$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$: $\mathbb{C}$-dual to $dz, d\bar{z}$.

$dz = dx + i\,dy$. ; $d\bar{z} = dx - i\,dy$.

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial(x+iy)} = \frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial(x-iy)} = \frac{\partial}{\partial x} + \left(\frac{1}{-i}\right)\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

But these are not dual

$$(dx + i\,dy)\left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}\right) = \frac{1}{2} + i\left(\frac{-i}{2}\right)\cdot 1 = 1$$

$$(dx + i\,dy)\,\frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}\right) = \frac{1}{2} + \frac{i^2}{2}\,1 = \frac{1}{2} - \frac{1}{2} = 0$$

Mnemonics

- $\mathbb{C}$-dual are $dz, d\bar{z} \leftrightarrow \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$.

The "$\frac{1}{2}$"s in $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ make them dual bases to $dz, d\bar{z}$

Harmonic aside:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

$$= \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$= 4\frac{\partial}{\partial \bar z}\frac{\partial}{\partial z}$$

$$= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}$$

$\Rightarrow$ if $\frac{\partial f}{\partial \bar z} = 0$ then $\Delta f = 0$

ie if $f = u + iv$

$\Delta u = 0$

$\Delta v = 0.$

similarly for $\frac{\partial f}{\partial z} = 0.$

eg. $\bar z$; $\bar z^5$ sat

$\Delta f = 0$, But $\bar z z^2$ does

Not..

Now $\quad J^T dx = -dy$

$\quad\quad J^T dy = +dx$

since, eg

$$(J^T dx)\left(\frac{\partial}{\partial x}\right) = dx\left(J\frac{\partial}{\partial x}\right) = dx\left(\frac{\partial}{\partial y}\right) = 0$$

$$(J^T dx)\left(\frac{\partial}{\partial y}\right) = dx\left(J\frac{\partial}{\partial y}\right) = dx\left(-\frac{\partial}{\partial x}\right) = -1$$

etc.

$$J^T(a\,dx + b\,y) = (-a\,dy + b\,dz.$$

Extend J, J^T to $\mathbb{C}$..

$$TM \otimes \mathbb{C}, \qquad T^*M \otimes \mathbb{C}$$

$J^2 = -I \Rightarrow$ evalues are $\pm i$.

~~evectors~~:

$\frac{\partial}{\partial z}, \; dz \quad$ for $+i$

$\frac{\partial}{\partial \bar{z}}, \; d\bar{z} \quad$ for $-i$

check: $J^T(dx + i\,dy) = -dy + i\,dx$

$\quad\quad = i(dx + i\,dy)$ ✓

$$J\left(\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right) = \frac{1}{2}J\frac{\partial}{\partial x} - \frac{i}{2}J\frac{\partial}{\partial y}$$

$$= +\frac{1}{2}\frac{\partial}{\partial y} - \frac{i}{2}\left(-\frac{\partial}{\partial x}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial y} + \frac{i}{2}\frac{\partial}{\partial x}$$

$$= i\left(\frac{1}{2}\frac{\partial}{\partial x} - i\frac{1}{2}\frac{\partial}{\partial y}\right)$$ ✓

linear algebra generalities.

$\mathbb{V}$. real vector space

$\mathbb{J} : \mathbb{V} \to \mathbb{V}$ w/ $\mathbb{J}^2 = -\mathbb{I}$

define $\mathbb{C}$-struc on $\mathbb{V}$ by

$(\alpha + i\beta) v = \alpha v + \beta \mathbb{J} v.$

verify sat axioms of scalar mult

so: $\mathbb{V} \simeq \mathbb{C}^N$ if V is finite dim'l.

$\mathbb{J} : \underbrace{\mathbb{V} \otimes \mathbb{C}}_{\mathbb{V}_{\mathbb{C}}} \circlearrowleft$ extend by usual cx mult.

$$\mathbb{V}_{\mathbb{C}} = \underbrace{\mathbb{V}^{1,0}}_{\frac{\partial}{\partial z}\text{'s}} \oplus \underbrace{\mathbb{V}^{0,1}}_{\frac{\partial}{\partial \bar{z}}\text{'}}$$

$$\mathbb{V}^*_{\mathbb{C}} \simeq \underbrace{\mathbb{V}^{*1,0}}_{dz\text{'s}} \oplus \underbrace{\mathbb{V}^{*0,1}}_{d\bar{z}\text{''}}$$

&

$$\mathbb{V}^* \simeq \mathbb{V}^{*1,0}.$$

$$\alpha \longrightarrow \frac{1}{2}(\alpha - i\mathbb{J}\alpha)$$

So: $T^*M \simeq T^*M^{1,0}.$

the operator

$$* = -J^* : \quad dx \to dy, \quad dy \to -dx.$$

its the Hodge * operator basic to analysis of de Rham complex

used in Hodge de Rham Theory

see lecture notes 8

reprinted from Marsden on,

Introduced in FK word

p. 25 -27

surface
Riem struc $\longleftrightarrow$ conformal struc.

since if z, w two coord
$$|dw|^2 = |h'(w)||dz|^2.$$

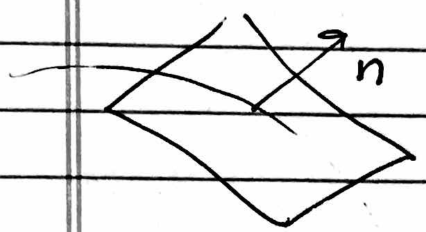
J; $*$ on 1-forms depend only on conformal structure (not on metric details.

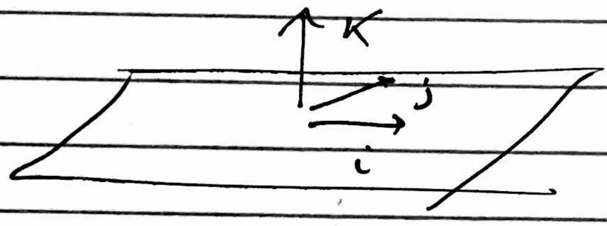
Conformal struc: angles only.
$\langle\ ,\ \rangle$, $\langle\ ,\ \rangle'$ two inner products called conformally equivalent iff $\exists\ \lambda^2 > 0$,
$$\langle v, w\rangle' = \lambda^2 \langle v, w\rangle.$$
$$\cos(\measuredangle(v,w)) = \frac{\langle v,w\rangle}{|v||w|} = \frac{\langle v,w\rangle'}{|v|'|w|'}$$

J: rotate $\circlearrowleft$ counterclockwise by $90°$ so - -

$v \perp J v$.

$\{v, Jv\}$ an oriented basis whenever $v \neq 0$.





$$J v = K \times v, \qquad v \in \mathbb{R}^2.$$

$$J_p v = n(p) \times v.$$

$M \subset \mathbb{R}^3$ oriented surface

$T_p M = V \ldots$

Anyhow: $K = T^* M^{1,0}$
$\simeq T^* M$.

$$\Omega^{1,0} = \Gamma^{\text{holo}}(K)$$
$= f(z)\,dz$'s locally