Proof. This is a consequence of the formula

$$\frac{d}{dt}F_t^*g = \underbrace{F_t^*}_{?}(L_X g)$$

from Sect. 2.2. ■

We now define the volume element on a Riemannian manifold.

2.7.11 Definition. *Let M be an oriented Riemannian n-manifold. If $v_1,\ldots,v_n \in T_xM$ are positively oriented, set*

$$\mu(v_1,\ldots,v_n) = (\det\langle v_i, v_j\rangle)^{1/2}$$

This is possible as $\det\langle v_i, v_j\rangle$ is $\geqslant 0$ for all $v_1,\ldots,v_n \in T_xM$ since the metric is positive-definite. Now define μ on all n-tuples by skew symmetry.

Clearly μ is a volume form on M. Locally, $\mu = \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n$. The definition is motivated from the fact that the volume spanned by vectors $v_1,\ldots,v_n \in \mathbf{R}^n$ is $(\det v_i \cdot v_j)^{1/2}$.

Since we have μ, we can use it to define the divergence of a vector field, $\operatorname{div} X$. From the expression for μ above, and the definition $L_X\mu = (\operatorname{div} X)\mu$ from Sect. 2.5, we find, locally, if $V = \sqrt{\det g_{ij}}$,

$$\operatorname{div} X = \frac{1}{V}\frac{\partial}{\partial x^i}(VX^i)$$

For $f: M \to \mathbf{R}$, $\operatorname{grad} f$ is the vector field defined by

$$\langle \operatorname{grad} f(x), v_x\rangle = df(x)\cdot v_x \quad \text{for all} \quad v_x \in T_xM$$

In coordinates,

$$(\operatorname{grad} f)^i = g^{ij}\frac{\partial f}{\partial x^j}$$

The *Laplace–Beltrami operator* on functions is defined by

$$\nabla^2 = \operatorname{div}\cdot\operatorname{grad}$$

so

$$\nabla^2 f = \frac{1}{V}\frac{\partial}{\partial x^k}\left(g^{ik}V\frac{\partial f}{\partial x^i}\right)$$

From Stokes' theorem we find that d and $-\operatorname{div}$ are adjoints and ∇^2 is

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symmetric:

$$\int_M df \cdot X \, d\mu = -\int_M f \, div X \, d\mu$$

$$\int_M f \nabla^2 g \, d\mu = -\int_M \langle grad f, grad g \rangle d\mu = \int_M g \nabla^2 f \, d\mu$$

for X, f, g having compact support.

Next we consider the *Laplace–de Rham Operator*.

2.7.12 Definition. *Let M be a Riemannian n-manifold and let β be a k-form. Define an $(n-k)$-form $*\beta$ by*

$$(*\beta)(v_{k+1}, \ldots, v_n) = \beta(v_1, \ldots, v_k)$$

where $v_1, \ldots, v_n$ are oriented orthonormal vectors in T_xM. We call $*$ the ***Hodge star operator.***

For example, on $\mathbf{R}^3$, $*dx = dy \wedge dz$, $*dy = dx \wedge dz$, and so forth. One can then verify that

$$\langle \alpha, \beta \rangle_x \mu_x = \alpha_x \wedge * \beta_x$$

defines an inner product on k-forms.

2.7.13 Definition. *Set $(\alpha, \beta) = \int_M \alpha \wedge * \beta$, which gives an L^2 inner product on the sections of $\Omega^k(M)$. Also, define the* ***codifferential operator*** $\boldsymbol{\delta} = (-1)^{n(k+1)+1} * \mathbf{d} *$.

It is easily checked that $\boldsymbol{\delta}$ is adjoint of $\mathbf{d}$:

$$(\boldsymbol{\delta}\gamma, \beta) = (\gamma, \mathbf{d}\beta)$$

[Use the fact that $\int_M \mathbf{d}(\gamma \wedge * \beta) = 0$ and $** \beta = (-1)^{k(n-k)}\beta$.]

2.7.14 Definition. *The* ***Laplace–deRham*** *operator is defined by*

$$\boldsymbol{\Delta} = \mathbf{d}\boldsymbol{\delta} + \boldsymbol{\delta}\mathbf{d}$$

The operator $\boldsymbol{\Delta}$ is symmetric, and nonnegative

$$(\boldsymbol{\Delta}\alpha, \beta) = (\alpha, \boldsymbol{\Delta}\beta), \quad (\boldsymbol{\Delta}\alpha, \alpha) \geqslant 0$$

A k-form α satisfying $\boldsymbol{\Delta}\alpha = 0$ is called *harmonic*. On functions, $\boldsymbol{\Delta}$ differs in sign from the Laplace–Beltrami operator ∇^2. (See, for instance, Nickerson-Spencer and Steenrod [1959].)

The operator $\boldsymbol{\Delta}$ is at the basis of "Hodge–De Rham Theory." The central result states that on a compact manifold M without boundary, the kernel of $\boldsymbol{\Delta}$

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