1. Jacobian; canonical embeddings

X is a compact Riemann surface of genus $g \geq 1$. The space $H^0(K) = \Omega^{(1,0)}$ of holomorphic differentials on X has dimension g. Choose a basis $\omega_1, \ldots, \omega_g$ of holomorphic differentials. View the ω_i as sections of the complex line bundle $K = T^{(1,0)}X$.

Theorem 1.1. *The ω_i can never simultaneously vanish: for any $p \in X$ there is an $i, i = 1 \ldots, g$ such that $\omega_i(p) \neq 0$*

It follows that

$$p \mapsto [\omega_1(p), \omega_2(p), \ldots, \omega_g(p)] \tag{1.1}$$

is a well-defined map $\Phi_K : X \to \mathbb{CP}^{g-1}$. We call this map the *canonical embedding.* If we start with a different basis $\tilde{\omega}_i, \ldots, \tilde{\omega}_g$ for the space of holomorphic one-forms, we would have another canonical embedding, $\tilde{\Phi}_K$, related to the old Φ_K by $\tilde{\Phi}_K = L \circ \Phi_K$ where $L : \mathbb{CP}^{g-1} \to \mathbb{CP}^{g-1}$ is the projective transformation induced by the linear map $\tilde{L} : \mathbb{C}^g \to \mathbb{C}^g$ corresponding to the change of basis matrix.

Theorem 1.2. *For $g > 2$ the canonical embedding is either an embedding of X into $\mathbb{CP}^{g-1}$ or it maps X in a $2:1$ fashion onto a $\mathbb{CP}^1$ embedded in $\mathbb{CP}^{g-1}$.*

Remark The exception X of the theorem are the hyperelliptic Riemann surface . These can be represented in an affine chart by $y^2 = p(x)$.

1.1. The terminology of base-point free linear systems.

Recall $L(D)$.

Recall $L(D) \cong \Gamma(L_D) = H^0(L_D)$.

Definition 1.1. A linear system on X is a linear subspace $V \subset L(D)$ for some divisor D on X.

The linear system $V \subset L(D)$ is called "base-point free" if there is no point $p \in X$ such that for all $s \in V$ we have $s(p) = 0$.

A base-point free linear system defines a a canonical map

$$\Phi_V : X \to \mathbb{P}(V^*)$$

as follows. For $p \in X$ we have the map

$$ev_p : V \to L_p \qquad ev_p(s) = s(p).$$

The base-point free assumption is that ev_p is onto for all p, thus $ev_p^* : L_p^* \to V^*$ has one-dimensional image. The canonical embedding associated to V then sends x to $[ev_x^* L_x] \in \mathbb{P}(V^*)$. It is a holomorphic map. A basis s_i for V defines linear coordinates on V^*. Consequently, the coordinate representation of Φ_V is

$$p \mapsto [s_1(p), \ldots, s_k(p)].$$

In this terminology, theorem **??** asserts that the complete linear system K is base point free. And the canonical embedding is the associated Φ_K.

Proof of theorem 1.1. By Riemann-Roch. Suppose there is such a point p. Then every holomorphic one-form, being a linear combination of the ω_i, vanishes at p. It follows that $L(K-p) = L(K)$ so $\ell(K-p) = g = \ell(K)$. But since $g > 0$ we have $L(p) = \mathbb{C}$ (no constant meromorphic functions with a single simple pole). Thus $\ell(p) = 1$. Also $deg(K-p) = deg(K) - 1 = 2g - 3$. Riemann-Roch says $\ell(K-p) - \ell(p) = deg(K-p) - g + 1$ which implies that $\ell(K-p) = g - 1$ contradicting $\ell(K) = \ell(K-p)$.

2. Jacobian

Now view the ω_i as integrands. Integrate the homogeneous coordinates ω_i of the canonical embedding Φ_K to get a g-vector of indefinite integrals

$$(\int^x \omega_1, \dots, \int^x \omega_g).$$

Locally, each component is a holomorphic function on X, but globally it is not a function, but rather a multi-valued function, for if we let x vary around a closed cycle c in X we will find that $\int^x \omega_i \mapsto \int^x \omega_i + \int_c \omega_i$. We call the integrals $\int_c \omega^i$ 'periods'. By appropriately dividing out by periods, the integration map will become well-defined.

