Mumford (Tata Lectures on Theta, II. pp: 3.12-3.13) describes a slick trick to put coordinates at infinity on the standard hyperelliptic curve. We skated the issue earlier. Here I would like to understand this completion. We expressed the surface as $w^2 = p(z)$ where all the roots of p are simple. We will switch to Mumford's notation so s replaces w and t replaces z and $f(t)$ replaces $p(z)$.

The hyperelliptic curve is the subvariety of $\mathbb{C}^2$ defined by

$$s^2 = f(t) \tag{1}$$

with

$$f(t) = (t - a_1)(t - a_2)\dots(t - a_N)$$

a degree N polynomial whose roots a_i are all distinct. Define the integer k by

$$N = \begin{cases} 2k & \text{for} N \text{ even} \\ N = 2k - 1 & \text{for} N \text{ odd} \end{cases}$$

Then the genus of C_1 is

$$g = k - 1.$$

for the hyperelliptic curve.

We showed on the first day of class that the 'no multiple roots conditions" implies that C_1 is non-singular - i.e. a RS, when viewed as a submanifold of $\mathbb{C}^2$. The function t forms a coordinate at points for which $s \neq 0$. In a neighborhood of poitns with $s = 0$ the function s restricted to C_1 forms a holomorphic coordinate. The issue is how to smoothly compactify C_1.

I have skated over the issue - really avoiding it.

Here's the slick change of variables I learned from Mumford. Make the change of variables:

$$\tilde{t} = \frac{1}{t} \tag{2}$$

$$\tilde{s} = \frac{s}{t^k} \tag{3}$$

Note that $\tilde{t} = 0$ corresponds to $t = \infty$.

The t change, eq (2) is the standard change to bring infinity to the origin, forming coordinates centered at infinty. The s change is mysterious at first, but we can derive it from the t change. First, partially re-express $f(t)$ in terms of $\tilde{t}$:

$$f(t) \qquad = \Pi_i(t - a_i) \tag{4}$$

$$= \Pi_i\{t(1 - a_i/t)\} \tag{5}$$

$$= t^N \Pi_i\{(1 - a_i\tilde{t})\} \tag{6}$$

Now split up N as either $2k$ or $2k - 1$. In the even case we see that the equation (1) for the curve is equivalent to

$$(\frac{s}{t^k})^2 = \Pi_i\{(1 - a_i\tilde{t})$$

while in the odd case it becomes

$$(\frac{s}{t^k})^2 = \frac{1}{t}\Pi_i\{(1 - a_i\tilde{t})$$

Thus, with $\tilde{s}$ defined as in eq (3) our curve becomes

$$(\tilde{s})^2 = \Pi(1 - a_i\tilde{t}), \text{ if } N \text{ even}$$

and

$$(\tilde{s})^2 = \tilde{t}\Pi(1 - a_i\tilde{t}),\ \text{if } N \text{ even.}$$

Added points at infinty. To understand the points we've added at infinity by this process, simply set $\tilde{t} = 0$ and solve. In the even case we get $\tilde{s}^2 = 1$, so we have added two points at infinity, $\tilde{s}, \tilde{t} = (\pm 1, 0)$. The curve continues to be 2:1 branched over infinity. In the odd case the equation reads $\tilde{s}^2 = 0$: $\tilde{t} = 0$ or $t = \infty$ is a branch point for the curve– with $(s, t) \mapsto t$ viewed as a map from the curve to the Riemann sphere.

Notation We will call this compactified curve, with points at infinity added the hyper-elliptic curve and denote it by C. When we write $C_1 \subset C$ we mean the affine part sitting inside $\mathbb{C}^2$, so C minus its one or two points at infinity.

0.1. **A basis for the Abelian differentials.** An important theorem asserts that if $g \geq 1$ then the space of global holomorphic one-forms, historically known as "Abelian differentials " has complex dimension g. One nice thing about hyperelliptic curves is we can write a basis down by hand.

Claim

$$\omega_0 = \frac{dt}{s}, \omega_1 = \frac{tdt}{s}, \ldots, \omega_{k-2} = \frac{t^{k-2}dt}{s}$$

form $k - 1 = g$ linearly independent globally holomorphic one-forms on C.

That they are linearly independent is clear, I think. If it is not, try to prove it, by imagining how it could be that $\Sigma c_i\omega_i = 0$ identically in some neighborhood .

That none of these forms has poles is less clear. It sure looks like $1/s$ has poles when $s = 0$. But $s = 0 \iff f(t) = 0$. We have, from the defining equation, that $2sds = f'(t)dt$ or $\frac{dt}{s} = \frac{2ds}{f'(t)}$. Now, at points where $f(t) = 0$ we must use s as a coordinate instead of t. Also $f'(t) \neq 0$ when $f(t) = 0$, since we've assumed no multiple roots. Re-expressed then our one-form is $\frac{2ds}{f'(t)}$ which is perfectly well defined and has no pole. We have, over points with $t \neq \infty$ both s and t are holomorphic functions on C, and that $t^k \frac{dt}{s}$ are also all holomorphic functions on the affine part $C_1 \subset C$.

