1. .

We continue with X compact Riemann surface .

Proposition 1.1. *If f is a meromorphic function on X then the number of zeros of f equals the number of poles, both counted with multiplicity*

Proof. View f as a map $X \to \mathbb{CP}^1$. Then both numbers are equal to the degree of f, which is a topological invariant of f and equals the number of points in the typical preimage of f.

Example 1.2. *A polynomial $p(z)$, $z \in \mathbb{C}$ has n zeros. Viewed as a meromorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$ it has a pole of order n at ∞.*

In Morse theory the following fundamental theorem is proved: If X is a closed (= compact without boundary) connected manifold and $f : X \to \mathbb{R}$ is a smooth function with exactly two critical points, both of which are nondegenerate, then X is homeomorphic to an n-sphere.

Compare with:

Proposition 1.3. *If our compact Riemann surface admits a meromorphic function with exactly one pole and that pole is simple, then X is the sphere.*

Let $p \in X$ be the pole. Being simple, and unique, the degree of f is 1, Hence away from p, f maps $X \setminus \{p\}$ in a 1;1 holomorphic fashion onto $\mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$. Add back in p which is sent to infinity, to see that f is a holomorphic isomorphism $X \to \mathbb{CP}^1$.

Corollary 1.4. *If X has genus greater than zero and f is a meromorphic function with exactly one pole, then that pole is not simple.*

For example, on the elliptic curves $X = \mathbb{C}/\Lambda$ there is a meromorphic function with exactly one pole at the origin. That pole is of order 2 and that function is unique up to scale and is the Weierstrass $\mathfrak{P}$ function.

2. Differentials

After meromorphic functions, the meromorphic one-forms comprise the next most basic space for analysis on a Riemann surface . In local coordinates z such a one-form has the shape $f(z)dz$ where $f(z)$ is a meromorphic function. If w were to be another holomorphic coordinate with domain overlap with the coordinate z then we would have $dz = \frac{dz}{dw}dw$. It follows that $g(w)dw = f(z)dz$ if and only if, on the overlap $f(z(w))\frac{dz}{dw} = g(w)$. There are more intrinsic perhaps slicker ways to define meromorphic one-forms, but let us hold off on them for a moment.

Proposition 2.1. *The only holomorphic one-form on the sphere $\mathbb{CP}^1$ is* 0.

We ask the reader to contrast this with the space of holomorphic functions on $\mathbb{CP}^1$, which is one-dimensional, consisting of the constants.

Proof. The overlap relation $w = \frac{1}{z}$ shows us that the apparently global differental form dz has a pole of order 2 at ∞, since $d(1/w) = -dw/w^2$. To cancel this pole, we have to multiply dz by a polynomial in w of order 2 at least and $w^2 = \frac{1}{z^2}$ any such attempt to cancel the pole will lead to new poles in the finite z plane

On the other hand we have:

Proposition 2.2. *The space of holomorphic one-forms on a torus $\mathbb{C}/\Lambda$ has complex dimension* 1*.*

Proof. Indeed the expression dz is invariant under translations and so descends as a one-form on the torus and spans the space of holomorphic one-forms.

The previous two propositions concerned genus $g = 0$ and $g = 1$ and are special cases of (A) of the following theorem. More generally

Theorem 2.3. *If X is a compact Riemann surface of genus g then*

A). The space of holomorphic one-forms on X forms a finite dimensional vector space of complex dimension g

B). If ω is a meromorphic differential on a Riemann surface X then the number of zeros of ω minus the number of poles, counted with multiplicity is $2g - 2$.

Note: $2g - 2 = -2$ for $\mathbb{CP}^1$ and $2g - 2 = 0$ for the torus, fitting the data we have so far.

3. Divisors

A divisor is a formal finite sum of points on X: $D = \Sigma n_p p$ where the $n_p = 0$ for all but a finite number of points of X. Here are two equivalent definitions of a divisor. (A) A divisor is an element in the free Abelian group generated by X. (B) A divisor is a function $X \to \mathbb{Z}$ which is zero at all but a finite number of points.

If f is a meromorphic function we write (f) or $div(f)$ for the sum of its zeros minus its poles, counted with multiplicities

$$(f) = \Sigma ord_p(f)p.$$

Recall that $ord_p(f) = 0$ if $f(p) \neq 0, \infty$, that in $\mathbb{C}$ we have $ord_0(z^k) = k$ and generally

$$ord_p(fg) = ord_p(f) + ord_p(g).$$

a pole $1/z^k$ counts as order $-k$.

If α is a meromorphic differential then we define its divisor similarly. The order of dz is taken to be zero, at any point p in the domain of the holomorphic chart z and we use the local formula $ord_p(gdz) = ord_p(g) + ord_p(dz) = ord_p(q)$. This gives (α) a well defined meaning as a divisor.

Definition 3.1. *For any divisor D set*

$$deg(D) = \Sigma n_p \in \mathbb{Z}.,$$

thus defining a homomorphism from the group of divisors to the integers.

Example 3.2. *Proposition 1.1 asserts that if $D = (f)$ is the divisor of a meromorphic function then $deg(D) = 0$.*

Part B of Theorem 2.3 asserts that if $D = (\alpha)$ is the divisor of a meromorphic one-form then $deg(D) = 2g - 2$.

We have a natural ordering on the divisors induced by the ordering of the integers: $\Sigma n_p p \geq \Sigma m_p p$ if and only if $n_p \geq m_p$ at all points. The ordering definition is even simpler when we think of divisors as an integer valued functions: $D \geq E$ if and only if for all $p \in X$ $D(p) \geq E(p)$.

