Projective Geometry

of proj plane.

Take a 3-d vector space $\mathbb{V}$ to start.

$\mathbb{P}\mathbb{V} = \{1\text{-d linear subspaces of } \mathbb{V}\}$

<u>A line in $\mathbb{P}\mathbb{V}$</u>

the collection of all 1-d subspaces of $\mathbb{V}$ lying in a fixed plane.

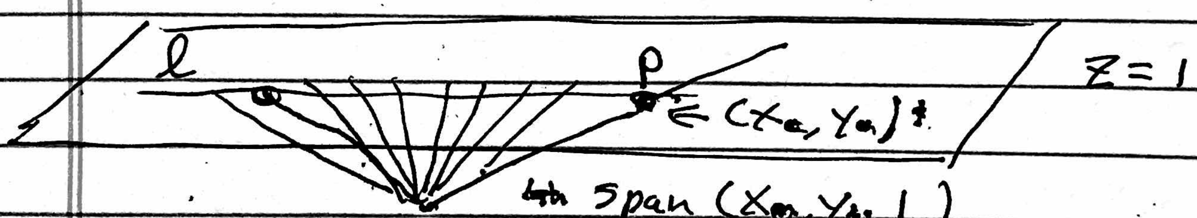
affine & homog. coord.

Basis e_1, e_2, e_3 for $\mathbb{V}$

So vectors $v = Xe_1 + Ye_2 + Ze_3$

$$\begin{array}{ccc} \mathbb{V}\setminus 0 & := & \mathbb{V}^\times \\ k^\times \downarrow \pi & & \downarrow \\ \mathbb{P}\mathbb{V} = \mathbb{V}^\times / k^\times & & \end{array} \qquad \begin{array}{c} v = (X, Y, Z) \\ \downarrow \\ [v] = [X, Y, Z] \end{array}$$

homog coord.



So point = image of 1-d linear subspace under Π

line = image of 2-d linear subspace under Π

specifying a pt:

$p_0 = [X_0, Y_0, Z_0] = [x_0, y_0, 1]$

$x = \frac{X}{Z}, \; y = \frac{Y}{Z}$.

specifying a line:

$AX + BY + CZ = 0$.

or $ax + by + c = 0$.

Axiomatics:

1) any two distinct points are incident to a unique line. ← "join" or span

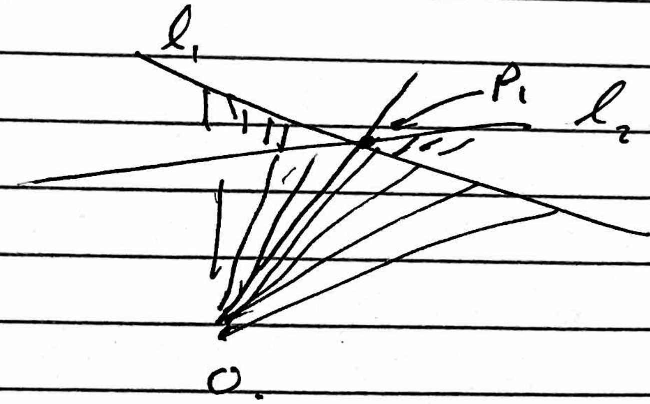
2) any two distinct lines are incident w/a unique pt.
↑ "intersect" ...

are satisfied & are translations of the following linear algebra assertions

1') : Any two distinct 1-dim linear subspaces span a unique 2-d. subspace.

2') Any two distinct planes* thru 0 (2-d linear subspaces) intersect in a unique 1-d linear subspace

Picture



* special to dim $V = 3$
or dim $\mathbb{P}^2 V = 2$.

Can restate in terms of projective duality

$$\begin{array}{cc} V & V^* \\ \downarrow & \downarrow \\ \mathbb{P}V & \mathbb{P}V^* \\ \text{pts} & \text{lines} \end{array}$$

incidence reln

$p = [v] = [X, Y, Z]$
$\ell = [\alpha] = [A, B, C]$

p incident to ℓ
$\iff \alpha(v) = 0$
$\iff AX + BY + CZ = 0.$

DO QUIZ problem A now.

Also answers B.

someone writes:
"True (?) - unless parallel.

what about the lines
$x = 0$
$x = 1$
where do they intersect?

