Its universal cover is $\mathbb{M} = \mathbb{C}$.

For *Arcsin* the one-form is $dx/y$, integrated paths on the Riemann surface for $y = \sqrt{1 - x^2}$ , which is to say, $x^2 + y^2 = 1$. [Use homogeneous coordinates, $x^2 + y^2 = w^2$ to better understand global structure] We have seen that, completed at infinity the latter is $\mathbb{C}P^1$ and the integrand has two poles , namely "$\pm i\infty$.

## 7.3. **Abelian differentials.**

**Definition 7.1.** *An Abelian differential is a meromorphic one-form on a Riemann surface*

Historically, they are divided into three types "the first kind", the "second kind" and the "third kind".

First kind: no poles. A holomorphic differential.

Big theorem. The space of holomorphic differentials on a compact Riemann surface is a finite dimensional complex vector space of dimension $g$, the genus of the Riemann surface .

Second kind: no simple poles, only higher order poles. As far as integration is concerned, we can ignore the poles and take the domain of the indefinite integral to be the universal cover of $\Sigma$.

Third kind: some simple poles. We will need the residues around the poles and the universal cover of the punctured $\Sigma$ to understand the indefinite integral as a global object.

EXAMPLES HERE?? or ...

SKIP TO Abel-Jacobi theory in its modern presentation ??

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# 8. Abel-Jacobi map! Periods!

# 9. Divisors, Line bundles, spaces of meromorphic functions

One of the big theorems in the subject is the Riemann-Roch theorem which gives a count for the number (dimension) of the space of meromorphic functions having a specific pole-and-zero structure. It has a series of generalizations all based on the viewing of a meromorphic functions as a sections of a complex line bundle over $\Sigma$.

## 9.1. **RR and the count.**

Exercise. p. 169 [4] Use RR to show that every RS is realized as a 'space curve": i.e. holomorphically embeds in $\mathbb{C}P^3$.

This is a realization theorem. Here is another, obtained by using perspective to projection the previous realization ot $\mathbb{C}P^2$.

Resolution of singularities and realization: Singularities of plane curves can be resolved, eg. by repeated blow-up or by Puiseux expansion. Any singular algebraic curve in $\mathbb{C}P^2$ has a unique (up to bi-holomorphism) desingularization which is a RS.

Take the above realization theorem suggested by Fisher. Project that realization to $\mathbb{C}P^2$. Result: Every RS is realized as a singular algebraic curve in $CP^2$.

## 9.2. **Examples.**

## 10. Bibliography

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