If $f \in C^\infty(M \times \mathbb{R})$ is a time-dependent function (0-form), then

$$\mathcal{L}_X f = \frac{df}{dt} = X^a \frac{\partial f}{\partial x^a} + \frac{\partial f}{\partial t},$$

where d/dt denotes the derivative along the integral curves of X.

Riemannian geometry. The Riemann tensor $R^a{}_{bcd}$ of a Riemannian or pseudo-Riemannian metric g_{ab} is defined by

$$\nabla_c \nabla_d X^a - \nabla_d \nabla_c X^a = R^a{}_{bcd} X^b,$$

where ∇ is the Levi-Civita connection. The Ricci tensor and the scalar curvature are defined by $R_{ab} = R^c{}_{acb}$ and $R = R^a{}_a$. The metric volume element is

$$\frac{1}{n!} \sqrt{|g|}\, \epsilon_{ab\ldots c} \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \cdots \wedge \mathrm{d}x^c,$$

where ϵ is the alternating symbol.

A.2 Vector bundles

Definitions. The theory of fibre bundles is described in several books. Accounts which emphasize physical applications are given by Choquet-Bruhat, DeWitt-Morette, and Dillard-Bleick 1982, Schutz 1980, Nash and Sen 1983, and Crampin and Pirani 1986. Most (but not all) of the bundles in this book are vector bundles.

Let M be a real manifold and let $\mathbb{F}$ denote either the real or the complex numbers. A *vector bundle* over M is a smooth manifold V (the *total space*) together with a smooth map $\pi : V \to M$ (the *projection*) such that the following hold.

(VB1) For each $m \in M$, $V_m = \pi^{-1}(m)$ has the structure of an N-dimensional vector space over $\mathbb{F}$ (V_m is called the *fibre* over m; N is constant throughout M: it is called the *fibre dimension* or *rank* of the bundle).

(VB2) There is a system of local trivializations (U_i, τ_i) such that the Us cover M. A local trivialization is an open set $U \subset M$ together with a diffeomorphism $\tau : U \times \mathbb{F}^N \to \pi^{-1}(U) \subset V$ which, on restriction to $\{m\} \times \mathbb{F}^N$, becomes a linear map $\mathbb{F}^N \to V_m$, for each $m \in U$.

With varying degrees of sloppiness, one denotes the bundle by V, $V \to M$, or $\pi : V \to M$.

The basic operations on vector spaces extend in an obvious way to vector bundles: given V and V', one can form the *direct sum* (Whitney sum) $V \oplus V'$, the *tensor product* $V \otimes V'$, the *dual bundle* V^*, and, in the complex case, the *complex conjugate bundle* $\overline{V}$. We say that a vector bundle $W \to M$ is a sub-bundle of V whenever the fibres of W are subspaces of the fibres of V.

A map $s : U \subset M \to V$ such that $\pi \circ s$ is the identity is called a *section* over U. A local section of V is a section over some open subset of M. The space of smooth sections over U is denoted $C^\infty_V(U)$. If s is a section of V and s' is a section of V', then $s + s'$, ss' (more properly $s \otimes s'$), and $\overline{s}$ are, respectively, sections of $V \oplus V'$, $V \otimes V'$ and $\overline{V}$.

A p-form with values in V is a map α that assigns a smooth section $\alpha(X, Y, Z, \ldots)$ to each collection of p vector fields $X, Y, Z, \ldots$. It must be linear in each entry and skew-symmetric under permutations of $X, Y, Z, \ldots$. For example, if α is a differential form on M (in the ordinary sense) and s is a smooth section of V, then αs is a form with values in V. Forms with values in V can be contracted with vector fields on M. One can also take their exterior products with ordinary differential forms on M.

Connections. A connection ∇ on a vector bundle V is an operator that assigns a 1-form ∇s with values in V to each (smooth) section s of V, such that for any function f and sections $s, s' \in C^\infty_V(M)$,

$$\nabla(s + s') = \nabla s + \nabla s' \quad \text{and} \quad \nabla(fs) = (\mathrm{d}f)s + f\nabla s.$$

If X is a vector field, we denote $X \lrcorner \nabla s$ by $\nabla_X s$.