Outline: We first make this into a multi-valued function on X by fixing a base point p_0. Then we make it into an honest-to-god function by dividing out by the periods. Then we make it into a function on divisors by linearity. Then we show it provides an isomorphism $Pic_0(X)$ to a torus.

Fix a $p_0 \in X$ as base point. Consider the vector function

$$p \mapsto \tilde{F}(p) := (\int_{p_0}^p \omega_1, \dots, \int_{p_0}^p \omega_g)$$

For p close to p_0 the integrals are well-defined holomorphic functions of p: we join p to p_0 by a path lying in a small contractible neighborhood of p_0. But as p moves away, we have to make a choice of path joining p_0 to p. The difference of two such paths is closed: that is, an element of $H_1(X, \mathbb{Z})$. We call the integrals $\int_c \omega_i$ the periods of the differentials. Because $d\omega_i = 0$ and $\partial c = 0$ these integrals are well-defined, independent of the choice of curve used to represent a homology class c. Any vector in $\mathbb{C}^g$ which can be written as $\lambda = (\int_c \omega_1, \dots, \int_c \omega_g)$ we say is in the period lattice. Write $\Lambda(X) = \Lambda(X; \{\omega_i\})$ for the collection of all such vectors.

Proposition 2.1. *The collection of vectors $\Lambda(X)$ forms a rank $2g$ lattice in $\mathbb{C}^g$.*

The multi-valued function $\tilde{F}$ yields a well defined once we mod out by this lattice:

$$F = \tilde{F}(mod \Lambda(X)) : X \to \mathbb{C}^g / \Lambda(X).$$

Exercise 2.2. *Verify that changing the basis ω_i changes the lattice by the corresponding change of basis matrix in such a way as to yield a holomorphically equivalent map of X into an equivalent torus.*

Exercise 2.3. *Verify that changing the base point p_0 of integration changes F by a translation:*

$$(\int_{p_1}^p \omega_1, \dots, \int_{p_1}^p \omega_g = \vec{d} + (\int_{p_0}^p \omega_1, \dots, \int_{p_0}^p \omega_g) \ mod \ \Lambda(X).$$

with $\vec{d} \in \mathbb{C}^g / \Lambda(X)$ independent of p.

Definition 2.1. We write $Jac(X)$ for the torus $\mathbb{C}^g / \Lambda(X)$.

Now, let D be any divisor; $D = \Sigma n_a p_a$. Set $\int_{p_0}^D \omega_i = \sum n_a \int_{p_0}^{p_a} \omega_i$ This multi-valued function has the same periods as before and so induces the ith component of a well-defined map

$$\Phi : Div(X) \to Jac(X); \Phi(D) = (\int_{p_0}^D \omega_1, \dots, \int_{p_0}^D \omega_g) \qquad \text{mod } \Lambda(X).$$

Lemma 2.4. *If $D = (f)$ is a principal divisor then*

$$\Phi(D) = 0 \quad mod\ \Lambda(X).$$

Abel's theorem is the converse of this lemma:

Theorem 2.5 (Abel's theorem)**.** *If D is a divisor with $\Phi(D) = 0$ and $deg(D) = 0$ then D is a principal divisor.*

This theorem asserts that the integration map Φ induces an injective map $\Phi : Pic_0(X) \to \mathbb{C}^g/\Lambda(X)$. Jacobi's map asserts the map is onto.

Theorem 2.6 (Jacobi's theorem)**.** *The integration map $D \mapsto \Phi(D)$ from degree zero divisors to $\mathbb{C}^g/\Lambda(X)$ is onto: every element of this torus is the integral of some degree zero divisor.*

Together, the Abel and Jacobi theorem yield

Theorem 2.7.

$$(\Phi, deg) : Pic(X) \to Jac(X) \times \mathbb{Z}$$

is an isomorphism of Abelian groups, and gives $Pic(X)$ the structure of an algebraic variety.

