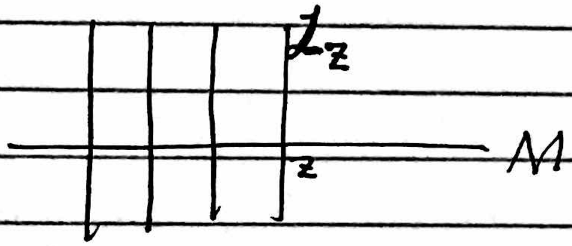
L6.21

Line bundles & RR

D, K, $K-D$ as line bundles

K: bundle of one forms

- basis at a point: dz
  where z a coord.
  ("uniformizing coord")



$\mathbb{C}^\times$

- What's a line bundle?
- A holomorphic complex line bundle?

How do divisors define line bundles?

Eg.

1) trivial line bundle:

$$\begin{array}{c} M \times \mathbb{C} \\ \pi \downarrow \\ M. \end{array}$$

global sections $\longleftrightarrow$ $\mathbb{C}$-valued fns on M.

$\uparrow s$

L6 L2

Eg 2 [Important]

A non-trivial line bundle:

$\mathbb{V} \simeq \mathbb{C}^2$ a cx 2-d vector space.

~~$\mathbb{C}P^2 \simeq$~~

$\mathbb{P}\mathbb{V} \simeq \mathbb{C}P^1$

$[v] \longleftrightarrow [z_1, z_2]$; $v = z_1 e_1 + z_2 e_2$.

e_1, e_2 basis for $\mathbb{V}$.

γ = Canonical line bundle.

$$\begin{array}{c} \gamma \\ \pi \downarrow \\ \mathbb{P}\mathbb{V} \end{array}$$

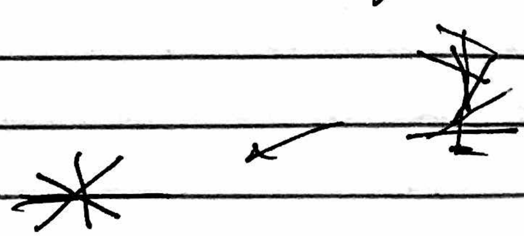
$\gamma_{[v]} = [v]$ - viewed as a cx 1-d subspace of $\mathbb{V}$ here a cx line.

in other words

$\pi^{-1}([v]) = \text{Span } v$

Another way to say:

$\gamma = \{ ([v], v) : v \in [v] \} \subset \mathbb{P}\mathbb{V} \times \mathbb{V}$



Works for any dim'l V.

yields
$$\begin{array}{ccc} \gamma & \subset & \mathbb{C}P^{n-1} \times \mathbb{C}^n. \\ \downarrow \mathbb{C} & & \\ \mathbb{P}V \cong \mathbb{C}P^{n-1} & & \end{array}$$

This canonical line bundle is "the mother of all line bundles"

<u>Prop</u> $\begin{array}{c} \mathcal{L} \\ \downarrow \\ X \end{array}$ a $\mathbb{C}$ line bundle

over <u>any</u> ~~cpt~~ top. space.

Then: $\exists$ ~~some~~ cts map

$F : X \to \mathbb{C}P^{n-1}$

for some ~~n~~ n s.t.

$\mathcal{L} \cong F^* \gamma$ (ctsly)

X cpt R.S. $n = 2$ does the trick!

Formal defs.

Cx line bundle over top space. X

A space $\mathcal{L}$ w a proj $\pi : \mathcal{L} \to X$
s.t. i) $\forall x \in X$ $\quad \pi^{-1}(x) := \mathcal{L}_x \underset{(*)}{\simeq} \mathbb{C}$

ii) this iso. $(*)$ is "cts in x"

Here is the standard way to say this: X is covered by opens U_α.

& $\exists$
iso's:

$$\begin{tikzcd} \psi_\alpha : \pi^{-1}(U_\alpha) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{C} \arrow[dl, "pr"] \\ & U_\alpha. & \end{tikzcd}$$

between $\pi^{-1}(U_\alpha)$ & the trivial bundle over U_α

iii) If $U_\alpha \cap U_\beta \neq \emptyset$

can anto compose:

L6~~7~~5

$$\begin{tikzcd}
U_\alpha \cap U_\beta \cancel{\times \mathbb{C}} \arrow[dr] & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[l, "\simeq", "\psi_\alpha^{-1}"'] \arrow[r, "\simeq"', "\psi_\beta"] & (U_\alpha \cap U_\beta) \times \mathbb{C} \arrow[dl] \\
& U_\alpha \cap U_\beta &
\end{tikzcd}$$

we require:

$\psi_\beta \psi_\alpha^{-1}$ is "fiber linear"

$$\psi_\beta \psi_\alpha^{-1}(x, z) = (x, g_{\alpha\beta}(x) z)$$

$$g_{\alpha\beta}(x) \in \mathbb{C}^* := \mathbb{C} \setminus 0.$$

$\nearrow$

- like overlap maps for manifolds -

This concludes the def of a cx line bundle.

