Riemann-Roch

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Overview and background. Every math class has its mandatory theorem. Calculus has its fundamental theorem of. A 2nd run through linear algebra must talk about Jordan normal form. The Riemann-Roch [RR] theorem is the mandatory theorem for a Riemann surface class. In its original form (R) it bounds the dimension of the space of meromorphic functions having a given set of poles in terms of the genus g of the RS and the number d of poles. This dimension is $\leq d + 1 - g$. The key new concepts around RR are DIVISOR and LINE BUNDLE.

RR kept getting generalized over ensuing decades. Hirzebruch's name got attached in replacing RSs with smooth algebraic varieties. Grothendieck's name got attached at the front of RR when he established a 'morphism' version of HRR and along the way introduced K-theory and schemes to the world. These generalizations merged with Chern's generalization of the Gauss-Bonnet theorem to blossom into the Atiyah-Singer index theorem in the 1960s.

Chern-Gauss-Bonnet expresses the Euler characteristic of a manifold as an integral of a top-dimensional form built out of the curvature tensor of a Riemannian metric on the manifold.

X : compact RS,
$\mathcal{M}(X)$ its field of meromorphic functions,
g = genus of X.

We saw:
$g = 0$.
$\exists f \in \mathcal{M}(X)$ with a single simple pole $\iff g = 0 \iff X \cong \mathbb{CP}^1$.

$g = 1$. Elliptic curves. (thank you Amethyst!)
The Weierstrass $\mathfrak{P}$ function has a pole of order 2 at $0 \in \mathbb{C}/L$.

QUESTION
$g > 1$ general. $X = X_g$. $p \in X$.
¿ Does $\exists f \in \mathcal{M}(X)$
(i) whose only pole is at p and has order $g + 1$?
(ii) with exactly $g + 1$ simple poles $p_1, \ldots, p_{g+1} \in X$?
(iii) with only g simple poles?

5.

Associate to a meromorphic function $f : X \dashrightarrow \mathbb{C}$ the formal sum

$$(f)_\infty = \Sigma m_i p_i$$

where the $p_i \in X$ are the poles of f and the $m_i \in \mathbb{N}$ are the orders of the poles. The object $\Sigma m_i p_i$ is a divisor on X. This particular divisor is the divisor at infinity for f. The integer

$$d = \Sigma m_i.$$

is the degree of the divisor.

We flip the logic around and suppose a formal finite sum:

$$D = \Sigma m_i p_i$$

given. Set

$$d = \Sigma m_i$$

assuming, for now, all m_i in the sum are positive integers.

ANSWERS to all but that last question:
Cor. 1 to RR.
$d \geq g + 1 \implies$ *there is a mero. fn* f *on* X *with* $(f)_\infty = D$.

Answering (i). Yes: Set $D = (g+1)p = p + p + \ldots + p$ $(g+1$ times$)$.
Answering (ii). Yes : Set $D = p_1 + p_2 + \ldots + p_g + p_{g+1}$.
Question (iii) I am not sure of the answer. It has to do with "Weierstrass points" and I have a guess at an answer.

Another wonderful corollary of RR.

Recall the notion of 'index' or 'degree" of one field inside another. If $\mathbb{K} \subset \mathbb{K}'$ are fields this index, written $[\mathbb{K}' : \mathbb{K}]$, is the dimension of $\mathbb{K}'$ *as* a $\mathbb{K}$-vector space. Equivalently, $[\mathbb{K}' : \mathbb{K}] = dim_{\mathbb{K}}(\mathbb{K}'/\mathbb{K}) + 1$.

EXAMPLES. The index of $\mathbb{R}$ over $\mathbb{R}$ is 1, not 0.
The index of $\mathbb{C}$ over $\mathbb{R}$ is 2: $[\mathbb{C} : \mathbb{R}] = 2$.

Cor. 2 to RR
If f is a non-constant mero fn on X then $[\mathcal{M}(X) : \mathbb{C}(f)] = d$ where $d = deg(f)_\infty$ as above.

REF. Miranda p 176, prop 1.21.

Example.

$f = z : \mathbb{C} \cup \{\infty\} \dashrightarrow \mathbb{C}$. Its only pole is at $p = \infty$. Using $w = 1/z$ the chart at infinity we see that this is a simple pole so:

$$(f)_\infty = 1(\infty),$$

a $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ with degree 1. Cor. 2 asserts $[\mathcal{M} : \mathbb{C}(z)] = 1$ i.e.

$$\mathcal{M} = \mathbb{C}(z)$$

- the dimension of $\mathcal{M}$ over $\mathbb{C}(z)$ is 1. Any meromorphic function on $\mathbb{CP}^1$ is a rational function.

BREAK OUT?

What's 'Cor. 2' theorem say about $[\mathcal{M}(X) : \mathbb{C}(f)]$ where $f = \mathfrak{P}$ and X an elliptic curve as per Amethyst?