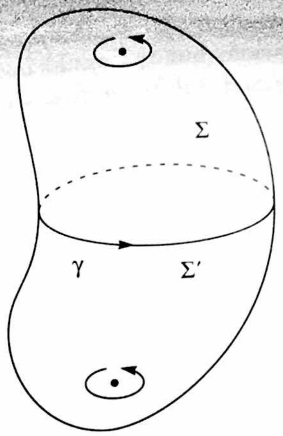

Fig. 8.1. The integrality condition.

8.3 The integrality condition

By comparing (8.2.2) with (A.3.2), it is clear that what is needed is a Hermitian line bundle $B \to M$ and a connection ∇ on B with potentials of the form $\hbar^{-1}\theta$; that is, with curvature $\hbar^{-1}\omega$. Such a bundle and connection exist if and only if ω satisfies Weil's integrality condition (Weil 1958, Kostant 1970a).

The integrality condition is closely related to the quantization rule in the old quantum theory (Messiah 1961, Chapter I, §15, Simms 1973). Its geometric origin can be understood by considering parallel transport with respect to ∇ around a closed curve γ through $m \in M$. Suppose that γ can be spanned by a surface Σ, and that Σ is contained in the domain of θ. Then the result is equivalent to the linear transformation $B_m \to B_m$ given by multiplication by

$$\exp\left(\frac{\mathrm{i}}{\hbar}\oint_\gamma \theta\right) = \exp\left(\frac{\mathrm{i}}{\hbar}\int_\Sigma \omega\right).$$

By dividing Σ into small pieces, it is easy to see that the factor is still given by the expression on the right-hand side even when Σ is not contained in the domain of a single potential.

We must get the same result when we span γ by a second 2-surface Σ'. Hence, taking account of the relationship between the orientations of γ, Σ, and Σ' (Fig. 8.1), the integral of $\hbar^{-1}\omega$ over $\Sigma \cup \Sigma'$ must be an integer multiple of 2π. So if $\hbar^{-1}\omega$ is to be the curvature of a connection, then ω must satisfy the following form of the integrality condition.

> **IC1.** The integral of ω over any closed oriented 2-surface in M is an integral multiple of $2\pi\hbar$.

This is necessary for the existence of B and ∇. When M is simply connected, we can show that it is also sufficient by reconstructing ∇ from its holonomy. Suppose that (IC1) holds. Pick a base point $m_0 \in M$ and consider the set of all triples (m, z, γ), where $m \in M$, $z \in \mathbb{C}$ and γ is a piecewise smooth path from m_0 to m. Two triples (m, z, γ) and (m', z', γ') are defined to be equivalent whenever $m = m'$ and

$$z' = z \exp\left(\frac{\mathrm{i}}{\hbar}\int_\Sigma \omega\right),$$

where Σ is any oriented 2-surface with boundary made up of γ (oriented from m_0 to m) and γ' (oriented from m to m_0). Because M is simply connected, such a surface exists; and because the integrality condition holds, it does not matter which surface is chosen. The set of equivalence classes has the structure of a line bundle $B \to M$.

Addition and scalar multiplication within the fibres are defined by

$$[(m, z, \gamma)] + [(m, z', \gamma)] = [(m, z + z', \gamma)] \quad \text{and} \quad c[(m, z, \gamma)] = [(m, cz, \gamma)],$$

where $c \in \mathbb{C}$ and the square brackets denote equivalence classes. The local trivializations are determined by symplectic potentials. Suppose that θ is a potential on some simply-connected open set $U \subset M$. Pick a point m_1 in U and a curve γ_0 from m_0 to m_1; and define a section s of B in U by taking $s(m)$ to be the class of

$$\left(m, \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{\gamma_1} \theta\right), \gamma\right)$$

where γ_1 is any curve from m_1 to m in U and γ is the curve from m_0 to m made up of γ_0 and γ_1 (Fig. 8.2). Since $\mathrm{d}\theta = \omega$, a different choice of γ_1 gives an equivalent triple and hence the same value of $s(m)$. A different choice of m_1 or γ_0 gives the same section, but multiplied by a constant of modulus one. The effect of replacing θ by $\theta' = \theta + \mathrm{d}u$, where $u(m_1) = 0$, is to replace s by $s' = \mathrm{e}^{-\mathrm{i}u/\hbar}s$. We can therefore define a global connection and a Hermitian structure on B by

$$\nabla s = -\frac{\mathrm{i}}{\hbar}\theta s \quad \text{and} \quad (s, s) = 1,$$

independently of the local choices of m_1, γ_1 and θ. The curvature is $\hbar^{-1}\omega$.