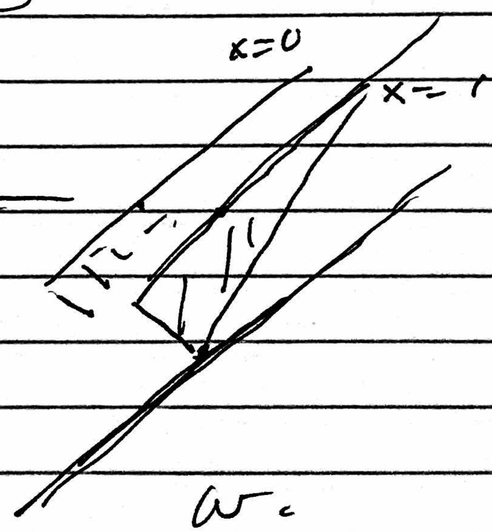
Homogenize: $X = 0$
$X = Z$

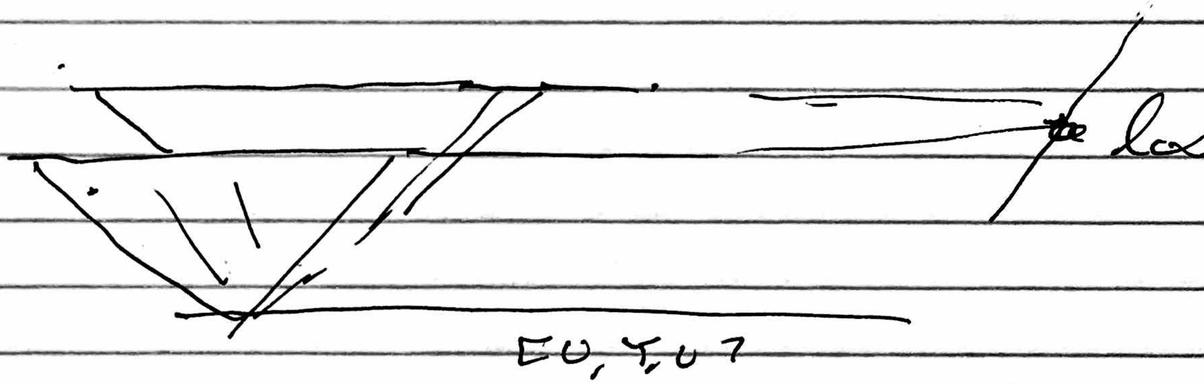
solve:
Y arbitrary, $Z, X = 0$

$[0, Y, 0] = [0, 1, 0]$

" line @ ∞: $Z = 0$ "

picture





$$\mathbb{P}V = k^2 \cup \text{los}.$$

$k = \mathbb{R}$: $\quad \mathbb{R}P^2 = \mathbb{R}^2 \cup \mathbb{R}P^1 \quad 2, S^1$

$k = \mathbb{C}$ $\quad \mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P^1 \quad 2, S^2$

$k = \mathbb{H}$ $\quad \mathbb{H}P^2 = \mathbb{H}^2 \cup \mathbb{H}P^1 \quad 2, S^3$

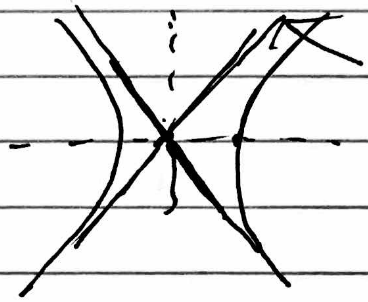
$\uparrow$ $\mathbb{R}^4$

but works for any field!

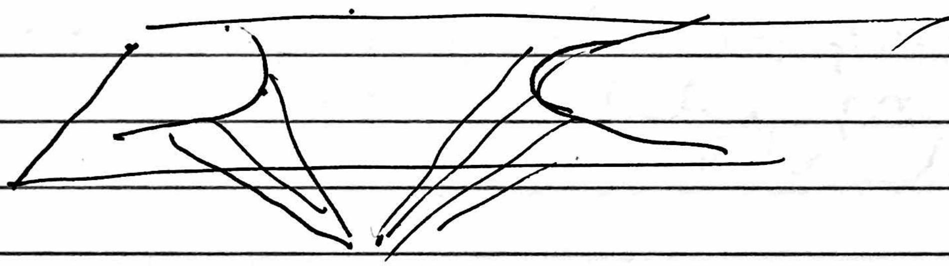
Quiz Problem C

C. Conics

$$x^2 - 3y^2 = 1$$



asymptotes rep. ∩ of lines.



take cone over conic:
ie set $z=1$ & join by all lines thru $(0,0,0)$

eqn: $X^2 - 3Y^2 = Z^2$.

line @ ∞: $Z = 0$.

