Application of Riemann-Roch to transcendence degree.

I saw this in Miranda I think, but could not relocate it.

Proposition 0.1. *If X is a compact Riemann surface then the transcendence degree of its function field over the base field $\mathbb{C}$ is 1.*

Recall transcendence degree.

Say $K \subset L$ are fields. (We have in mind $\mathbb{C} \subset \mathcal{M}(X)$.) We say that an element $\alpha \in L$ is 'transcendent' over K if there is no nonzero polynomial with coefficients in K satisfied by α. In this case the map $K[x] \to L$ sending a polynomial p to $p(\alpha)$ is injective and so we can identify $K(\alpha) \subset L$ with $K[x]$. And in this case we say that L is transcendental over K. If every element of L is algebraic over $K(\alpha) \subset L$, then we say that "L has transcendence degree 1. Otherwise, we have that the transcendence degree is greater than 1, and we can repeat the construction. There would have to be another element $\alpha_2 \in L$, $\alpha_2 \notin K(\alpha)$ such that when we consider the inclusion of fields $K(\alpha) \subset L$ we have that α_2 is transcendent over $K(\alpha)$. One verifies without difficulty that this is true if and only if there is no non-trivial two-variable polyomial $p \in K[x, y]$ such that $p(\alpha_1, \alpha_2) = 0$, and in this case we say that "α, α_2 are algebraically independent over K". Continuing in this way, one can define the notion of a "transcendence basis for L over K". The transcendence degree of L over K is the number of elements in such a basis. In analogy with linear algebra the transcendence degree is also equal to the maximal number of elements of L which are algebraically independent over K.

Proof of Prop. The transcendence degree of $\mathcal{M}(X)$ over $\mathbb{C}$ is at least one: any non-constant meromorphic function is transcendental over $\mathbb{C}$.

To show that the transcendence degree is exactly one we must show that any two non-trivial meromorphic functions on X satisfy a polynomial relation. So, let f, g be two such polynomials, and suppose there is no polynomial relation among them. Then the monomials $f^i g^j$ are all linearly independent. We will get a contradiction to Riemann-Roch from this assertion.

Let $(f) = P_f - N_f, (g) = P_g - N_g$ be the decomposition of the divisors of f and g into positive and negative parts. Thus, for example, N_f is a sum over the poles of f. Then $(f) \geq -N_f, (g) \geq -N_g$, Also $-N_f, -N_g \geq -(N_f + N_g)$ so that $f, g \in L(D)$ if we set $D = N_f + N_g > 0$. Now the poles of $f^i g^j$ are contained within the poles of f union the poles of g, and by looking at orders we see that $f^i g^j \in L((i + j)D)$. Now recall that

$$dim(L(nD)) = n\ell(D) + C$$

for n large.

On the other hand, if f, g are algebraically independent, we have seen that the monomials $f^i g^j$ are linearly independent. How many of them are there with $i+j \leq n$. Well, this is the dimension of the space of degree n polyonomials in two variables x, y which is , by the 'stars and bars" argument, equal to $\binom{n+2}{2} = \frac{1}{2}n^2 + O(n)$. Thus we have from independence that

$$dim(L(nD)) = \frac{1}{2}n^2 + O(n)$$

and this contradiction implies that indeed, it is impossible that f and g are algebraically independent

QED

Wow! That is a surprising proof!

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Stars and bars counting aside...

Degree 5 polynomials in 2 variables. Dimension? Basis: monomials $x^k y^m$, $k + m \leq 5$. How many? Counting $xy^3 = *|***|*$. $x^2y^2 = **|**|*$ etc. We have $7 = 5 + 2$ slots. In these slots , put two bars. Fill the rest with stars. The dividing bars tell us then how to convert the starts to a monomial. There are $\binom{5+2}{2}$ such monomials.

$dim(V(d;n) =$ space of polynomials of degree d in n variables is $\binom{d+n}{n}$.