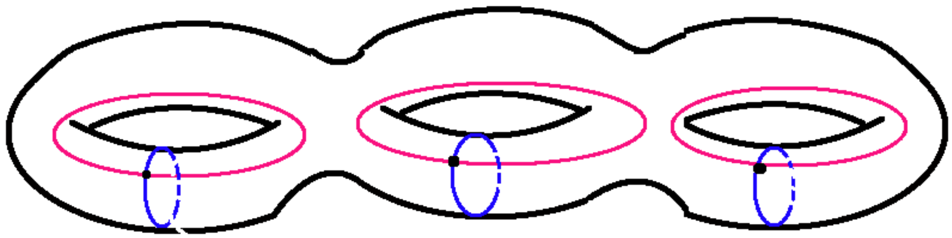

(a) intersection matrix for a genus 3 curve: The a curves are in pink, the blue curves in blue.

1. Topological Background for Riemann-Roch

We continue with a X compact Riemann surface . Let g be the genus of X. The Euler characteristic of X is

$$\chi(X) = 2 - 2g.$$

Now

$$\chi(X) = dim H_0(X, \mathbb{R}) - dim H_1(X, \mathbb{R}) + dim H_2(X, \mathbb{R})$$

and the dimension of H_0 and H_2 are both one, so that

$$2g = dim H_1(X, \mathbb{R})$$

Now the middle homology comes with the intrinsic intersection form:

$$H_1(X, \mathbb{R}) \otimes H_1(X, \mathbb{R}) \to \mathbb{R}$$

which is a non-degenerate symplectic form and is integer on the integer lattice $H_1(X, \mathbb{Z}) \subset H_1(X, \mathbb{R})$. We can choose a basis of $2g$ closed curves denoted $a_1, \ldots, a_g, b_1 \ldots b_g$ for $H_1(X, \mathbb{Z})$. called the a and b cycles which are adapted to the decomposition of X as the connected sum of g torii – so that

$$a_i \cdot b_i = 1 = -b_i \cdot a_i$$

$$a_i \cdot b_j = b_j \cdot a_i = 0, i \neq j$$

$$a_i \cdot a_j = b_i \cdot b_j = 0, \text{ all } i, j$$

See figure for the case $g = 3$.

1.0.1. *..to holomorphic one-forms.* Now we have $H_1(X,\mathbb{C}) = H^1(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{2g}$. Using the deRham interpretation, $H^1(X,\mathbb{C})$ is interpreted as closed complex-valued one-forms, modulo exact complex valued one form.

We have the further split:

$$H^1(X,\mathbb{C}) = H^{(1,0)}(X,\mathbb{C}) \oplus H^{(0,1)}(X,\mathbb{C}) = \text{ holomorphic} + \text{anti-holomorphic}.$$

1.0.2. *Intersection Pairing on a Riemann surface.* On one forms we have: $\alpha \wedge \beta = -\beta \wedge \alpha$ so that the the intersection pairing

$$H^1(X,\mathbb{R}) \times H^1(X,\mathbb{R}) \to \mathbb{R}$$

is symplectic: $\int_X \alpha \wedge \beta = -\int_X \beta \wedge \alpha$. Moreover, it restricts to an integer valued symplectic form on the integer lattice $H^1(X,\mathbb{Z}) \subset H^1(X\mathbb{R})$. In this way, the group of integer valued $2g$ by $2g$ symplectic matrices becomes a crucial algebraic object in the study of families of genus g Riemann surfaces.

When $g = 1$ this is the group $SL(2,\mathbb{Z})$.

Since non-degenerate symplectic forms only exist in even dimensions, we see that , indeed we must have that $H^1(X,\mathbb{R})$ is even.

1.1. **1, 0 and 0,1 forms.** First on $\mathbb{C}$. Let $z = x + iy$ be the coordinate. Set $dz = dx + idy$ and $d\bar{z} = dx - idy$. Then we can write any one-form on $\mathbb{C}$ with values in $\mathbb{C}$ as a complex linear combination of dz and $d\bar{z}$. For example $dx = \frac{1}{2}(dz + \bar{d}z)$. It is worth taking a moment to be careful about the linear algebra here. We view $T_p\mathbb{C}$ as a *real* vector space. Then the space of *complex* valued real linear functionals $T_p\mathbb{C} \to \mathbb{C}$ is canonically identified with $T_P\mathbb{C}^*) \otimes_{\mathbb{R}} \mathbb{C}$ which is a *two-dimensional* complex vector space. It breaks up into two components, one spanned by dz the other spanned by $d\bar{z}$. Now write $J(x,y) = (-y,x)$ for the real linear operator which is multiplication by i on $T_P\mathbb{C}$. The element dz is characterized by $dz(Jv) = idz(v)$ while the element $\bar{d}z$ satisfies $dz(Jv) = -idz(v)$. Note: if $v = (v_1, v_2)$ then $dz(v) = v_1 + iv_2$.

The span of dz is called the space of $(1,0)$ forms. The span of $d\bar{z}$ is the space of $(0,1)$ forms.

Now we can write any $(1,0)$ differential form

$$\alpha = f(z,\bar{z})dz = f(x,y)dz$$

A simple computation shows that

$$d\alpha = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz +$$

If z is a local holomorphic coordinate then $dz = dx + idy$ is a basis for the '(1,0) forms, while $d\bar{z} = dx - idy$ is a basis for the (0,1) forms. Any complex valued one form is a sum of a (1,0) and (0,1) form ...

