"The ~~great~~ Amazing synthesis"

Mumford, p1.

<u>Algebra</u>. K a field.

$K \supset \mathbb{C}$, of transcendence degree 1

<u>Means</u> 1) $\mathrm{Dim}_{\mathbb{C}} K/\mathbb{C} = \infty$

2) $\exists x \in K$ s.t $\dim K/\mathbb{C}(x) < \infty$

(Alg.)Geometry: Curves $C \subset \mathbb{P}^2$

specified by irred homog poly

α in x, y, z.

<u>Analysis</u> [geometry] X cpt Riem Surf.

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are all equivalent <u>categories</u>; ie "$\exists$ can. bijections between iso. classes of objects"

$K \simeq C \simeq X$

what are the isomorphisms (of categories!)

Lets see how this goes for elliptic curves.

$X = \mathbb{C}/L$

May assume $L = \mathbb{Z}\text{-span}\{1, \omega\}$ where $\operatorname{Re}\omega > 0$.

Curve C.

use $\wp, \wp'$

$[1, \wp, \wp']$ defines a map

$$X \xrightarrow{(\wp)} \mathbb{C}P^2$$

By looking at $\lim z \to \infty$
$z \to 0$:

$[L] \in X \longmapsto [0, 0, 1]$

i.e. $\frac{\wp}{\wp'}$ $\wp, \wp' \to \infty$ as $z \to 0$.

But $\frac{\wp}{\wp'} \to 0$.

image C is

$\wp$ sat.

$\cancel{(\wp')^3}$

$(\wp')^2 = x^3 + g_2 x + g_3$

g_2, g_3 fns of ω

a 10 Ahlfors.

$K(C) = K(x, y : y^2 = x^3 + g_2 x + g_3)$

$= K(\wp, \wp')$

finite extension of $K(x)$ or $K(\wp)$.

'<u>Anal</u>' $\to$ <u>Alg</u>

$X \longmapsto K(X) = \mathcal{M}(X)$

$= $ mero-fns on X.

what's to prove?

1) $\exists$ non const. $f \in \mathcal{M}(X)$,

2) If $g \in \mathcal{M}(X)$, $g \neq F(f)$

then $\exists$ poly.

R, $R(f(z), g(z)) \equiv 0$.

on X.

"Geom" $\to$ Alg.

$C \subset \mathbb{P}^2$;

$K(C) = K(x, y)$

x, y affine ~~fns~~ coord

so $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$

$[X, Y, Z]$ homog. coord.

we may have to use "$PGL_3(\mathbb{C})$ to 'rotate'" C for these fns to be good.

In affine coord

$$C = \{f(x,y) = 0\}$$

so K has trans. deg 1.

"Geom" $\longrightarrow$ Analysis.

Must resolve singularities!

<u>Thm</u> If $X \hookrightarrow \mathbb{P}^2$ is an ~~smoo~~ smooth alg curv then $g = \frac{1}{2}(d-1)(d-2)$.

$\cancel{= \frac{1}{2}d^2}$

Not all genuses are triangular numbers!

d=

g	d
0	1, 2.
1	3,
2	Nope!
.	

$\frac{1}{2}(3-1)(3-2) = 1$

$\frac{1}{2}(4-1)(4-2) = 3$

Always possible, But "a whole thing"

A whole involved theory going back to Newton.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

"Newton - Puiseux expansions"

↳ Justin Lake

Jurecne Sue : $\boxed{\text{resolution of singularities}}$

End result:

given $C \subset \mathbb{P}^2$ irred

$\exists!$ X smooth R.S. w

a holo. immersion $X \hookrightarrow \mathbb{P}^2$

onto C; reg at all but a

finite # of points, the resolved singu's.