Riemann Roch:

Let's write

$$\ell(D) = \dim L(D)$$

$$\ell(D) - \ell(K-D) = d + 1 - g$$

$$K = \text{can. bundle} = T^*M \simeq T^*M^{(1,0)}$$

$$L(D) = \{ f \in \mathcal{M}er(M) : D + (f) \geq 0 \}.$$

take $D = 0$, $L(0) \simeq \mathbb{C}$

so $d = 0$ $\ell(0) = 1.$

$$1 - \ell(K) = 0 + 1 - g.$$

or

$$\Rightarrow \ell(K) = g.$$

indeed:

$L(K) \simeq$ holo. one-forms.

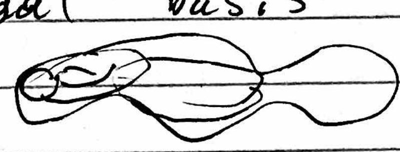
Hodge de Rham theory.

$H^1(M, \mathbb{R})$ via $\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2$

$\cong$

$H_1(M, \mathbb{R})^*$ has real dim $2g$.

dual basis



a_i, b_i

$$\omega \mapsto \int_{a_i} \omega.$$

~~Hodge~~:

$$H^1_{\mathbb{C}} \simeq H^{1,0} \oplus H^{0,1}$$

$f(z)dz$'s $\quad \overline{f(z)dz}$'

~~d.~~

$\dim_{\mathbb{C}}$: $\quad 2g = g + g$

Next

$D = K, \qquad K - K = 0.$

$\ell(K) = \ell(0) = d + 1 - g.$

~~$\ell($~~

$g - 1 = d + 1 - g.$

$\Rightarrow 2(g-1) = d$

Recall: $\chi(M) = V - E + F$
$= 2 - 2g$
$= -2(g-1)$

Reason: $\chi(M)$ is 'd' for TM.

$T^*M = (TM)^*$.

In general:

Divisors $\longleftrightarrow$ Line bundles.
This correspondence, let's write

$$D \longleftrightarrow \mathcal{L}_D.$$

satisfies

$$-D = \mathcal{L}_D^*.$$

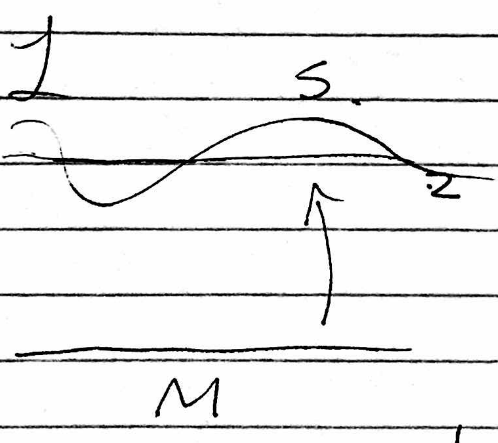
$$D+E \longleftrightarrow \mathcal{L}_D \otimes \mathcal{L}_E.$$

$$\deg(D) = \text{chern}\ \#\ \mathcal{L}_D.$$

$$\deg(-D) = -\deg D.$$

so $\underbrace{\deg(T^*M)}_{2g-2} = -\underbrace{\deg(TM)}_{2-2g}$

$\Rightarrow$ a top. meaning of chern #:



perturb zero section to keep ⋔ to Z.
ie choose a ⋔ section.

$$\text{chern}\ \# = \sum_{p:\, s(p)=0} (\pm)$$

$\leftarrow$ dep on ule

I : depends on whether orient is preserved or reversed:

in $T_{\varphi} \mathcal{L} \simeq T_p M \oplus ds_p(T_p M)$.

Poincare - Hopf : $\pm$ = index (vector field)

A canonical embedding.
[Mumford p. 9]

$\boxed{g=3}$

$\omega_1, \omega_2, \omega_3$ basis for $L(K)$.

$p \mapsto [\omega_1(p), \omega_2(p), \omega_3(p)]$.

in loc. coord: $\omega_i = f_i(t)\,dt$

$[\omega_1, \omega_2, \omega_3] = [f_1, f_2, f_3]$.

change coord: $z = h(t)$; $t = g(z)$,
$dt = g'(z)\,dz$

$= [f_1 g', f_2 g', f_3 g']$

This map is indep of coord.
well defined $\longrightarrow$

provided : $\nexists\, p$ s.t.
$$\varphi_1(p) = \varphi_2(p) = \varphi_3(p).$$

$$L(K-p) = \{ f : (f) + K - p \geq 0 \}$$

K rep by any mero one fn.