$\rightarrow$ $X^2 - 3Y^2 = 0$

$X = \pm\sqrt{3}\, Y, \quad Z = 0$

points: $[\sqrt{3}, 1, 0]$, $[-\sqrt{3}, 1, 0]$

lines w/ slope $\frac{1}{\sqrt{3}}$, $-\frac{1}{\sqrt{3}}$

where from here?

Bezout?

Mner?

Q. Mech.

stories! Poncelet in prison
Dirac studying pure QM foundation.

Relation to Rieman surfaces.

affine algebraic curve: $p(x,y)=0$. ← max deg. d,

eg $x^2-3y^2=1$

homogenize:

eg $\underbrace{X^2-3Y^2=Z^2}$

$\underbrace{F(X,Y,Z)}=0$

homog. deg d.

defines a subset of $\mathbb{P}^2$
called an "algebraic curve.

if F is irreducible
(does not factor)
&

0 is a reg value:
$dF \neq 0$ wherever $F(X,Y,Z)=0$
but $(X,Y,Z) \neq 0$

then over $\mathbb{C}$ this is a
Riem surface.

<u>Surprise</u>: Its genus is
$$g = \binom{d-1}{2}$$
Not all R.S., even
topologically can be realized

d	$\binom{d-1}{2}$	
1	0	$Ax+By+Cz=0$.
2	0	all nondeg quadrics are $\mathbb{P}^1$'s
3	1	elliptic curves
4	3	genus 3, eg $X^4+Y^4+Z^4=0$.
5	6	

$g=2$ is missing! So is 4, 5, ...

Returning

a) Bezout,

$\#(\text{line} \cap \cancel{\ \ } V_F) = d.$

$\#(V_F \cap V_G) = mn$

$\deg F = m$

$\deg G = n.$

Projectivities & Symmetries

Def: A map : $\mathbb{P}^2 \circlearrowleft$
which maps lines to lines
& preserves incidence
is a projectivity.

Prop over $\mathbb{R}$ all projectivities
are induced by linear transformation
of $\mathbb{V} \simeq \mathbb{R}^3$.

* Similarly for $\mathbb{P}^n$ over k.
projectivity: induced map
from a linear map
$k^{n+1} \to k^{n+1}$

Fund Thm of 2-d proj geometry:

Any projective transformation is uniquely determined by ~~the~~ its image on 4 pts in general position.

'general position': No 3 in a line.

Pf p_1, p_2, p_3, p_4

p_1', p_2', p_3', p_4'

Then if $p_i = \pi(v_i)$
we have v_1, v_2, v_3 a basis
so are v_1', v_2', v_3'

Let g be the linear map sending $v_i \to v_i'$ $i = 1, 2, 3$

~~what about~~ ~~So~~ Then $[g]p = [gv_i]$ sends p_i to p_i' $i = 1, 2, 3, 4$

What about $p_4 \to p_4'$?

$g^{-1}v_4' = Xv_1' + Yv_2' + Zv_3'$, & $X \neq 0$ & $Y \neq 0$ & $Z \neq 0$.

$\pi v_4 = p_4$, $\pi v_4' = p_4'$

While

$$v_{p_4} = X v_1 + Y v_2 + Z v_3.$$

Choose h diag in the v_1, v_2, v_3 basis

$$h = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix},$$

$$\lambda_1 X = X'$$
$$\lambda_2 Y = Y'$$
$$\lambda_3 Z = Z'$$

Then $h v_i = \lambda_i v_i \qquad i = 1, 2, 3.$

so

$[gh] p_i = p_i \qquad i = 1, 2, 3$

$$[gh] p_4 = g(X' v_1 + Y' v_2 + Z' v_3)$$
$$= g(g^{-1} v_4')$$
$$= v_4'$$

&

$$g h v_4 = v_4'$$

so

$$[gh] p_4 = p_4'$$

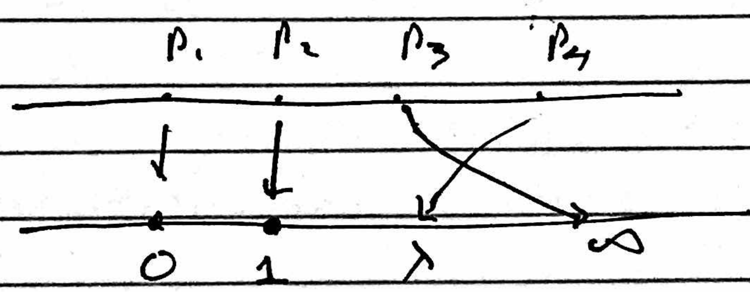
Finally: check:
if $[g]$ fixes
~~$v_1, v_2, v_3,$~~
p_1, p_2, p_3, p_4