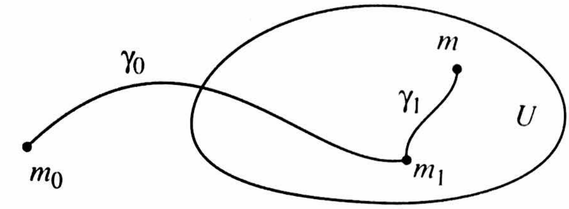

Fig. 8.2. The construction of B.

It is harder to deal in the same direct way with the general case in which M is not simply connected since it turns out that B and ∇ are not unique—there are different possibilities for the holonomy of ∇ around closed loops that cannot be contracted to points. Instead, we shall take as our starting point a different form of the integrality condition. See Weil (1958, p. 87) and, for example, Vaisman (1973).

IC2. The class of $(2\pi\hbar)^{-1}\omega$ in $H^2(M,\mathbb{R})$ lies in the image of $H^2(M,\mathbb{Z})$.

In other words, there is an open cover $U = \{U_j\}$ such that the class of $(2\pi\hbar)^{-1}\omega$ in $H^2(U,\mathbb{R})$ contains a cocycle z in which all the z_{ijk}s are integers (see §A.6).

Proposition (8.3.1). There exists a Hermitian line bundle $B \to M$ and a connection ∇ on B with curvature $\hbar^{-1}\omega$ if and only if IC2 holds. When IC2 holds, the inequivalent choices of B and ∇ are parametrized by $H^1(M,T)$, where $T \subset \mathbb{C}$ is the circle group.

Proof. Suppose that the condition holds. Then there is a contractible open cover $\{U_j\}$ of M, a collection of symplectic potentials $\theta_j \in \Omega^1(U_j)$, and a collection of functions $u_{jk} \in C^\infty(U_j \cap U_k)$ such that

(1) $\mathrm{d}u_{jk} = \theta_j - \theta_k$ whenever $U_j \cap U_k \neq \emptyset$; and

(2) $(2\pi\hbar)^{-1}(u_{jk} + u_{kl} + u_{lj}) \in \mathbb{Z}$ whenever $U_j \cap U_k \cap U_l \neq \emptyset$.

Put $c_{jk} = \exp(\mathrm{i}u_{jk}/\hbar)$. Then

$$\frac{\mathrm{d}c_{jk}}{c_{jk}} = \frac{\mathrm{i}}{\hbar}(\theta_j - \theta_k),$$

whenever $U_j \cap U_k \neq \emptyset$; and

$$c_{jk}c_{kl}c_{lj} = \exp\left(\frac{2\pi \mathrm{i}}{\hbar}(u_{jk} + u_{kl} + u_{lj})\right) = 1$$

whenever $U_j \cap U_k \cap U_l \neq \emptyset$ (there is no summation over the repeated indices). It follows that the cs are the transition functions of a line bundle $B \to M$ and that the $\hbar^{-1}\theta$s determine a connection on B with curvature $\hbar^{-1}\omega$ (see §A.3). Since the potentials are real and the transition functions are of unit modulus, there is also a compatible Hermitian structure $(\cdot,\cdot)$.

Suppose, conversely, that we are given B and a connection on B with curvature $\hbar^{-1}\omega$. Let $\{c_{jk}\}$ be the transition functions of B relative to some open cover. For each nonempty triple intersection, put

$$z_{jkl} = \frac{1}{2\pi \mathrm{i}}(\log c_{jk} + \log c_{kl} + \log c_{lj}).$$

Then, by (A.3.4), z_{jkl} is an integer, and hence constant. The zs satisfy the cocycle condition. There is an ambiguity in the definition of the logarithms since $(2\pi \mathrm{i})^{-1}\log c_{jk}$ is defined only up to the addition of an integer x_{jk}. However, the cohomology class $[z]$ of z in $H^2(M,\mathbb{Z})$ is independent of the choice of branches. It is called the *Chern class* of B. It follows from (A.3.6) that $2\pi\hbar z$ is a representative cocycle of the class in $H^2(M,\mathbb{R})$ determined by ω. Hence IC2 is also a necessary condition.

