Riemann-Roch

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Riemann-Roch, the Canonical Class and LIne bundles

Goals: Understand K. How K and divisors D can be viewed as line bundles.
Topological and holomorphic classification of line bundles.
The space of holo. one-forms.
DIVISORS. A divisor is an element of the free Abelian group generated by the points $p \in X$. We write $Div(X)$ for the group of all divisors on X.
Example: $D = p - q$ where $p, q \in X$.

Equivalently, a divisor is a map $D : X \to \mathbb{Z}$ whose support is finite with addition being pointwise addition.

We write

$$D = \Sigma m_i p_i$$

or sometimes $D = \Sigma D(p)p$.

Degree. The degree map is the homomorphism

Divisors and Principal Divisors

For $f \in \mathcal{M}(X)$ set

$$(f) = (f)_0 - (f)_\infty \tag{1}$$
$$= \Sigma ord_p(f)p \tag{2}$$

$(f)_0 = \Sigma m_i p_i$ where the sum is over the zeros of f and where the integers m_i are the multiplicities of these zeros.
$(f)_\infty$ we just saw.
In the last equality above for (f) the integer "ord_p"' denotes the order of f at a point p. If $f(p) \neq 0, \infty$ then $ord_p(f) = 0$. If $f(p) = 0$ then $ord_p(f)$ is the usual multiplicity of that zero p. If $f(p) = \infty$ then $ord_p(f) = -m$ where m is the order of the pole, or, what is the same, the order of the zero at p for the function $1/f$.

On order.

Order is a multiplicative homomorphism

$$ord_p : \mathcal{M}(X)_p \setminus \{0\} \cong \mathbb{C}(z) \setminus \{0\} \to \mathbb{Z}.$$

Thus $ord_p(1) = 0, ord_p(f/g) = ord_p(f) - ord_p(g)$.
$\mathcal{M}_p$ denotes the germs of meromorphic functions at $p \in X$.

REFS. Miranda p 26 re 'order".
Donaldson section 11.1.2. on 'Valuations" (or wiki for same)
The order is the valuation associated to the field $\mathbb{C}(x)$ at the origin.

"The number of poles equals the number of zeros"

Lemma

If $f \in \mathcal{M}(X)$ *then* $deg(f) = 0$.

PROOF. This lemma follows directly from the characterization of degree of a map $F : X \to Y$ in terms of multiplicities of the points in the pre-image of any point in the target, independent of the point. Apply to $F = f : X \to \mathbb{CP}^1$. Write c for the degree of F.
Then $c = deg((f)_0) = -deg((f)_\infty)$.
Cf also Miranda p 49 prop 4.12.

Definition

$$L(D) = \{f \in \mathcal{M} : D + (f) \geq 0\} \subset \mathcal{M}(X)$$

a vector subspace over $\mathbb{C}$ of $\mathcal{M}(X)$.

Example. $D = 3P - Q$. Then
$f \in L(D) \iff (f) \geq -D = -3P + Q \iff f$ has a pole of order 3 or less at P a zero at Q and no other poles. f is allowed to have other zeros besides Q.

Ordering the space of divisors.

Define $D = \Sigma m_i p_i \geq 0$ iff all $m_i \geq 0$

This relation induces a partial order on the set of all divisors:
$D \geq E \iff D - E \geq 0$.

BASIC FACTS.

a) $D \geq 0 \implies \mathbb{C} = \mathbb{C}1 \subset L(D)$ so $dim(L(D)) \geq 1$.
b) $D \leq E \implies L(D) \subset L(E)$.
c) $deg(D) < 0 \implies L(D) = 0$.
d) $dim(L(D + p))/L(D))$ is 1 or 0. (Here $p \in X$.)
e) $deg(D) = 0 \implies dim(L(D))$ is 0 or 1.
f) $dim(L(D)) \leq deg(D) + 1$.
g) $dim(L(0)) = 1$

BREAK OUTS

SOLVE THESE

Proofs.

