Moduli space.

11 of genus g Riem surfaces.

<u>Def</u>: The moduli space of Riem surfaces of genus g is the ~~sp~~ space of equivalence classes of Riem surfaces of this genus:

$\Sigma \sim \Sigma'$

$\iff \exists$ a biholo: $\varphi: \Sigma \to \Sigma'$

~~Pro~~ Notation: $\mathcal{M}_g$.

<u>Thm</u>: ~~Moduli s~~

$g=1$: $\mathcal{M}_1$ is 1 cx dim.

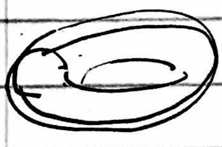
$g=0$: $\mathcal{M}_0$ is a point (0 dim'l)

$g \geq 2$ $\mathcal{M}_g$ is ~~a~~ ~~sm~~ $3g-3$ cx dim'l.

To be compared w/ the space of compact oriented surfaces of genus g:

Declare $\Sigma \sim \Sigma'$ if $\exists$ a differ $\Sigma \to \Sigma'$.

Then each g has a single pt in its "moduli" case $g=1$ $\qquad \Sigma \simeq T^2 \simeq S^1 \times S^1$



choose a basis for $H_1(\Sigma, \mathbb{Z})$

a, b

$$a \cdot b = 1$$
$$b \cdot a = -1.$$

in $S^1 \times S^1$ model

$S^1 \times pt$

$pt \times S^1$.

Space of holo diff'ls : 1 dim'l.

$$\omega$$

Def~~ine a map~~

$L \subset \mathbb{C}$ by

$$L = \left\{ \int_\gamma \omega : \gamma \in H_1(\Sigma, \mathbb{Z}) \right\}$$

then $L = m\omega_a + n\omega_b$.

$$\omega_a = \int_a \omega, \quad \omega_b = \int_b \omega.$$

Define

$$\varphi : \Sigma \longrightarrow \mathbb{C}/L.$$

$$\varphi(z) = \left(\int_{z_0}^{z} \omega \right)$$

Fact: φ is a biholomorphism.

Pf o: $d\varphi(z)$ "$= \omega$"

M3

RR.: ω has no zeros.

if ω had a zero then
$[\omega] = p_1 \cancel{p_2}$. so

$\deg[\omega] > 0$.

But $\deg(K) = 2g-2$
$= 2\cdot 1 - 2$
$= 0$ for $g=1$,

so φ is a covering map.
finite to 1,

~~Now~~ if $\Phi(z_1) = \varphi(z_2)$
then $\int_\gamma^{z_1} \omega = \int_\gamma^{z_2} \omega$.

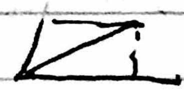
so $\int_{z_1}^{z_2} \omega \equiv 0 \mod L$.

$\Rightarrow z_1 = z_2$ (why?)

ah. we n (? !

pf: put toral coordinates T^2
θ_1, θ_2, Then $\omega = a\,d\theta_1 + b\,d\theta_2 + df$
$a, b \in \mathbb{C}$

lattice:
$2\pi a n_1 + 2\pi b n_2$.



$z_i = (\theta_1^i, \theta_2^i)$

$$\int_{z_1}^{z_2} \omega = a(\theta_1^2 - \theta_1^1) + b(\theta_2^2 - \theta_2^1) + f(z_2) - f(z_1)$$

M4

<u>Reason 2.</u>

Uniformization.

$\tilde{\Sigma} = H$. $\supset \pi_1(\Sigma) \subset Sl(2, \mathbb{R})$.

Σ

presentation of $\pi_1(\Sigma)$: $\{A_i, B_j :$

$\prod_i [A_i, B_i] = 1$

where $[A_i, B_i] = A_i B_i A_i^{-1} B_i^{-1}$

$(*)$ $\prod_i [A_i, B_i] = [A_1, B_1][A_2, B_2] \cdots [A_g, B_g]$

$\rho(a_i) = A_i$

3 real dim'l.

so: $2g \times 3$ matrices A_i, B_i

1 relation in $Sl(2, \mathbb{R})$: $(*)$

<u>Mod</u> conjugation: subtract 3

$\mathbb{R}$-dim $6g - 3 - 3 = 6(g-1) = 2 \times \underbrace{3(g-1)}_{\mathbb{C}}$

Move in this dir'n

M cpt connected Riem mfd. of dim ≥ 1.

$p_1, \ldots, p_N \in M$, $p_i \neq p_j$.

$p_1', \ldots, p_N' \in M$ $p_i' \neq p_j'$.

Q: Does $\exists$ isotopy or diffeom. $\varphi : M \to M$ s.t. $\varphi(p_i) = p_i'$?

[N-fold transitivity.]

(Yes!) !

Now: M a Riem surface.

$p_1, \ldots, p_N \in M$, $p_i \neq p_j$ $i \neq j$

$p_1', \ldots, p_N' \in M$ $p_i' \neq p_j'$ $i \neq j$.

does $\exists$ holo invertible $\varphi : M \circlearrowleft$ $\varphi(p_i) = p_i'$?

Typically: No!

M6

$M = \mathbb{C}P^1$. $n > 3$

No!

cross ratio

find the proj gen.

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similarly for $\mathbb{R}P^1$ & projectivities vs generic diffeos

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$M = \Sigma_g$?

No. Indeed if $g > 2$ $\exists$ special pts $w_1, \ldots, w_{2g}$ called "Weierstrass pts"

generic pts are NOT Weier.

& if $\varphi: M \circlearrowleft M$ holo.

then $\varphi(\text{Weier}) = \text{Weier}$.

so

$\mathcal{M}_{g,n}$; $\dim_{\mathbb{C}} \mathcal{M}_{g,n} = 3(g-1) + n$ $g>1$

but

$\mathcal{M}_{1,1} \simeq \mathcal{M}_1$ due to translations.

M7

$\dim_{\mathbb{C}} \mathcal{M}_{0,n} = n-3$.

eg $\mathcal{M}_{0,4}$ is param by cross ratio.

topologically: No $\mathcal{M}_{0,4}$: $\mathrm{Diff}(S^2)$ acts on field transitively!