There is freedom in the construction of B and ∇ from ω since we can replace u_{jk} by $u_{jk} + y_{jk}$, where the ys are real constants chosen so that

$$y_{jk} = -y_{kj} \quad \text{and} \quad \frac{1}{2\pi\hbar}(y_{jk} + y_{kl} + y_{lj}) \in \mathbb{Z},$$

whenever $U_j \cap U_k \cap U_l \neq \emptyset$. The effect is to replace B by $B \otimes F$, where F is the Hermitian line bundle with transition functions[3]

$$t_{jk} = \exp(\mathrm{i}y_{jk}/\hbar) \in T.$$

Since these are constant, F has a connection with vanishing curvature, and so $B \otimes F$ has a connection with the same curvature as ∇.

Conversely, if (B,∇) and (B',∇') both have curvature $\hbar^{-1}\omega$, then $F = B^{-1} \otimes B'$ is a Hermitian line bundle with flat connection. Hermitian line bundles with flat connections are labelled by elements of $H^1(M,T)$, where T is the circle group: the correspondence is given by the transition functions of local trivializations in which the connection forms vanish. Thus the various choices for B and ∇ are also parametrized by $H^1(M,T)$. ∎

Note that when M is simply connected, $H^1(M,T) = 0$, and B and ∇ are unique up to equivalence.

We end this section with some remarks about trivializations of B and complex symplectic potentials. Any non-vanishing local section s determines a local trivialization and also a symplectic potential by $\theta s = \mathrm{i}\hbar\nabla s$. If $(s,s) = 1$, then θ is real. In general, however, θ is complex, with imaginary part equal to $\mathrm{d}(\frac{1}{2}\hbar\log(s,s))$. This follows from

$$\mathrm{d}(s,s) = (\nabla s, s) + (s, \nabla s) = \frac{\mathrm{i}}{\hbar}(\bar{\theta} - \theta)(s,s).$$

Conversely, if we are given a (real or complex) symplectic potential θ on a simply-connected neighbourhood $U \subset M$, then we can define a non-vanishing section $s : U \to B$ by picking any $m_0 \in U$ and $b_0 \in B_{m_0}$ and putting

$$s(m) = b\exp\left(-\frac{\mathrm{i}}{\hbar}\int_\gamma \theta\right),$$

where γ is a curve from m_0 to m and b is obtained from b_0 by parallel transport along γ. The value of $s(m)$ is independent of γ. Moreover, $\mathrm{i}\hbar\nabla s = \theta s$, so s determines a local trivialization in which the connection form is $\hbar^{-1}\theta$.

8.4 Prequantization of canonical transformations

A symplectic manifold (M,ω) is said to be *quantizable* whenever ω satisfies the integrality condition. There then exists a Hermitian line bundle $B \to M$ and a connection ∇ on B with curvature $\hbar^{-1}\omega$. We shall call B the *prequantum bundle*.

The Hilbert space of prequantization is the space $\mathcal{H}$ of square integrable sections $s : M \to B$, with the inner product

$$\langle s, s' \rangle = \int_M (s, s')\,\epsilon.$$

The operator $\hat{f}$ corresponding to $f \in C^\infty(M)$ is

$$\hat{f}s = -\mathrm{i}\hbar\nabla_{X_f}s + fs,$$

where s lies in some appropriate subset of $\mathcal{H}$. It is always symmetric; and when X_f is complete, it is actually self-adjoint. The definition also makes sense for complex f, but $\hat{f}$ is then no longer symmetric.

There is another way of looking at the 'quantum observable' $\hat{f}$ that may make the definition appear more natural. Let V_f be the vector field on B given in a local trivialization by[4]