$= \cancel{\{f\omega}$.
$$(f\omega) = (f) + (\omega).$$

$$= \{ f\omega : (f\omega) - p \geq 0 \}$$

$$= \{ \omega \in \Omega^{1,0} : \omega(p) = 0 \}$$

R.R:
$$\ell(K-p) - \ell(\cancel{(K-p)-p})$$
$$- \ell(K-(K-p)) = \cancel{2g-2}\, 1$$

$$= d + 1 - g$$
$$= (2g-2-1) + (1-g)$$
$$= g-2$$

Now:
$$\ell(p) = 1 \text{ or } 0.$$

$$\Rightarrow \ell(K-p) = \begin{cases} g-2 \\ \text{or } g-1 \end{cases} < g.$$

Not all one forms can vanish at p.
if $g \geq 1$.

($g = 1$: $\overset{\text{holo}}{1\text{-forms}}$: dz
do not vanish anywhere.)

This trick works in any genus, $g > 1$

$$p \longmapsto [\omega_1(p), \omega_2(p), \ldots, \omega_g(p)]$$

$$M \longleftrightarrow \mathbb{C}P^{g-1}$$

$\star\star$ $\boxed{\text{Mumford, p. 9}}$: "As is well known, there are two types of C's (= M's) those for which Φ is an embedding (one injective & $\Phi(C)$ non singular), & those for which Φ is 2:1 in which case $\Phi(M) \cong \mathbb{C}P^1$ & M is hyperelliptic"

I did not know this!
[proofs? refs.?]

- - -

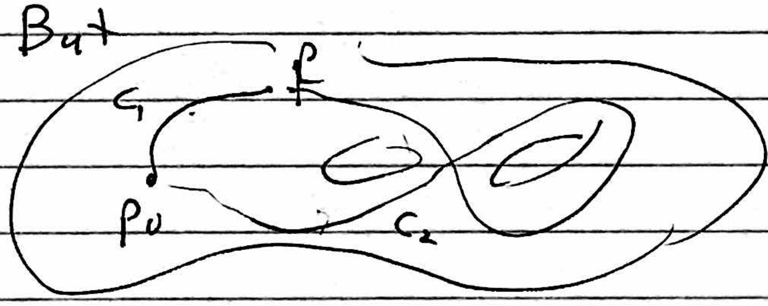
We can integrate this canonical map Φ:

Fix a base point $p_0 \in M$.

$$p \mapsto \left(\int_{p_0}^{p} \omega_1, \int_{p_0}^{p} \omega_2, \dots \int_{p_0}^{p} \omega_g \right)$$

takes values in $\mathbb{C}^g$.

But



$$\int_{p_0}^{p} \omega_1 - \int_{p_0}^{p} \omega_1 = \oint_{\text{cycle}} \omega_1$$

let $\Lambda \subset \mathbb{C}^g$ be the results of $\left(\oint_\gamma \omega_1, \dots \oint_\gamma \omega_g\right)$.

as γ varies over closed loops.

$$\Rightarrow \text{“}\int \Phi\text{”} : M \longrightarrow \mathbb{C}^g/\Lambda$$

(If we change basis $(\omega_1, \dots, \omega_g)$

$\Lambda \longrightarrow B\Lambda$, $B \in Gl(g, \mathbb{C})$

&

$$\begin{array}{ccc} \mathbb{C}^g & \xrightarrow{B} & \mathbb{C}^g \\ \downarrow & & \\ \mathbb{C}^g/\Lambda & \longrightarrow & \mathbb{C}^g/B\Lambda \end{array}$$

) .

derive of Φ:

$$d(\int\Phi) : T_pM \longrightarrow T_{\Phi(p)}(\mathbb{C}^g/\Lambda)$$

$\cong$ can

$T_{\Phi(p)}\,\mathbb{C}^g$.

a 1-dim space.

view projectively: $L \subset \mathbb{C}^g$.

yields $\qquad$ homog coord

$$p \longmapsto \overbrace{[\omega_1(p), \dots, \omega_g(p)]}$$