Connections on V and V' induce connections on $V \oplus V'$, $V \otimes V'$, V^*, and $\overline{V}$ in the obvious way. For example, in the case of $V \otimes V'$, $\nabla(ss') = (\nabla s)s' + s(\nabla s')$.

Hermitian structures. A Hermitian structure on a complex vector bundle V is a Hermitian inner product $(\cdot, \cdot)$ on the fibres which is smooth in the sense that $V \to \mathbb{C} : v \mapsto (v, v)$ is a smooth function. It is *compatible* with a connection ∇ if for all sections s, s' and all real vector fields X,

$$X \lrcorner \mathrm{d}(s, s') = (\nabla_X s, s') + (s, \nabla_X s').$$

Equivalence. Two vector bundles V and V' are equivalent if there exists a diffeomorphism $\iota : V \to V'$ which restricts to a linear isomorphism $V_m \to V'_m$ at each $m \in M$. If they have connections or Hermitian structures, then they are equivalent as bundles with connection or as Hermitian vector bundles if ι can be chosen so that in addition

$$\nabla_X(\iota \circ s) = \iota \circ \nabla_X s \quad \text{or} \quad (\iota \circ s, \iota \circ s) = (s, s)$$

for all sections s and vector fields X. It is important in prequantization (Chapter 8) that it is possible to have inequivalent connections on equivalent vector bundles.

Pull-backs. Let $V \to M$ be a vector bundle and let $\rho : M' \to M$ be a smooth map. Then

$$\rho^* V = \{(m', v') \mid m' \in M',\ v' \in V_{\rho(m')}\}$$

is a vector bundle over M', called the *pull-back* of V. If (U, τ) is a local trivialization of V, then

$$\rho^{-1}(U) \times \mathbb{F}^N \to \rho^* V : (m', \mathbf{z}) \mapsto \big(m', \tau(\rho(m'), \mathbf{z})\big),$$

where $\mathbf{z} \in \mathbb{F}^N$, is a local trivialization of $\rho^* V$.

A section s of V determines a pull-back section $\rho^* s$ of $\rho^* V$ by

$$(\rho^* s)(m') = \big(m', s(\rho(m'))\big).$$

A connection or Hermitian structure on V induces a connection or Hermitian structure on $\rho^* V$ in the obvious way.

When M' is a submanifold of M and ρ is the inclusion map, the pull-back is the *restriction* of V to M', and is denoted by $V|_{M'}$ or $V_{M'}$.

In the next section, we shall look in more detail at the special case of complex vector bundles with one-dimensional fibres, which play a central role in geometric quantization. We shall not develop further the general theory of vector bundles since it is required only in particular examples.

A.3 Line bundles

Curvature. Let $L \to M$ be a line bundle, that is, a complex vector bundle with one-dimensional fibres. Strictly, we should call L a 'complex line bundle', but real line bundles do not appear in this book, so the qualification is unnecessary. Let ∇ be a connection on L.

A local trivialization (U, τ) is determined by its *unit section* $s = \tau(\cdot, 1)$, which is a nonvanishing element of $C_L^\infty(U)$. The *potential 1-form* $\Theta \in \Omega^1_{\mathbb{C}}(U)$ of the connection in this trivialization is characterized by

$$\nabla s = -\mathrm{i}\Theta s. \tag{A.3.1}$$

If s' is any other section, then $s' = \psi s$, where ψ is a complex-valued function, and

$$\nabla_X s' = \big(X(\psi) - \mathrm{i} X \lrcorner \Theta \psi\big) s. \tag{A.3.2}$$

By dropping the distinction between s' and its local representative ψ, we can write this as $\nabla = \mathrm{d} - \mathrm{i}\Theta$.

The 2-form $\Omega = \mathrm{d}\Theta$ is the *curvature*. For any $X, Y \in V(M)$,

$$\Omega(X, Y)s' = \frac{\mathrm{i}}{2}\left(\nabla_X \nabla_Y s' - \nabla_Y \nabla_X s' - \nabla_{[X,Y]} s'\right).$$

Since this holds for any smooth section s', the curvature 2-form is independent of the local trivialization.