Given any real symplectic vector space $\mathbb{V}$ we can choose a complex structure on $\mathbb{V}$ and a Hermitian metric on the resulting complex vector space such that such that $\langle v,w \rangle = (v,w) + i\omega(v,w)$. This is done by defining the metric $\langle v,w \rangle_{\mathbb{R}} = \omega(v,Jw)$ and ...

1.2. **(Co)homological basics.** Let X be a compact connected oriented n-manifold. We review some of the basic facts of the homology and cohomology of X. Eqs **??** hold. Moreover:

$$dim H_k(X,\mathbb{R}) = dim(H^k(X,\mathbb{R}) = dim H^{n-k}(X,\mathbb{R}).$$

in case that $H_k(X,\mathbb{Z})$ is torsion-free. We think of elements Σ of $H_k(X,\mathbb{R})$ as k-dimensional finite simplicial complexes $\Sigma \subset X$ with real coefficients, and with two such equivalent if they bound a $k+1$-dimensional simplicial complex Y: $\Sigma \sim \Sigma' \iff \Sigma - \Sigma' = \partial Y$. Instead of a simplicial complex, we can think of Σ as a cell complex or even a k-dimensional manifold. (For general X it may not be be possible to represent every homology class by that of a k-dimensional manifold.) We think of elements of $H^k(X,\mathbb{R})$ as k-dimensional integrands, that is to say, degree k closed differential forms ω on X modulo exact differential forms: $\omega \sim \omega' \iff \omega - \omega' = d\beta$. By a small homotopy we can perturb any k-dimensional simplicial complex to one for which the faces are smoothly embedded, so that we can integrate over it. Then the dimensional equalities above are induced by integration-defined nondegenerate pairings

$$(H^k(X,\mathbb{R}) \times H^k(X,\mathbb{R}) \to \mathbb{R}; ([\Sigma],[\omega]) \to \int_\Sigma \omega$$

and

$$H^k(X,\mathbb{R}) \times H^{n-k}(X,\mathbb{R}) \to \mathbb{R}; (\alpha,\beta) \to \int_X \alpha \wedge \beta$$

Stokes' theorem asserts that these pairings are well-defined, independent of representatives Σ of $[\Sigma]$, ω of $[\omega]$, etc.

It is often helpful to view the last pairing geometrically as the intersection between a k-dimensional cycle and an n-k dimensional cycle. We can wiggle the two cycles so they intersect transversally, in which case they intersect in a finite number of points p and these points may be taken to be the interiors of the faces of the simplices. Each simplex is oriented, so we can assign a $+$ or $-$ sign to the points according to whether or not the direct sum orientation agrees with the orientation of T_pX.

1.2.1. *Lattice structure.* We can define homology and cohomology with coefficients in any ring. The universal coefficient theorem asserts that taking the ring to be the integers is 'universal': if we know the homology over $\mathbb{Z}$ then we know the homology and cohomology over any other ring. When that ring is a field $\mathbb{F}$ of characteristic zero we get that $H_k(X,\mathbb{F}) = H_k(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}$ so $H_k(X,\mathbb{Z})/(torsion$ embeds in $H_k(X,\mathbb{F})$ as a lattice. This lattice structure is quite useful and gives a certain rigidity to homology and cohomology.

In the case of Riemann surface we do not need to worry about torsion: there is no torsion in any of the homology groups. We can suppose $H_1(X,\mathbb{Z})$ embedded in $H_1(X,\mathbb{R})$ which in turn embeds in $H^1(X,\mathbb{C})$. Similarly we have

$$H^1(X,\mathbb{Z}) \subset H^1(X,\mathbb{R}) \subset H^1(X,\mathbb{C}) = H^1(X,\mathbb{R}) \oplus iH^1(X,\mathbb{R})$$

It is useful to look at the intersection pairing on homology:

$$H_k(X,\mathbb{Z}) \times H_{n-k}(X,\mathbb{Z}) \to \mathbb{Z}$$

To see this pairing geometrically , take a k-dimensional cycle Σ and an $n-k$ dimensional cycle W. We can wiggle the two cycles so that their faces are smooth and intersect each other transversally in interior points. In that case the number of these intersection points is finite. Each simplex is oriented, so we can assign a $+$ or $-$ sign to the points according to whether or not the direct sum orientation agrees with the orientation of T_pX. Summing these numbers gives the intersection pairing.

Finally, the lattice structure of $H_k(X,\mathbb{Z}) \subset H_k(X,\mathbb{R})$ induces a lattice structure $H^k(X,\mathbb{Z}) \subset H^k(X,\mathbb{R})$: a closed differential form is in the integer lattice if its integral over any integral chain $\Sigma \in H_k(X,\mathbb{Z})$ is an integer.

1.2.2. *Intersection Pairing on a Riemann surface.* On one forms we have: $\alpha \wedge \beta = -\beta \wedge \alpha$ so that the the intersection pairing

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In our case, the complex structure is the Hodge $*$ operator. ....

We may think of $H^1(X,\mathbb{R})$ as formal sums of loops $S^1 \to \mathbb{R}$ where two loops a,b are equivalent if there is a map of an annulus $S^1 \times [0,1] \to X$ whose boundary is $a - b$, and more generally $\Sigma a_i = 0$ if there is an oriented surface S with boundary ∂S and a map $h : S \to X$ such that $h(\partial S) = \Sigma a_i$.