L2. B1

Uniformization theorem: [Hard]
Up to biholomorphism there are
exactly 3 connected ~~simply connected~~
& simply connected
Riemannian manifolds.
1) $\mathbb{CP}^1$
2) $\mathbb{C}$
3) open unit disc $\mathbb{D} \simeq$ upper half plane $\mathbb{H}$.
"models $\mathbb{M}$"

Cor: Every cpt R.S is
either:
1) $\simeq \mathbb{CP}^1$ $g=0$
2) covered by $g=1$
$\mathbb{C}$ & hence $\simeq \mathbb{C}/\Lambda$
(elliptic curves), $\Lambda \subset \mathbb{C}$ a lattice.

3) covered by $\mathbb{H}$ $g>1$
hence iso. to
$\mathbb{H}/\Lambda$
$\Lambda \subset PSL_2(\mathbb{R})$ a discrete
'lattice' subgroup, iso
to $\pi_1(X)$ as an abstract
group.

U. Thm

Let's accept ~~cor~~ & try to understand cor.

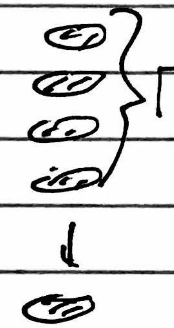
generalities: : X smooth connected mfd, Then it universal cover $\tilde{X}$ is also a smooth connected mfd, which is simply connected.

$\Gamma = \pi_1(X) \curvearrowright \tilde{X}$

&

$\tilde{X}/\Gamma \simeq X$

$$\tilde{X} \xrightarrow{\pi} X$$



any geometric structure (metric, v-fld, one form, ..) lifts to $\tilde{X}$

any Γ-invariant geom struc on $\tilde{X}$ pushes down to X.

L2B3

Automorphisms of the 3 models $\mathbb{M}$. (biholomorphisms)

1) $\mathbb{M} = \mathbb{CP}^1$; $\mathrm{Aut}(\mathbb{M}) = \mathbb{P}GL_2(\mathbb{C})$
= cx proj group.

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right]$$

why the "$\mathbb{P}$" ?

6 dim'l real Lie gp; 3 cx dim'l.

2) $\mathbb{M} = \mathbb{C}$, $\mathrm{Aut}(\mathbb{M})$ = cx affine maps

$z \mapsto az + b$

$a \neq 0$.

4 real dim'l Lie group.
2 cx dim'l.

3) $\mathbb{M} = \mathbb{H}$ or $\mathbb{D}$.
$\mathrm{Aut}(\mathbb{H}) = PSL_2(\mathbb{R}) \subset \mathbb{P}SL_2(\mathbb{C})$

explain!

3 dim'l real Lie group!

__Pfs__: → See Exer, 8 (a) (b) (c) ~~and~~ Donaldson ch 3, L2 B4

How the classif of automorphisms + U. Thm $\Rightarrow$ Cor.

Take X
$\tilde{X} = \mathbb{M}$.
$\Gamma \circlearrowleft \mathbb{M}$ so $\Gamma \subset Aut(\mathbb{M})$

But $X \simeq \mathbb{M}/\Gamma$.

$g=0$; Now $\mathbb{M} = \mathbb{C}P^1$, alredy cpt s. conn
$g=0$ ✓.

$g=1$ $\mathbb{M} = \mathbb{C}$,
do some work on subgrps of $az+b$ grp s.t. $\mathbb{M}/\Gamma$ is cpt, smooth.

$\Rightarrow$ Γ a lattice: "abstract copy of $\mathbb{Z}^2 \subset \mathbb{C}$.

all such: generated by τ_1, τ_2
$\Gamma = \{m\tau_1 + n\tau_2 : m, n \in \mathbb{Z}\}$
τ_1, τ_2 indep over $\mathbb{R}$.

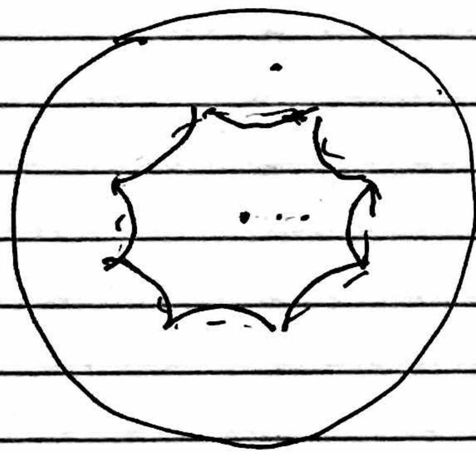
use "a" of $az+b$.
to turn $\tau_1 = 1$, so $\tau_2 \notin \mathbb{R}$
....

L2B5

$g \geq 2$. All other X w genus g's, $g \geq 2$ have
$$\mathbb{M} \simeq \mathbb{H} \simeq \mathbb{D}.$$

Then there $\pi_1(X) \hookrightarrow \Gamma \subset PSL_2(\mathbb{R})$

the classic $g = 2$ picture via octagons



<u>Need to Know:</u>

hyperbolic metric

geod.

something about gluing hyperbolic isometries

Remarkable fact:

$\mathbb{M} = \mathbb{D}$ or $\mathbb{H}$

admits a complete Riem metric ds^2 unique up to scale s.t.

$$\mathrm{Isom}_+(\mathbb{M}) = \mathrm{Aut}(\mathbb{M}).$$

That is: every bi-holo- from $\mathbb{M}$ to $\mathbb{M}$ is an isometry for this metric & every orient. preserving isom of $\mathbb{M}$ is a bi-holo. of $\mathbb{M}$.

5–10 min breakouts.

see if someone in yr group

a) knows the fmla for the standard hyperbolic metric on $\mathbb{H}$, the upper $\frac{1}{2}$ plane.

b) knows its geodesics

c) can explain sthng about $\mathrm{Isom}_+(\mathbb{H})$.