It remains to verify that none of these forms have poles at infinity. We switch to the other chart. Thus $t = \frac{1}{\tilde{t}}$ so

$$dt = -\frac{d\tilde{t}}{\tilde{t}^2}$$

while

$$s = t^k\tilde{s} \text{ or } s = \frac{1}{\tilde{t}^k}\tilde{s}$$

It follows that, in the charts at infinity we have

$$\frac{dt}{s} = -\tilde{t}^{k-2}\frac{d\tilde{t}}{\tilde{s}}.$$

Let us do the case of N even. Then there are two points at ∞ corresponding to the two roots $\tilde{s} = \pm 1$ to $(\tilde{s})^2 = \Pi(1 - a_i\tilde{t})$. Neither is zero, so that $d\tilde{t}/\tilde{s}$ has no pole at either. Since $t^\ell = \tilde{t}^{-\ell}$ we have that, expressed near infinity:

$$\omega_\ell = -\tilde{t}^{k-\ell-2}\frac{d\tilde{t}}{\tilde{s}}$$

which has no pole at $\tilde{t} = 0$ as long as $k - \ell - 2 \geq 0$ which is to say, for $\ell = 0, 1, 2, \ldots, k-2$. We have shown that all these ω_ℓ are holomorphic over the entirety of C.

Exercise Do the case $N = 2k - 1$ is odd.

Due to the dimension count and the claim we can state the result as follows: An altervative way to state the claim is that any holomorphic differential (as opposed to a meromorphic differential) on the hyperelliptic curve is of the form $\omega = \frac{R(t)dt}{s}$ where $R(t)$ is a polynomial of degree at most $k - 2$. Note $k - 2 = g - 1$ and the space of such polynomials has dimension g.

Where'd this change of variables at infinity come from? What's it mean? Mumford's magical change of coordinates smelled of weighted projective space. The weighted projective space denoted $\mathbb{P}(1, 1, k)$ is the quotient of $\mathbb{C}^3 \setminus \{0\}$ by the action of $\mathbb{C}^*$ which sends (X, Y, Z) to $(\lambda X, \lambda Y, \lambda^k Z)$. Indeed if we rethink of $f(t)$ as a homogeneous polynomial by setting $f(t_1, t_2) = \Pi_i(t_1 - a_i t_2)$ then

$$s^2 = f(t_1, t_2)$$

is weighted homogeneous of degree $2k$ in $\mathbb{P}(1, 1, k)$. C_1 is obtained by setting $t_2 = 1$ while C_2 is obtained by setting $t_1 = 1$. To work out the change of variables set $[X, Y, Z] = [t, 1, s] = [1, \tilde{t}, \tilde{s}]$ where, as per usual projective space $[X, Y, Z]$ denotes the $\mathbb{C}^*$ orbit through X, Y, Z. The C_1 part is thus in the chart $Y \neq 0$ while the C_2 part is in the chart $X \neq 0$. We first work out t, s. We must have that $t = X/Y, s = Z/Y^k$ we need as coordiantes $\mathbb{C}^*$ invariant functions. Similarly we see that $\tilde{t} = Y/Z, \tilde{s} = Z/X^k$. So $\tilde{t} = 1/t$ while $\tilde{s} = Z/(Y^k)(X^k/Y^k) = s/t^k$, explaining Mumford's magical change of coordinates. I found this trick by stumbling about the web, after googling "hyperelliptic curve in weighted homogeneous " and getting the following reference of Miles Reid from the early 1990s: `https://homepages.warwick.ac.uk/~masda/surf/more/grad.pdf` The example is mostly worked out on p. 2. There he writes: ". Note that it is not a wise move to take the projective closure of C_1 in straight $\mathbb{CP}^2$ - it leads to a complicated singularity at infinity, and general confusion."

Notes on impossibility of embedding in straight $\mathbb{CP}^2$. This unwise move is **exactly** what most texts seem to do when dealing with the points at infinity on the hyperelliptic curve! The standard trick is to homogenize the polynomial thus defining a compact (irreducible) curve in $\mathbb{CP}^2$ containing C_1 as its affine part. See Brieskorn-Knorrer, top p. 620, example (5). This yields a (rather deep and non-generic!) singularity at the added point at infinity.

Indeed, there is NO WAY to smoothly embed 'most hyperelliptic curves in $\mathbb{CP}^2$. A smooth curve in $\mathbb{CP}^2$ is defined as the zero locus of some homogeneous polynomial in three variables. If d is the degree of the variety then

$$g = \frac{(d-1)(d-2)}{2}$$

This is Lemma 4, p. 611 of Breiskhorn-Knorrer and follows fairly simply from the Riemann-Hurwitz. See what B-K call 'Theorem 5' a bit after lemma 4, or see the proof of lemma 4 there. Kirwan is another good source for this.