Some Missing pieces / Loose ends

• ~~regard~~ $T^*M \simeq (T^*M)^{1,0} = K$

re: w/ local sections dz,
z loc. holo. coords.

operators
$$\begin{cases} J : TM \to TM & \text{fiber linear end.} \\ * : T^*M \to T^*M & \text{" " "} \\ \partial & \text{"} \\ \Omega^1_{\mathbb{C}} = \Omega^{1,0} \oplus \Omega^{0,1} \end{cases}$$

$$\partial f = \frac{\partial f}{\partial z}\, dz; \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}}\, d\bar{z}$$

$$\boxed{d = \partial + \bar{\partial}} \quad \text{on } \Omega^0_{\mathbb{C}} \to \Omega^1_{\mathbb{C}}$$

f loc. holo $\iff \bar{\partial} f = 0.$

with $z = x + iy$
$dz = dx + i\,dy$
$d\bar{z} = dx - i\,dy$

Recall Cauchy Riemann D.E.

$$f(z) = u(x,y) + iv(x,y)$$
$$z = x+iy.$$

$$f'(z) = \lim_{\varepsilon \in \mathbb{R} \to 0} \frac{f(z+\varepsilon) - f(z)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{f(z+\varepsilon i) - f(z)}{\varepsilon i}$$

$$\Rightarrow \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon, y) - f(x,y)}{\varepsilon}.$$

$$= \lim_{\varepsilon \to 0} \frac{f(x, y+\varepsilon) - f(x,y)}{\varepsilon i}$$

$\Rightarrow$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= +\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

so

$$u_x = v_y$$
$$v_x = -u_y$$

Using conformal struc & Hodge $*$ to define ∂, $\bar{\partial}$, dz ...

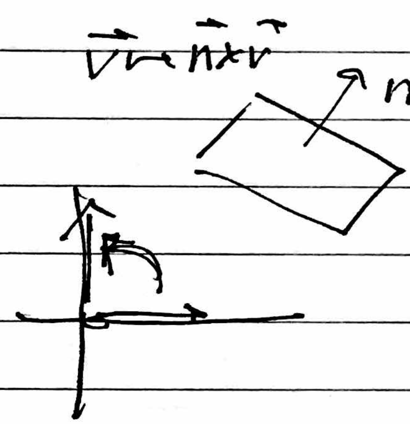
Almost cx structures.

$\mathbb{C} \simeq \mathbb{R}^2$ basis $\Rightarrow (1, i) \leftrightarrow (x,y) \leftrightarrow \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$.

$i(a+ib) = \cancel{ia}\ ia - b.$

or

$(a,b) \overset{J}{\longmapsto} (-b,a)$



rotates by $90°$.

call operator J ; $J : \mathbb{R}^2 \to \mathbb{R}^2$.

verify.

$\ni\ \mathbb{C}\ (a+ib)v = av + bJv.$

when write: $v = v_1 + i v_2$.

Properties: $\boxed{J^2 = -I}$; so $\boxed{\langle v, Jv \rangle = 0}$

& $\langle Jv, Jw \rangle = \langle v, w \rangle$;

$\{v, Jv\}$ pos oriented $\forall\ v \neq 0$.

Let J act on $\mathbb{C}$ by mult by i

verify: $\underbrace{}_{}$ rotate $90°$ mult. by i

$$df \circ J_{TM} = J\, df$$

$\iff$ CRDE ~~& holds for f~~ (!).

Hints: look at $dF\left(J\frac{\partial}{\partial x}\right)$

$$= (du + i\,dv)\left(\frac{\partial}{\partial y}\right).$$

&

cf with $J\,df\left(\frac{\partial}{\partial x}\right) = i(du + i\,dv)\left(\frac{\partial}{\partial x}\right)$

~~So: → for a $\mathbb{C}$-valued one form~~

~~α I locally~~

~~$\alpha = df$ w/ f holo.~~

~~$\iff$ $\alpha \circ J = i\alpha$.~~

&

Set $df \circ J = -{*}df$

or $J^t = -{*}$

Breakout

verify: $f = u + iv$ a fn on $\mathbb{C}$ (or on R.S.)

satisfies $df \circ \mathbb{J} = \mathbb{J}_{\mathbb{C}}\, df$

where $\mathbb{J}_{\mathbb{C}}$ = mult by i

$\Leftrightarrow$ f sat. the C.R.D.E.