Let $\gamma : [0,1] \to M$ be a smooth curve with tangent vector $T = \dot{\gamma}$ and let s' be a section over γ; i.e. $s' : [0,1] \to L$, with $\pi(s'(t)) = \gamma(t)$. Then $\nabla_T s'$ is well defined by

$$\nabla_T s' = \left(\frac{\mathrm{d}\psi}{\mathrm{d}t} - \mathrm{i} T \lrcorner \Theta \right) s.$$

We say that s' is *parallel* along γ if $\nabla_T s' = 0$. There is a unique parallel section for each value of $s'(0)$. If γ is closed (i.e. $\gamma(0) = \gamma(1)$) and s' is parallel, then $s'(1) = \xi s'(0)$ for some $\xi \in \mathbb{C}$; ξ is the *holonomy* of the connection around the curve. The holonomy is independent of s', and also of the parametrization of the curve, as well as of the choice of the base point $\gamma(0)$. If γ can be spanned by a smooth surface S, and if S is given the appropriate orientation, then

$$\xi = \exp\left(\mathrm{i} \int_S \Omega \right).$$

The connection on $\overline{V}$ is given by $\nabla = \mathrm{d} + \mathrm{i}\Theta$ and has curvature $-\Omega$. The tensor product of two line bundles with connection is again a line bundle with connection, and its curvature is the sum of the curvatures of the individual connections.

Transition functions. Let (U_1, τ_1) and (U_2, τ_2) be two local trivializations of L, and let s_1 and s_2 be the corresponding unit sections. The *transition function* from (U_1, τ_1) to (U_2, τ_2) is the function $c_{12} \in C^\infty_{\mathbb{C}}(U_1 \cap U_2)$ such that $s_2 = c_{12} s_1$ on $U_1 \cap U_2$.

It is possible to find a collection $\{(U_j, \tau_j)\}$ of local trivializations such that $\{U_j\}$ is a contractible open cover of M (the subscripts are in some indexing set). That is, each U_j and each nonempty finite intersection $U_j \cap U_k \cap \cdots$ is contractible to a point. Let s_j be the unit section of (U_j, τ_j) and let $c_{jk} \in C^\infty_{\mathbb{C}}(U_j \cap U_k)$ be the transition function from (U_j, τ_j) to (U_k, τ_k). Then $s_k = c_{jk} s_j$ and

$$c_{jk} c_{kj} = 1 \quad \text{whenever} \quad U_j \cap U_k \neq \emptyset \tag{A.3.3}$$

$$c_{jk} c_{kl} c_{lj} = 1 \quad \text{whenever} \quad U_j \cap U_k \cap U_l \neq \emptyset \tag{A.3.4}$$

(there is no summation over the repeated indices). These are the *cocycle relations.*

Conversely, given a contractible cover $\{U_j\}$ and a collection of nonvanishing smooth complex functions $\{c_{jk}\}$ on the nonempty intersections such that (A.3.3) and (A.3.4) hold, it is possible to construct a line bundle L with the cs as transition functions. The total space is the disjoint union of the sets $U_j \times \mathbb{C}$ modulo the equivalence relation $(m_j, z_j) \sim (m_k, z_k)$ whenever $(m_j, z_j) \in U_j \times \mathbb{C}$, $(m_k, z_k) \in U_k \times \mathbb{C}$ and

$$m_j = m_k \quad \text{and} \quad z_j = c_{jk}(m_j) z_k .$$

If L and L' are line bundles with transition functions c_{jk} and c'_{jk} (relative to the same open cover), then

$L \otimes L'$ has transition functions $c_{jk} c'_{jk}$;

L^* has transition functions $(c_{jk})^{-1}$;

$\overline{L}$ has transition functions $\overline{c}_{jk}$.

If there exists a collection of nonvanishing functions $f_j \in C^\infty_{\mathbb{C}}(U_j)$ such that

$$c'_{jk} = \frac{f_j}{f_k} c_{jk} \quad \text{whenever} \quad U_j \cap U_k \neq \emptyset, \tag{A.3.5}$$

then L and L' are equivalent. The equivalence $\iota : L \to L'$ is given by $\iota(\tau_j(m, z)) = f_j \tau'_j(m, z)$ on U_j, where the (U, τ)s and the (U, τ')s are the respective local trivializations of L and L'. The converse is also true.