Proofs: Following Donaldson. On Thurs Feb 21, 2013: Will fix up computation on the integral of that $\frac{\bar{\partial} f}{f} \wedge \theta$ which led to $2\pi i \int_\gamma \theta$.

2.1. **Canonical embedding as differential of Abel-Jacobi map.** Map X into $Pic(X)$ by sending p to the divisor $\{p\}$. Then the Abel-Jacobi map becomes our original integration map:

$$F(p) = \Phi(p).$$

$Jac(X)$ is a torus – an Abelian Lie group – and as such has a canonical translation of tangent spaces back to the origin 0. Differentiating F with respect to p and translating back to the origin, we get a family of lines $[d\Phi(p))/dp] \subset T_0 Jac = \mathbb{C}^g$ depending on p. By the fundamental theorem of calculus, this line is the one defined by the canonical embedding.

We can add points together and compose to get maps

$$\Phi^{(d)} : S^d X \to Pic(X)$$

where

$$S^d(X) = (X \times X \times \ldots \times X)/S_d$$

is the d-fold symmetric product of X by itself. The symbol $/S_d$ means divide out by the action of the symmetric group acting by interchaning indices of points: $(x_1, \ldots, x_d) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(d)})$. Thus a point of $S^d(X)$ is an unordered collection of d points $p_1, \ldots, p_d \in X$ with multiplicity allowed. We can send such a collection to its sum: $p_1 + p_2 + \ldots p_d \in Pic(X)$. Then compose with the Abel-Jacobi map to give the map $\Phi^{(d)}$ above: s a map

$$\Phi^{(d)} : S^d(X) \to Jac(X) \quad \{p_1, \ldots, p_d\} \mapsto \Phi(\Sigma p_a).$$

Proposition 2.8. *The symmetric product $S^d(X)$ has a natural structure of a complex manifold of dimension d.*

The maps Φ^d are holomorphic. For $d \geq g$ the map $\Phi^{(d)}$ are onto.

The fibers of $\Phi^{(d)}$ are projective spaces.

For $d > 2g-2$ the map $\Phi^{(d)}$ is a fibration, fibers all isomorphic to a projective space of dimension $d-g$, this being the dimension of a projective space $\mathbb{P}(V)$ where $V \cong H^0(D)$, D any divisor of degree d, according to Riemann-Roch.

3. Topological perspective

Let X be a smooth compact manifold. We can form the Picard variety $H_1(X, S^1) = H_1(X, \mathbb{R})/H_1(X, \mathbb{Z})$ and the Jacobian $H^1(X, S^1) = H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$. These are dual torii of dimension $b_1(X)$. The latter torus can be viewed as $Hom(\pi_1(X), S^1)$ and as such forms the moduli space of flat complex line bundles (or circle bundles) over X.

Choose a basis $\omega_1 \ldots, \omega_{b_1}$ of closed one-forms for $H^1(X, \mathbb{R})$. The integration map

$$\Phi^{(d)} : Sym^d(X) \to Jac(X)$$

continues to make sense.

In general, there is no natural isomorphism $H^1 \cong H_1$ so these tori are different. In the case of an oriented surface, the intersection pairing (or Poincare duality) yields a natural isomorphism between the two torii.

4

... traditional integrals and group laws following Mumford.

5

Flat line bundles, following Atiyah ... and

OVERFLOW:

by sending A line bundle $L \to X$ is called *base point free very ample* if (a) its sections separate points: meaning for any distinct pair $xy \in X$ there is a section with $s(x) = 0$ and $s(y) \neq 0$.

A very ample line bundle induces an embedding $X \to \mathbb{P}(H^0(L))$ by $x \mapsto [s_1(x), \ldots, s_d(x)]$ where the sections s_i are a basis for $H^0(L)$. Most line bundles L over X are ample. [Not all: hyperelliptic curves..]

Theorem 5.1. *For generic X, the canonical bundle K is an ample line bundle, so that the canonical embedding is indeed an embedding $X \to \mathbb{P}^{g-1}$.*