Maps between RSs

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Maps between Riemann Surfaces

Definition

A **holomorphic map** between Riemann surfaces is a map whose expressions relative to some (and hence any!) pairs of local coordinate systems on the two surfaces is holomorphic.

Examples

Example

- Any meromorphic function on X defines a holomorphic map $X \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$.
- Conversely, any holomorphic map $X \to \mathbb{CP}^1$ defines a meromorphic function on X.
- The coordinate function z is holomorphic on $\mathbb{C}$. It extends to a meromorphic function on $\mathbb{CP}^1$ with a simple pole at infinity. The coordinate at infinity is $w = 1/z$ and in these coordinates $f(z) = z$ has the local expression $z = 1/w$

Example

- The exponential map $z \mapsto e^z := exp(z)$. Since $exp(z) = exp(z + 2\pi ik)$ we have that $exp(z)$ is a well defined map on the quotient Riemann surface $\mathbb{C}/(2\pi i\mathbb{Z})$. This quotient space is a cylinder. A fundamental domain consists of a horizontal strip bounded by two parallel lines separated by 2π. Thus *exp* maps the cylinder biholomorphically onto the punctured plane $\mathbb{C} \setminus \{0\}$.

Theorem

If X is a compact RS then every holomorphic function $f : X \to \mathbb{C}$ is constant.

Proof. Maximum modulus principle.

The above theorem shows the necessity of allowing functions to have poles in order to have a useful function theory on RSs.

We can divide, add, subtract and multiply meromorphic functions. It follows that the space $M(X)$ of meromorphic functions on X forms a field, indeed it is a field extension of $\mathbb{C} = \mathbb{C}1$. A main current in the development of the theory is subject is reconstructing X from $M(X)$.

Two surprises

Theorem

If the cpt connected X admits a meromorphic function f with a single simple pole then $f : X \cong \mathbb{CP}^1$ is a biholomorphism.

ref: D cor 1, p 45.

Below X and Y are compact connected RSs of genus g_X and g_Y.

Theorem

$F : X \to Y$ is a non-constant holomorphic map then $g_X \geq g_Y$. If, in addition $g_Y > 1$ then equality holds if and only if the map F is a bi-holomorphism.

Tools to prove theorems: topological degree, multiplicity of a point relative to a function.

DEGREE. See, eg Milnor, "Topology from the Differentiable Viewpoint"

MULTIPLICITY. Let $p \in X, q = F(p) \in Y$, z loc coord for X centered at p, w loc coord for Y centered at q.

F has the local coord expression $w = h(z)$ where $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ holo in a neighborhood of $z = 0$.

Thus:

$$h = Az^m + Bz^{m+1} + \dots .$$

Definition

This positive integer m is called the 'multiplicity of the point p' for the map F, denoted

$$m = mult_p(F)$$

Exercise

Show that m is independent of the choice of centered coordinates.

Notational Footnotes

Coordinates z "Centered" at p : means $z(p) = 0$. Equivalently $z : (X, p) \dashrightarrow (\mathbb{C}, 0)$.

Notations of broken arrows; pointed spaces, maps between pointed spaces; ... germs of maps.

Lemma ("Local Normal Form Lemma")

If $m = mult_p(F)$ then we can put F into the local normal form

$$w = z^m.$$

rel. coordinates z, w centered at $p, q = F(p)$.

Notational warnings. FK (p.11-12) talks about the branching number $b_p(F)$ instead of the multiplicity of a point:

$$b_P(F) = mult_P(F) - 1$$

D. (p.44) writes k_x for $mult_P(F)$, $x = P$.
Note: $mult_p(F) = 1$ iff p is a regular point for F.

Corollary

The set of points $p \in X$ at which $mult_p(F) > 1$ is a discrete point set of X. In particular, if X is compact, this locus of points is finite.

Theorem

The degree d of F and the multiplicities of points are related by

$$d = \Sigma_{p:F(p)=y} mult_p(F).$$

In particular, the sum is independent of the point $y \in Y$.

PROOF. If y is a regular point of F then all the integers $mult_p(F) = 1$ and these +1s agree with the ± 1's in one of the definitions of degree of a map between oriented compact manifolds found in Milnor, because holomorphic maps are orientation preserving.

If y is not a regular point, then it has some critical points above it. These have $mult_p(F) = m > 1$. Cover all the points p over F with 'normal form' nbhds. Then, any y' sufficiently close to y will be a regular value and, within each nbhd U_p there will be precisely m_p inverse images of y' lying within U_p, again by the normal form lemma. Since the degree of the map is computed by counting the total number of inverse images of such y' the result follows.

Write

$$B = \Sigma_p b_f(p) := \Sigma_{p \in X}(mult_p(F) - 1))$$

for the total branching number of F. Then we have the following amazingly simple and useful

Theorem (Riemann-Hurwitz relation)

If $F : X \to Y$ is a map between compact connected RSs whose whose genuses are g_X and g_Y and for which the map has degree d then

$$g_X = d(g_Y - 1) + 1 + B/2.$$