We denote the tensor product of L with itself by L^2; we also use the notation L^{-1} for L^*. By taking tensor products and duals, we can construct arbitrary integral powers L^n (L^0 is the trivial bundle $M \times \mathbb{C}$). We can also try to construct non-integral powers by taking roots of the transition functions. For example, $\sqrt{L}$ has transition functions $\sqrt{c_{jk}}$. It may not be possible to choose the square roots so that the cocycle relations hold, in which case $\sqrt{L}$ does not exist; when it does exist, there may be inequivalent ways of constructing it. The existence of $\sqrt{L}$ requires the vanishing of an obstruction in $H^2(M, \mathbb{Z}_2)$, and the equivalence classes of square roots are labelled by elements of $H^1(M, \mathbb{Z}_2)$ (see §A.6). If M is contractible, then $\sqrt{L}$ exists and is unique up to equivalence.

Connection potentials. Let ∇ be a connection on L and let Θ_j be the potential 1-form in the local trivialization (U_j, τ_j). On each nonempty $U_j \cap U_k$,

$$\Theta_k - \Theta_j = \mathrm{i} \frac{\mathrm{d} c_{jk}}{c_{jk}}. \tag{A.3.6}$$

Given a collection of 1-forms satisfying these relations, we can recover ∇ by using (A.3.1).

Without upsetting (A.3.6), we can replace the cs by $c'_{jk} = d_{jk} c_{jk}$, where the ds are constants satisfying the cocycle relations, while keeping the Θs unchanged. This gives another line bundle with connection with the same curvature, which is not equivalent to (L, ∇) unless there exist complex *constants* f_j such that (A.3.5) holds.

If L also has a Hermitian structure compatible with ∇, then, by rescaling if necessary, we can choose the local trivializations (U_j, τ_j) so that the unit sections satisfy $(s_j, s_j) = 1$. In this case, the transition functions are of unit modulus and the potential 1-forms are real, since for any $X \in V(M)$,

$$0 = X \lrcorner \mathrm{d}(s_j, s_j) = (\nabla_X s_j, s_j) + (s_j, \nabla_X s_j) = \mathrm{i}(X \lrcorner \overline{\Theta}_j - X \lrcorner \Theta_j).$$

Conversely, if the cs have unit modulus and the Θs are real, then L has a Hermitian structure compatible with ∇.

Connection forms. A useful way to describe a connection on a line bundle is in terms of the connection form on the total space (less the zero section).

Let $\pi : L \to M$ be a line bundle with connection ∇ and let $L^\times$ be the complement of the zero section in L. The connection form $\alpha \in \Omega^1_{\mathbb{C}}(L^\times)$ is defined by

$$\alpha = \pi^*(\Theta) + \mathrm{i}(\tau^{-1})^* \left(\frac{\mathrm{d}z}{z}\right),$$

where $\tau : U \times \mathbb{C} \to \pi^{-1}(U)$ is a local trivialization, Θ is the connection potential, and $z \in \mathbb{C}$. If (U', τ') is another local trivialization, then

$$(\tau^{-1})^* \left(\frac{\mathrm{d}z}{z}\right) - (\tau'^{-1})^* \left(\frac{\mathrm{d}z}{z}\right) = \frac{\mathrm{d}c}{c},$$

where c is the transition function from (U, τ) to (U', τ'). However the corresponding potential 1-forms are related by $\Theta' - \Theta = \mathrm{i}\,\mathrm{d}c/c$, so α does not depend on the choice of local trivialization.

The connection can be recovered from α either by using

$$\nabla s = -\mathrm{i}(s^*\alpha)s,$$

which holds for any nonvanishing smooth section s, or by using $\Theta_j = s_j^*\alpha$, where s_j is the unit section of (U_j, τ_j).

If L also has a Hermitian structure compatible with ∇, then

$$\alpha - \overline{\alpha} = \mathrm{i}\frac{\mathrm{d}H}{H}$$

where $H : L \to \mathbb{R} : l \mapsto (l, l)$. When M is connected, this determines H, and hence $(\cdot, \cdot)$, uniquely up to a constant factor. A necessary and sufficient condition for (L, ∇) to admit a compatible Hermitian structure is that the imaginary part of α should be exact (Kostant 1970a, Proposition 1.9.1).

Locally constant bundles. A locally constant line bundle is a line bundle with a flat connection; that is, with $\Omega = 0$. It has a preferred class of local trivializations with covariantly constant unit sections. The corresponding transition functions are locally constant (constant in a neighbourhood of