The theorem that no Riemann surface admits a global holomorphic function asserts that: if $D \geq 0$ then the only possible meromorphic functions f on X for which $(f) = D$ are the constant functions. With this in mind, we introduce a function space associated to a divisor:

$$L(D) := \{f : meromorphic, (f) \geq -D\}.$$

Exercise 3.3. *if $D < 0$ then $L(D) = 0$.*

Example 3.4. *a. For any compact X we have $L(0) = \mathbb{C}$, the space of constant functions.*

b1. For $X = \mathbb{CP}^1$ and $p_\infty = \infty \in \mathbb{CP}^1$, in one of the HWs we asked to show: $L(p_\infty) =$ space of linear functions on $\mathbb{C}$. More generally, in the same context:

b2. For $X = \mathbb{CP}^1$, $p_\infty = \infty \in \mathbb{CP}^1$ and $d > 0$ an integer we have that $L(dp_\infty) \cong$ the space of degree d polynomials on $\mathbb{C}$.

Let us pause to check part of example (b1). For $a \neq 0$ the function $az + b$ has for its divisor $(az + b) = p_0 - p_\infty \geq -p_\infty$, consequently, $az + b \in L(p_\infty)$

Definition 3.5. *Write $\ell(D) = dim L(D)$.*

The famous Riemann-Roch theorem concerns the behaviour of $\ell(D)$ as a function of D. Before we state it, let us content ourselves with a special case.

Theorem 3.6. *For $X = \mathbb{CP}^1$ and D any divisor on $\mathbb{CP}^1$ we have $\ell(D) = 0$ if $deg(D) < 0$ and $\ell(D) = deg(D) + 1$ if $deg(D) > 0$.*

The space of polynomials of degree d has dimension $d + 1$. And $deg(dp_\infty) = d$. Consequently b2. is a special case of this theorem.

4. Riemann-Roch statement

To state the Riemann-Roch theorem we need a fact regarding meromorphic one-forms α and their divisors (α).

Proposition 4.1. *If α, β are meromorphic one-forms, neither identically zero, then $\ell((\alpha)) = \ell((\beta))$*

Compare Miranda, p. 138, Lemma 2.3, part (b).

We write K for the divisor associated to any meromorphic one-form. The above proposition asserts that the integer $\ell(K)$ makes sense, if we define it to be $\ell((\beta))$.

The proposition immediately follows directly from the following lemma, and an observation.

Lemma 4.2. *(A) For any divisor D we have that $\ell(D)$ is a finite integer: i.e. $L(D)$ is finite dimensional.*

(B) If g is a nonzero meromorphic function then multiplication by g^{-1} defines a linear isomorphism: $m_g^{-1} : L(D) \to L(D + (g))$

Proof of lemma. (A) is hard and we put it off. For (B): Suppose that $f \in L(D)$. Thus $(f) \geq -D$. Consequently $(f) - (g) \geq -D - (g) = -(D + (g))$. But $(g^{-1}f) = -(g) + (f)$. So $(g^{-1}f) \in L(D + (g)$. To show the map is invertible multiply by g to map from $L(D + (g))$ to $L(D)$.

Proof of Proposition. Observe, by coordinate computation, that if α, β are meromorphic one-forms as in the proposition then their quotient $g = \alpha/\beta$ is a well-defined meromorphic function on X. Since $g^{-1}\alpha = \beta$ the lemma, part (B) implies that $L((\alpha)) = L((\beta))$ as complex vector spaces. QED

Inspired by the propositions we write K to represent the divisor of any meromorphic function.

Theorem 4.3 (Rieman-Roch)**.** $\ell(D) - \ell(K - D) = deg(D) - g + 1$

Corollary 4.4. $\ell(K) = g$

Proof of cor. Take $D = 0$. We have seen that $L(0) = \mathbb{C}$ so that $\ell(0) = 1$. The degree d of 0 is 0 and $K - 0 = K$. Apply Riemann-Roch.

This corollary gives us an analytic meaning for the genus: it is the dimension of the space of holomorphic differentials on X.

5. Jacobian; Linear equivalence; Picard group..

Definition 5.1. *A principal divisor is the divisor of a meromorphic function.*

Definition 5.2. *Two divisors are "linearly equivalent" if they differ by a principal divisor.*

Write the equivalence relation of being 'linearly equivalent' as $E \sim D$. The definition asserts that two divisors D, E satisfy $E \sim D$ if $E - D = (f)$ for some meromorphic function f.

Definition 5.3. *The space of equivalence classes of divisors under linear equivalence is called the "Picard group" of X and is denoted by $Pic(X)$*

Since the principal divisors form a subgroup of the Abelian group of divisors, the Picard group also forms an Abelian group, namely $Div(X)/(\text{principal divisors})$.

Proposition 1.1 asserts that the principal divisors lie in the kernel of the degree homormorphism. It follows that degree induces a well-defined homomorphism, which, by slight abuse of notation, we still call "degree":

$$deg : Pic(X) \to \mathbb{Z}.$$

Write $Pic(X)_d$ for those elements of the Picard group having degree d.

Theorem 5.4. *For each $d \in \mathbb{Z}$ we have that $Pic(X)_d$ has the structure of a complex torus $\mathbb{C}^g/\Lambda(X)$ of complex dimension g, g being the genus of X. The lattice $\Lambda(X)$ is sometimes called the "period lattice' . This torus is called the "Jacobian" of the Riemann surface X.*

Remarks. In HW 4 we ask you to prove the theorem in the case $g = 0$ of the Riemann sphere.

When $g = 1$ there is an almost canonical identification between X and its Jacobian.