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* Need to define what it means for two to be Bo etc.

1st: $[g_{\alpha\beta}] \in H^1(X; \text{"}\mathcal{C}(X)\text{"}]$

sheaf of cts cx valued fns on X

encodes everything about $\mathcal{L}$.

L6 L7

Before this

overlaps for K, γ.

For K

K: loc sections: dz.
$\downarrow$ Fiber: ~~T~~ $T^*_m M$ viewed
M as a cx vector space ~~of~~ dim 1.

U_α, U_β two coord nbhds.

~~a~~ dz_α, dz_β; $\underbrace{z_\beta = h_{\beta\alpha}(z_\alpha)}_{\text{coord overlap}}$.

$dz_\beta = h'(z)\, dz_\alpha$ $\quad h'(z) \neq 0$

$\uparrow$

yields the $g_{\beta\alpha}(p)$

$$\boxed{g_{\beta\alpha}(p) = h'_{\beta\alpha}(z_\alpha(p)).}$$

L628

For γ
$\downarrow$
$\mathbb{C}P^1$

~~Ea~~ would like to try to make a global section.

$$[z_0, z_1] \mapsto (z_0, z_1)$$

Does not work!

$$[z_0, z_1] = [\lambda z_0, \lambda z_1].$$

Insist. $z_1 \neq 0$ ÷

$$[z_0, 1] \mapsto (z, 1) = s_0([z_0, z_1])$$

defined on $z_1 \neq 0$.

$$[1, z] \mapsto (1, z) := s_1([z_0, z_1])!$$

over U_0: $v \in \gamma$; $v = \lambda_0 s_0(p)$.

define p. λ_0.

$$\pi^{-1}(U_0) \simeq U_0 \times \mathbb{C}$$

over U_1:

$$\pi^{-1}(U_1) \simeq U_1 \times \mathbb{C}$$

$$v = \lambda_1 s_1(p)$$

over lap: say $z_0 \neq 0$ & $z_1 \neq 0$.

$(v_0, v_1) \in \pi^{-1}(U_0 \cap U_1)$

so $v_0 \neq 0$, $v_1 \neq 0$.

$v = \lambda_0 (z, 1)$
$= \lambda_1 (1, w)$ $\qquad w = \frac{1}{z}$

$v_0 = \lambda_0$.

$v_0 = \lambda_1$?

Better! ~~or:~~ take case $\lambda_0 = 1$.

$v = v$

$s_1 = (1, w) = w(\frac{1}{w}, 1)$
$= w(z, 1)$
$= (\frac{1}{z})(z, 1)$

$g_{10}([z_0, z_1]$

$z = z_0/z_1$;

Game: $s_\beta = \underbrace{g_{\beta\alpha}}_{\text{over laps.}} s_\alpha$.

L6 Z10

s_α induces

$$\pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{C}$$

$$v \longleftrightarrow (p, \lambda)$$

$$\cancel{v} = \lambda_\alpha s_\alpha(p)$$

also:

$$v = \lambda_\beta s_\beta(p)$$

$$= \underbrace{\lambda_\beta g_{\beta\alpha}(p)}_{\lambda_\alpha} s_{\alpha}(p)$$

$$\lambda_\alpha = \lambda_\beta g_{\beta\alpha}(p)$$

or

$$U_\alpha \times \mathbb{C} \leftarrow - \; U_\beta \times \mathbb{C}$$

$$(p, \lambda_\alpha) \leftarrow - \; (p, \lambda_\beta)$$

inverse:

$$g_{\alpha\beta} = (g_{\beta\alpha}(p))^{-1}$$

Using, over & over:

(*) local nonvanishing sections

$\updownarrow$

local trivializiations

$$U_\alpha \times \mathbb{C} \underset{\psi_\alpha^{-1}}{\cong} \pi^{-1}(U_\alpha)$$

$$(p, 1) \longrightarrow \psi_\alpha^{-1}(p, 1).$$

&

$$s_\alpha : U_\alpha \to \mathcal{L}, \quad \pi \circ s_\alpha = 1_{U_\alpha}.$$

induces

$$\psi_\alpha : \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}$$

$$\boxed{v = \lambda_\alpha s_\alpha(p)}$$