A plane curve C here means the subset of the projective plane $\mathbb{CP}^2$ given as the vanishing of a homogeneous polynomial in 3 variables:

$$C := \{[X,Y,Z] \in \mathbb{CP}^2 : P(X,Y,Z) = 0\}$$

Write $\tilde{C} := \{(X,Y,Z) \in \mathbb{C}^3 : P(X,Y,Z) = 0, (X,Y,Z) \neq (0,0,0)\} \subset \mathbb{C}^3 \setminus \{0\}$ for this same solution set, viewed in $\mathbb{C}^3$ so that

$$C = \pi(\tilde{C}).$$

Here $[X,Y,Z]$ are homogeneous coordinates on $\mathbb{CP}^2$ so that we can write

$$\pi(X,Y,Z) = [X,Y,Z]$$

for the projection $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2$. We say C is smooth, or 'non-singular' if 0 is a regular value for P, viewed as a map $P : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}$. Then the holomorphic implicit function theorem tells us that $\tilde{C}$ and C are holomorphic manifolds, with C a Riemann surface. Remember what it means for 0 to be a regular value here: $dP(X,Y,Z) := P_X dX + P_Y dY + P_Z dZ \neq 0$ as a one-form on $\mathbb{C}^3 \setminus 0$ whenever $P(X,Y,Z) = 0$ and $(X,Y,Z) \neq (0,0,0)$.

Theorem 0.1 (Degree-Genus formula)**.** *If P has degree d then the smooth curve $C = \{P = 0\}$ has genus $g = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$.*

The space $\Omega^{(1,0)}(C)$ of all holomorphic one-forms on C consists of the one-forms

$$\omega_Q := Q\omega \tag{1}$$

on $\tilde{C}$ pushed down to C, where Q is any homogeneous of degree $d-3$ polynomial and ω is any one of :

$$\omega_1 = \frac{YdZ - ZdY}{P_X}, \omega_2 = \frac{ZdX - XdZ}{P_Y}, \omega_3 = \frac{XdY - YdX}{P_Z}$$

All three one-forms are equal along $\tilde{C}$ and annihilate the 'vertical distribution' $ker(d\pi)$.

As an important corollary we find that not all genuses -hence not all topological curves - can be realized as smooth planar curves for the simple reason that not all integers g have the form $\binom{d-1}{2}$. Indeed $g = 2$ cannot be so represented!

Let us see how the representation of one-forms, equation (1) arises. A one-form on $\mathbb{C}^3 \setminus \{0\}$ pushes down to $\mathbb{CP}^2$ if and only if it is "basic" and 'invariant'. *Basic* means that the kernel of $d\pi$ lies in the kernel of α. *Invariant* means invariant under scaling $\tau_\lambda^* \alpha = \alpha$ where $\tau_\lambda(X,Y,Z) = (\lambda X, \lambda Y, \lambda Z)$ for $\lambda \in \mathbb{C}^*$. The kernel of $d\pi$ at (X,Y,Z) is the $\mathbb{C}$-span of (X,Y,Z). The one-forms $YdZ - ZdY, ZdX - XdZ, XdY - YdX$ all kill this kernel and they span the space of of one-forms which kill the kernel. (But they are not a basis! There are too many of them!) Now these forms are homogeneous of degree 2 with respect to scaling:

$$\tau_\lambda^*(YdZ - ZdY) = \lambda^2(YdZ - ZdY), etc.$$

For a form to be invariant it must be homogeneous of degree 0 with respect to scaling. The derivatives P_Y etc are homogeneous of degree $d-1$ hence the forms ω_i are homogeneous of degree $2-(d-1) = 3-d$. Multiplying them by a homogeneous polynomial $Q(X,Y,Z)$of degree $d-3$ yields a form which is invariant.

Remark. This business about 'basic' plus 'invariant' implies projectable holds for any principal G-bundle. In our case $G = \mathbb{C}^*$.

