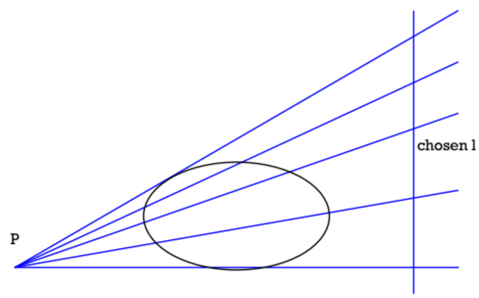

(a) Projection from a point P

We work out the geometry of the map $[X, Y, Z] \to [X, Y]$ and how it yields a map from a curve $C \subset \mathbb{CP}^2$ to the projective line.

Begin synthetically. In the projective plane $\mathbb{P}^2$ there is a unique line connecting any two distinct points, and there is a unique point shared by any two distinct lines [1]. Fix a point $P \in \mathbb{P}^2$. The space of all lines passing through P forms a projective line, $\mathbb{P}^1(P) \cong \mathbb{P}^1$, as we show momentarily. Take a point $Q \neq P$. Connect Q to P by the guaranteed unique line $\vec{P}Q \in \mathbb{P}^1(P)$. In this way we get a map $\mathbb{P}^2 \setminus \{P\} \to \mathbb{P}^1(P) = \mathbb{P}^1$. We follow the algebraic geometry tradition and write this map as

$$\mathbb{P}^2 - \to \mathbb{P}^1(P) \qquad (*),$$

This map is the projection from P. the dotted arrow here, and throughout algebraic geometry, indicating that the domain of the map is not the entirety of what it appears to be, ie. here it is not all of $\mathbb{P}^2$.

To visualize the map, it helps to choose an auxiliary line $\ell_0 \subset \mathbb{P}^2$ not passing through P. Since every line $\ell \in \mathbb{P}^1(P)$ intersects ℓ_0 in a unique point, we have a a bijection $\mathbb{P}^1(P) \cong \ell_0$, so by composition, projection from P induces a map

$$\mathbb{P}^2 - \to \ell_0 \qquad (*)$$

as indicated in the figure.

Exercise 0.1. *Use homogeneous coordinates $[X, Y, Z]$ for $\mathbb{P}^2$. Suppose that $P = [0, 0, 1]$ and that the line $\ell_0 = \{Z = 0\}$. Show that the projection (*) is given in homogeneous coordinates by $[X, Y, Z] \mapsto [X, Y, 0]$*

Exercise 0.2. *Show that the above map, in the right affine coordinates on domain and range, is given by $(x, y) \mapsto x/y$.*

Exercise 0.3. *Show that the linear projection $(x, y) \to x$ is achieved as the affine coordinate representation of a projection from a well-chose point P onto a well-chose line ℓ_0. Find the point P and line ℓ.*

Answer to exercise 0.3. The choice of affine coordinates tells us to represent points in $\mathbb{P}^2$ as $[x, y, 1]$ Take $P = [0, 1, 0]$ so as to 'project out' y. We want the projection to be $[x, y, 1] \mapsto [x, 1]$. Think of $[x, 1] = [x, 0, 1]$. Takin $\ell_0 = \{Y = 0\}$ does the trick.

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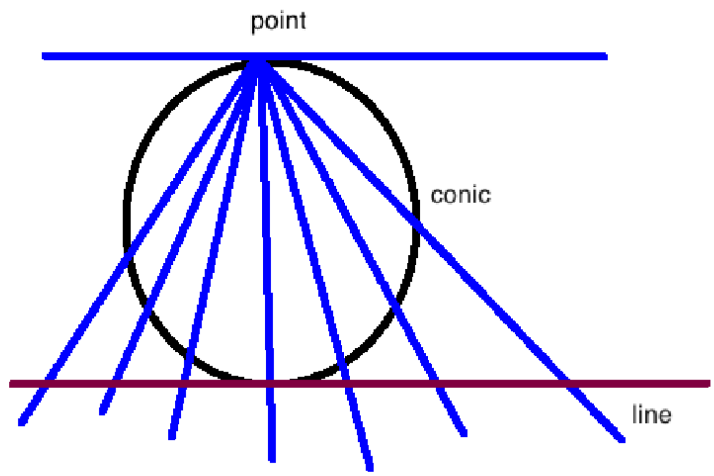
[1] These incidence relations, plus some axiomatization of Desargue's theorem forms the entirety of the axioms of projective geometry.



(a) Placing the viewpoint on the conic yields an isomorphism with the line.

Restrict the projection from P to a subset $\Sigma \subset \mathbb{P}^2$ disjoint from P. We get a map

$$\Sigma \to \mathbb{P}^1.$$

We are interested in the case in which Σ is an algebraic curve, meaning the zero locus of a homogeneous polynomial $F(X, Y, Z)$: $\Sigma = \{F = 0\} := V(F)$.

