Homework 1. Riemann surface s

Throughout $(x, y) \in \mathbb{C}^2$ denote standard coordinates on $\mathbb{C}^2$ while $[X, Y, Z]$ are homogeneous coordinates for $\mathbb{CP}^2$.

1. Show that the solution set to $xy = 0$ is not a Riemann surface.

2. Show that the locus $x^2 + y^2 = 1$ is a smooth Riemann surface and is diffeomorphic to the tangent bundle to the unit circle.

3. 0. Compactify example 2 by homogenizing the equation and viewing $\mathbb{C}^2$ as the affine chart $Z = 0$ in $\mathbb{CP}^2$. Denote the resulting curve in $\mathbb{CP}^2$ as Q (for "quadric"). This process adds points at infinity to the curve C of example 2. How many points are added at infinity to C in order to form Q?

3.1 Now show that Q is biholomorphic to $\mathbb{CP}^1$ by the following standard geometric trick from projective geometry.

a) Pick any point $p_0 \in Q$. Then the set of all lines in $\mathbb{CP}^2$ passing through p_0 forms a $\mathbb{CP}^1$.

b) Each point ℓ of this $\mathbb{CP}^1$, viewed as a line in $\mathbb{CP}^2$, except one intersects Q in exactly one other point p besides p_0. Map the line ℓ to that point p. The exceptional line ℓ_∞ is the tangent line to Q at p_0 and we map it to p_0 itself.

c) By using an appropriate basis and coordinates, find an explicit expression for the map of (b). Your expression will be a rational parameterization $t \mapsto [X(t), Y(t), Z(t)]$ of Q relative to an affine coordinatization of the lines ℓ of $\mathbb{CP}^1$. Compute either the degree of this map or its critical points to show that this map is diffeomorphism.

4. Let $p(x)$ be a multiplicity free polynomial: thus $p'(x) \neq 0$ whenever $p(x) = 0$. Let $y^2 = p(x)$ be the corresponding hyerelliptic curve. Show that dx/y is a holomorphic differential on C.