1. .

We continue with X compact Riemann surface and D a divisor on X.

Theorem 1.1.

- *1)* $deg(D) < 0 \implies L(D) = 0$
- *2)* $L(D - p) \subset L(D)$
- *3)* $dim(L(D)/L(D - p))$ *is either* 0 *or* 1

Corollary 1.2. $\ell(D) \leq deg(D) + 1$

Proof of theorem (1): Suppose that $deg(D) < 0$ and $f \in L(D)$ is a meromorphic function not identically zero. Then $div(f) \geq -D$. Observe that the degree mapping on divisors is order preserving: $E \geq F \implies deg(E) \geq deg(F)$. Therefore $deg(div(f) \geq deg(D) > 0$, contradicting the fact $div(f) = 0$. Thus $f = 0$.

(2). Say $f \in L(D-p)$. Then $div(f) \geq -(D-p) = p-D > -D$. So $div(f) \geq -D$ and $f \in L(D)$.

(3) For simplicity, we begin with the case where $D(p) = 0$. Consider the function $p \mapsto f(p)$. It is a linear $\mathbb{C}$-valued function on $L(D)$. Its kernel is $L(D - p)$, since, in this case $f \in L(D)$ and $f \in L(D - p)$ if and only if $f(p) = 0$. Thus $L(D - p) \subset L(D)$ is the kernel of a linear function. If that linear function is non-trivial then $L(D - p)$ has codimension 1 in $L(D)$ and so $L(D)/L(D - p)$ has dimension 1. If the linear functional is trivial, i.e. identically zero, then $L(D - p) = L(D)$ and so $L(D)/L(D - p)$ has dimension 0.

For the general case, choose a local holomorphic coordinate z centered at p. If $D(p) = -m$ and $f \in L(D)$ then we must have that $f = a_m z^m + a_{m-1} z^{m-1} + \ldots$. The function $f \mapsto a_m$ is a well-defined linear functional on $L(D)$. And $f \in L(D - p)$ if and only if $a_m = 0$. Thus again we have a linear functional whose vanishing defines $L(D - p)$.

Proof of the corollary. Start with the case of $deg(D) = 0$. $D - p$ has negative degree. Thus $\ell(D - p) = 0$. From (3) we have $\ell(D)$ is either 1 or 0.

For general D with $deg(D) > 0$ proceed by induction on the degree of D.

Corollary 1.3. *For D an effective divisor* $\ell(nD) \leq n deg(D) + 1$.

Recall a divisor is called "effective" if $D > 0$.

Proof. $deg(nD) = n deg D$. Now use the previous corollary.

1.1. Corollaries of the theorem plus Riemann-Roch.

We have the following immediate corollaries of the above discussion, and Riemann-Roch.

Corollary 1.4. *If* $deg(D) > 2g - 2$ *then* $\ell(D) = deg(D) + 1 - g$

Indeed, $deg(K - D) = deg(K) - deg(D) = 2g - 2 - deg(D)$ so in this case $deg(K - D) < 0$ and by (1) of the theorem $\ell(K - D) = 0$.

Corollary 1.5. *Let $p_1, p_2, \ldots p_k, p_{k+1}$ be a list of points on X, not necessarily distinct. Form the sequence of divisors $D^{(k)} = p_1 + p_2 \ldots + p_k$. Then the integers $\ell(D^{(k)})$ form a non-decreasing list starting from $\ell(D^0) = 1$, jumping by at most 1 as k increases until $k = 2g - 1$. For $k > 2g - 2$ we have $\ell(D^{(k)}) = k + 1 - g$ so that for k increasing from $2g - 1$ the jump in $\ell(D^{(k)})$ is exactly 1 at each step.*

Indeed: $deg(D^{(k)}) = k$ and $D^{(k-1)} = D^{(k)} - p_k$.

The corollary asserts that if we start by imposing exactly $2g - 1$ poles for our meromorphic functions, (so $f \in L(D^{(2g-1)}$) then each additional pole allowed will add exactly one new function to our function space.

1.2. **Transcendence degree from R.-R.** We are going to use R-R to prove that the transcendence degree of the function field of a Riemann surface is 1. The key is the observation that $\ell(nD) \leq n\ell(D)$ Indeed, $deg(nD) = ndeg(D)$ and for any divisor $\ell(E) \leq deg(E) + 1$. Thus, setting $C = deg(D) + 1$ we have $\ell(nD) \leq nC$.

Now, suppose that f, g are meromorphic functions which are everywhere algebraically independent.

NOW: draw out tables of d vs $L(D)$, $d = 0, 1, 2$, for various genuses g !

$g = 0, 1, 2, 3...$!