Exercise 0.4. *Let the degree of the homogeneous polynomial F be d. Let $\ell \subset \mathbb{P}^2$ be a line. If ℓ is not contained in $\Sigma = V(F)$ then $\ell \cap \Sigma$ consists of d or fewer points.*

Hint. Write the restriction of F to ℓ in terms of an affine parameterization of ℓ.

Remark: The Bertini theorems are algebraic-geometric versions of the Sard theorem which assert that for algebraically closed fields such as $\mathbb{C}$, we have that for almost every line ℓ [2], the number of points of the intersection $\ell \cap \Sigma$ is exactly d, at least in our case where the underlying field is $\mathbb{C}$. Thus, the restriction map yields a degree d map $\Sigma \to \mathbb{P}^1$.

Exercise 0.5. *In case the underlying field is $\mathbb{C}$, and $\Sigma \subset \mathbb{CP}^2$ is a smooth algebraic curve, show that the critical values of the map $\Sigma \to \mathbb{P}^1 = \mathbb{P}^1(P)$ are precisely the tangent lines to Σ which pass through P.*

We come to the guts of our discussion.

Proposition 0.6. *If Σ is a nondegenerate conic, and $P \in \Sigma$, then the projection from P defines an isomorphism of Riemann surface $\Sigma \to \mathbb{P}^1(P)$. (See figure (a) above.)*

Proof. Choose homogeneous coordinates so that in the associated affine coordinates the conic is the 'unit circle' $x^2 + y^2 = 1$ and the point P is the 'north pole' $(x, y) = (0, 1)$. (Compare Miranda p. 57. Note all nondegenerate conics are isomorphic since all nondegenerate quadratic forms on $\mathbb{C}^3$ are isomorphic.) Take

[2] in algebraic geometry, 'almost everywhere' becomes replaced, almost everywhere, by "outside of a Zariski closed set" which means off the zero locus of some set of polynomials

the line ℓ_0 which we project onto to be the line $y = -1$ in these coordinates, the line tangent to the circle at the 'south pole' $(0, -1)$. Then the projection of the conic from P to ℓ_0 in these coordinates is the standard stereographic projection (perhaps scaled by 2, depending on how you learned stereo projection), namely: $(x, y) \mapsto 2x/(1-y)$. Although the map is not formally defined at P its limit is, letting $y \to 1$, and by so extending the map $P \in \Sigma$ is sent to the point at infinity on the line.

Remark: The full story of this limit procedure involves the construction of a new algebraic variety, the blow-up of $\mathbb{CP}^2$ at the point P.

1. Linear algebraic underpinnings of the Projective Plane

Begin with a 3-dimensional vector space $\mathbb{V}$ over a field $\mathbb{K}$. Write $\mathbb{P}(V) \cong \mathbb{KP}^2$ for the corresponding projective plane/ Points of this projective plane are 1-dimensional subspaces of $\mathbb{V}$. If $v \in \mathbb{V}, v \neq 0$ we write $[v] = span(v) \in \mathbb{P}(\mathbb{V})$ for the corresponding point of $\mathbb{P}(\mathbb{V})$. Lines in $\mathbb{P}(V)$ correspond to 2-dimensional subspaces $S \subset \mathbb{V}$. Write $\ell = \ell_S$ for the line represented by S. Points of $[v] \in \ell_S$ correspond to 1-dimensional subspaces of $\mathbb{K}v \subset S$, so that $\ell_S = \mathbb{P}(S) \cong \mathbb{P}^1$.

A 2-dimensional subspace S is defined by the vanishing of a nonzero linear functional $f \in \mathbb{V}^*$, and f is uniquely defined by S, up to scale. Thus lines in $\mathbb{P}(\mathbb{V})$ are naturally in bijection with points of $[f] \in \mathbb{P}(\mathbb{V}^*)$. The incidence relation "point $P = [v]$ lies on line $\ell = \{f = 0\}$ is the duality relation $f(v) = 0$.

Any linear isomorphism $L : \mathbb{V} \to \mathbb{V}$ maps lines in $\mathbb{V}$ to lines in $\mathbb{V}$ and so induces a "projective isomorphism" $\mathbb{P}(\mathbb{V}) \to \mathbb{P}(\mathbb{V})$. The group of all such transformations is the projective linear group $PGL(\mathbb{V}) \cong Gl(\mathbb{V})/$ nonzero scalar multiples of Id.. When the field is $\mathbb{C}$ this group is precisely the group of biholomorphisms of $\mathbb{CP}^2$. An invertible linear transformation of $\mathbb{V}$ also maps two-planes to two-planes so defines a map $\mathbb{P}(\mathbb{V}^*) \to \mathbb{P}(\mathbb{V}^*)$. This joint action of $PGL(3)$ on $\mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V}^*)$ preserves the incidence relation.