To see that the forms are all equal on $\tilde{C}$ we find a complex basis for $T\tilde{C}$. Now the tangent space $T\tilde{C}$ is the kernel of $dP = P_X dX + P_Y dY + P_Z dZ$. One element of the basis is (X, Y, Z) which spans $ker(d\pi)$. Indeed $dP(X, Y, Z)(X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}) = XP_X + YP_Y + ZP_Z = (d)P$ by one of Euler's identities. On this element all of the ω_i are zero, hence agree. For the other basis element of $T\tilde{C}$ we can take any one of: $E_Z = (P_Y, -P_X, 0)$ - by which we mean

$$E_Z := P_Y \frac{\partial}{\partial X} - P_X \frac{\partial}{\partial Y}$$

or the obvious cyclic permutations thereof $(0, P_Z, -P_Y)$ or $(P_Z, 0, -P_X)$. Lets evaluate our three one forms

$$\omega_1 = \frac{YdZ - ZdY}{P_X}, \omega_2 = \frac{ZdX - XdZ}{P_Y}, \omega_3 = \frac{XdY - YdX}{P_Z}$$

on E_Z. We have that $\omega_3(E_Z) = \frac{-XP_X - YP_Y}{P_Z}$. But by Euler's identity , on C we have that $-XP_X - YP_Y = ZP_Z$ so that $\omega_3(E_Z) = Z$. And $\omega_2(E_Z) = \frac{ZP_Y}{P_Y} = Z$. Finally $\omega_1(E_Z) = \frac{-Z(-P_X)}{P_X} = Z$. All three one-forms are equal along a dense open subset of $\tilde{C}$, hence are equal on all of $\tilde{C}$.

We have established that all forms of the given type of equation (1) define meromorphic one -forms on $\tilde{C}$. They are in fact holomorphic, i.e. have no poles because, at any given point one of P_X, P_Y or P_Z is non-zero and we can choose the corresponding representative ω_i accordingly.

Genus computation. A. We count $\ell(K)$, the dimension of the space of holomorphic forms on C, which is the vector space of all forms given by equation (1). This space is in linear bijection with the homogeneous degree $d-3$ polynomials Q in 3 variables. so need the dimension of this space. We set $N = d - 3$ and proceed. Dehomogenization (setting $Z = 1$) defines a linear isomorphism between the space of all degree N homogeneous polynomials in 3 variables and the space of *all* polynomials of degree N *or less* in 2 variables. We can count the dimension of the latter space by arranging the monomials $x^i y^j$ in a triangular pattern:

$$\text{degree } N : x^N, x^{N-1}y, x^{N-2}y^2, \ldots, y^N$$

$$\text{degree } N-1 : x^{N-1}, x^{N-2}y, x^{N-2}y^2, \ldots, y^{N-1}$$

$$\ldots$$

$$\text{degree } 1 : x, y$$

$$\text{degree} 0 : 1.$$

We see the space of polynomials which are homogeneous of a given degree m in two variables has dimension $m + 1$, and, summing up, those of degree N form a space of dimension $N + 1$, those of degree $N - 1$ a space of degree N so that our space of polynomials in x and y of degree less than or equal to N has dimension $(N + 1) + N + (N - 1) + \ldots + 2 + 1 = \binom{N+2}{2}$. Now return to $N = d - 3$ so that $N + 2 = d - 1$ to get that the space of holomorphic forms has dimension $g = \binom{d-1}{2}$.

As an independent check, we can compute the degree of a nice ω_Q. Take for $Q = \ell(X,Y,Z)^{d-3}$ where ℓ is any linear function. For example $\ell = X$ would do. By the simplest form of Bezout, $C \cap \{\ell = 0\}$ consists of d points, when counted with multiplicity. By wiggling ℓ we can insure that these points are all distinct. Thus ω_Q vanishes at precisely d points. Each such point p is a zero of multiplicity $d-3$ for ω_Q, since $\omega_Q = \ell^{d-3}\omega_i$. So, $deg(\omega_Q) = d(d-3)$. Now $d(d-3) = d^2 - 3d = (d^2 - 3d + 2) - 2 = 2g - 2$ where $g = \frac{1}{2}(d^2 - 3d + 2) = \binom{d-1}{2}$.