Of (b). $D < E \iff -D > -E$. Thus $(f) \geq -D \implies (f) \geq -E$ or $f \in L(D) \implies f \in L(E)$.

Of (c): Say $deg(D) \leq -1$ and $f \in L(D)$. Then $(f) \geq -D$ so $deg(f) \geq deg(-D) \geq 1$. But for any constant or non-constant meromorphic function we have $deg(f) = 0$. So the only possible $f \in L(D)$ is the zero function.

Formally, the divisor of the zero functional is $\Sigma_{p \in X}(\infty)p$ since the zero functional has infinite contact with $z = 0$. Thus the divisor of the zero functional is formally infinitely greater than all other finite divisors. We insist $0 \in L(D)$ for any D. The smallest $L(D)$ can be is $\{0\}$.

Proofs ct'd

Of (e).
Use d): if $deg(D) = 0$ then by substracting a point from D we get $E = D - p$ with $deg(E) < 0$ so $dim(L(E)) = 0$. It follows from $D = E + p$ that $dim(L(D))$ is either 1 or 0.

Of (d).
Equivalent to assertion (d) is:

Theorem
$dim(L(D + p)) = dim(L(D)) + \epsilon$ where ϵ is one or zero

PROOF for the case when p is not in the support of D.
z –local coordinate centered at p.
$f, g \in L(D + p)$ but $f, g \notin L(D)$. Expand in z.
$f = a_{-1}/z + a_0 + \ldots; g = b_{-1}/z + b_0 + \ldots$
Solve a linear equation to find $\lambda_1, \lambda_2 \in \mathbb{C}$ so that
$F = \lambda_1 f + \lambda_2 g$
has no pole at p. ($\lambda_1 = b_{-1}, \lambda_2 = -a_{-1}$ work.)
Away from p both f, g have the pole and zero structure specified by D. Thus $F \in L(D)$ and f, g are linearly dependent MOD $L(D)$.
In other words, $dim(L(D + p)/L(D)) \leq 1$.

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ct'd

If the quotient dimension is 1 then $\epsilon = 1$. Otherwise $L(D + p) = L(D)$ and $\epsilon = 0$
QED

Of (e).

Keep subtracting points from D. Affter subtracting $d+1$ points $p_1, p_2, \ldots, p_{d+1}$ we get a divisor $E = D - \Sigma p_i$ of degree -1 so that, by (b) we have $dim(L(E)) = 0$. Now add the points p_i back in one by one. At each addition the dimension increases by at most one, so $dim(L(D) \leq dim(L(E)) + d + 1 = d + 1$.

Of (f). $L(0)$ consists of the meromorphic functions having no poles. These are the constant functions $L(0) = \mathbb{C} = \mathbb{C}1$.

Version 1, RR. [Riemann's version]

Theorem

For $d = deg(D) \geq 0$ we have

$$dim(L(D)) \geq d + 1 - g$$

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COR 3:

$$d + 1 - g \leq dim(L(D)) \leq d + 1$$

$g = 0 \implies dim(L(D)) = d + 1$ for $d \geq -1$.
In particular, if $D = (f)$ we have $d = 0$ and $dim(L(D)) = 1$: specifying the poles and zeros of a meromorphic function on $\mathbb{C}P^1$ specifies the function up to scale.

PROOF OF COR. 1 to RR.

Recall statement:
$D = p_1 + \ldots + p_N$ (repeated points allowed).
$N \geq g + 1 \implies \exists f \in L(D)$, f not constant..
Equivalently $(f)_\infty = D$.

Proof: $D \geq 0$ so $\mathbb{C} \subset L(D)$ and $dim(L(D)) \geq 1$.
So $dim(L(D)) \geq 2 \iff \exists f \in L(D)$ with f not constant.
Version 1 of RR says $N + 1 - g \leq dim(L(D))$ so
$N \geq g + 1 \implies 2 \leq dim(L(D))$.

Full-blown RR. There is a divisor of degree K called the “canonical divisor” and having degree $2g - 2$ such that

Theorem

$$dim(L(D)) - dim(L(K - D)) = d + 1 - g$$

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