then $g = \lambda \mathrm{Id}$.

Fund thm of n-dim proj geom
a proj. transf of $\mathbb{P}^n$
is uniquely determined by
its image on $\underline{\underline{n+2}}$
points in general position

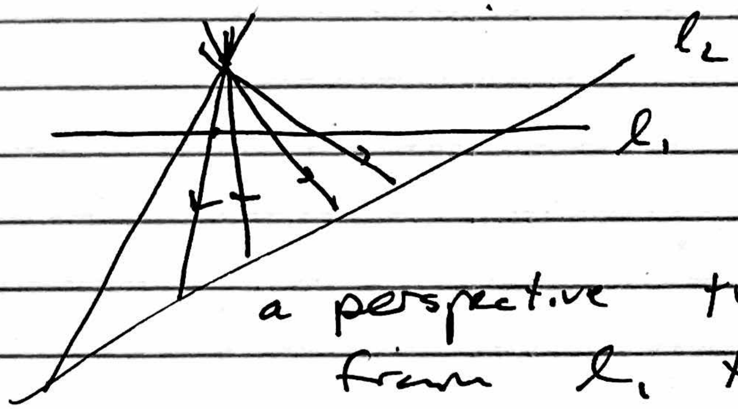
$n = 1$: Need 3 pts.

Can take



λ is the cross ratio of
$(p_1, p_2, p_3, p_4$

Any line in $\mathbb{P}^2$
is a $\mathbb{P}^1$.



a perspective from a pt
from l_1 to l_2

one writes
$$l_1 \underset{p}{\wedge} l_2.$$

can make a string

$$l_1 \underset{p}{\wedge} l_2 \underset{p'}{\wedge} l_3 \underset{p''}{\wedge} \cdots \wedge l_1$$

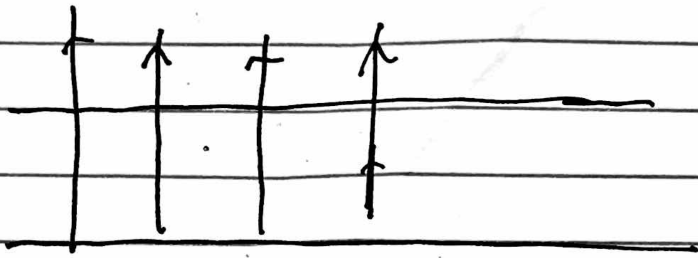
Prop: a composition of perspectivities
$l_1 \longrightarrow l_1$
is a projectivity. All
projectivities arise this way.

eg.

x-axis.

• ∞ in y dir.

$p = p_\infty = [0, 1, 0]$



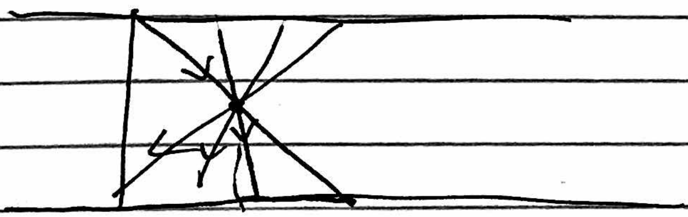
l_1 : $x = +1$

l_{-1} ← $x = -1$

$(x, -1) \longrightarrow (x, 1)$

N . $l_{-1} \underset{p_\infty}{\wedge} l_1$

Now take p_0 = origin in affine plane $= [0, 0; 1]$



$(x, -1) \longrightarrow (-x, -1)$.

Any $x \mapsto \frac{ax+b}{cx+d}$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

can be built this way,

Q: what is the minimal #, ...?

Etc etc etc.

Refs: Hilbert - Cohn Vossen.

Coxeter.

Stillwell. 4 pillars of geom.