Refs. Itzyk et al.
Thurston's orbifold notes.
Guardia and co-author.
McMullen's email.

1.

The nth Fermat curve F_n is the algebraic curve

$$F_n \text{ given by } X^n + Y^n + Z^n = 0 \text{ in } \mathbb{CP}^2.$$

for $n = 1, 2, 3, 4, \dots$,where $[X, Y, Z]$ are homogeneous coordinates for $\mathbb{CP}^2$.

Like any Riemann surface, the F_n's can be uniformized. The purpose of this note is to make this uniformization as explicit as I can. Driving the process is a tiling of F_n by equilateral triangles. The vertex angles of these triangles are π/n and there are $2n^2$ of them. They lift to a tiling of the universal cover. If we can compute the associated group of this tiling we will be a long ways towards the uniformization.

Exercise 1.1. *Show that the Fermat curves are smooth.*

Exercise 1.2. *Use the degree-genus formula to find the genus of F_n.*

Exercise 1.3. *Show that F_1 is biholomorphic to $\mathbb{CP}^1$.*

Exercise 1.4. *Show that F_2 is biholomorphic to $\mathbb{CP}^1$.*

Exercise 1.5. *Show that F_3 is an elliptic curve. Normalize the cubic defining it to find out which elliptic curve. Recall that the space of all elliptic curves is parameterized by a single complex variable τ varying in the upper half -plane. I'm asking you to find the τ.*

2. Uniformizing

When $n > 3$ the Fermat curves are hyperbolic: they are uniformized by the upper half plane. This means that there exists a commutative diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\Gamma_n"] & \tilde{F}_n \arrow[d, "\pi_1(F_n)"] \\ H/\Gamma_n \arrow[r] & F_n \end{tikzcd}$$

where H denotes the upper half plane, the right arrows are biholomorphisms, the down arrows are holomorphic projections - covering maps, and where $\Gamma_n \subset PSL_2(\mathbb{R})$ is a realization of $\pi_1(F_n)$ as a subgroup of the group $PSL_2(\mathbb{R})$ of orientation preserving isometries of H. Uniformization tell us that F_n admits hyperbolic structure whose overlap maps – elements of $PSL_2(\mathbb{R})$ – are holomorphic with respect to the Riemann surface structure on F_n. The game is to describe this structure, the group Γ_n, and its action on H as explicitly as possible.

Uniformization is hard. But it is also equivariant and the extra symmetries available on F_n due to the symmetric nature of its defining equation save the day and give us some purchase to perhaps explicitly compute the uniformizations of the F_n' s.

As we will see in the next section, F_n admits a decent sized automorphism group as a Riemann surface and under uniformization, these symmetries of the complex structure become isometries of its geometric structure. We can lift the symmetries all the way up to H where they yield a supergroup $D(n) \supset \Gamma_n$ in $PSL_2(\mathbb{R})$, a supergroup which 'plays well' with the Γ_n action on H. The automorphism group of F_n is a finite group of order $6n^2$ which contains $N = \mathbb{Z}_n \times \mathbb{Z}_n$ as a normal subgroup. As a result we get a quotient map $h_n : F_n \to F_1 = F_n/(\mathbb{Z}_n \times \mathbb{Z}_n)$. Thus the previous diagram extents to

$$\begin{tikzcd}
H \arrow[r] \arrow[d, "\Gamma_n"] & \tilde{F}_n \arrow[d, "\pi_1(F_n)"] \\
H/\Gamma_n \arrow[r] \arrow[d, "n^2"] & F_n \arrow[d, "h_n"] \\
(H/\Gamma_n)/N \arrow[r] & F_1 = \mathbb{CP}^1
\end{tikzcd}$$

The main step in our uniformization will be to construct the aforementioned supergroup. Specifically, we will motivate and show how to construct this supergroup of Γ_n as the "von-Dyck triangle group" $D(n) := D(n, n, n) \subset \mathbb{P}Sl(2, \mathbb{R})$ to be described below, a group based on an equilateral triangle whose vertex angles are all π/n. This triangle group contains Γ_n as a normal subgroup with quotient N. Thus we have an exact sequence of groups:

$$1 \to \Gamma_n \to D(n, n, n) \to N \to 1.$$

We thus have that $(H/\Gamma_n)/N = H/D(n, n, n)$. This later space is topologically the sphere but is the sphere as an orbifold, with three marked points, each of 'type n'. There are a whole host $D(n, m, \ell)$ of Von Dyck hyperbolic triangle groups and each one yields such an orbifold. Thurston denotes these orbifolds by $S^2_{n,m,\ell}$. To the extent that this group $D(n, n, n)$ can be explicitly described, so can Γ_n. Indeed, for me, one of the main remaining steps is to show that the commutator of $D(n, n, n)$ is Γ_n.

2.1. **Symmetries and branched covers.** Permuting the three homogeneous coordinates or multiplying them by independent nth roots of unity leaves the equation defining the Fermat curve untouched, hence this group acts by automorphisms on F_n.

Exercise 2.1. *Formalize the above sentence by showing that the group $S_3 \times_s (\mathbb{Z}/n\mathbb{Z})^3$ acts holomorphically on $\mathbb{CP}^2$ so as to map F_n to itself. Show that the ineffective kernel of the group – the part that does nothing to points of $\mathbb{CP}^2$ is a $\mathbb{Z}/n\mathbb{Z}$ embedded diagonally into $(\mathbb{Z}/n\mathbb{Z})^3$, ie by $\omega \mapsto (\omega, \omega\omega)$. Show that the quotient of the group by its ineffective kernel is $S_3 \times_s (\mathbb{Z}/n\mathbb{Z})^2$. Show that the order of this group is $6n^2$.*

This group forms the entire automorphism group of F_n and so will be denoted $Aut(F_n)$. That assertion requires more work to prove, but we do not need it, and simply write $Aut(F_n)$ for this group.

Footnote on the $\times_s$ notation By this notation "$\times_s$" I mean the semidirect product of groups. Here, S_3 acts on $(\mathbb{Z}/n\mathbb{Z})^3$ by permuting the three 'coordinates' -the three roots of unity. As a set, the semi-direct product of two groups G_1, G_2 is their product. To form the semi-direct product $G_1 \times_s G_2$ we need that G_1 acts by automorphisms of G_2 and we use this automorphism to define the group law. Another way to say the same thing is that G_2 sits in the semi-direct product by $g_2 \mapsto (1, g_2)$ as a normal subgroup, yielding an exact sequence $1 \to G_2 \to G_1 \times_s G_2 \to G_2 \to 1$ of groups.

Define the map $h = h_n : F_n \to F_1 = \mathbb{CP}^1$ by $[X, Y, Z] \to [X^n, Y^n, Z^n] := h_n([X, Y, Z])$. Since F_1 is a line in the projective plane, it is biholomorphic to $\mathbb{CP}^1$.

Exercise 2.2. *Show that h_n is a well-defined holomorphic map whose degree is n^2. Find its branch points - the critical values of h_n.*

Exercise 2.3. *Show that h_n realizes the quotient of F_n by the normal subgroup $N = \mathbb{Z}_n \times \mathbb{Z}_n$ of $Aut(F_n)$. Show that $S_3 = Aut(F_n)/N$ acts on the quotient by permuting the three branch points.*

We can parameterize F_1 by $t \in \mathbb{CP}^1$ so that the three branch points are $t = 0, 1, \infty$. The branch points lie on the 'equator' $\mathbb{RP}^1 \subset \mathbb{CP}^1$ parameteized by $t \in \mathbb{R} \cup \{\infty\}$. These three points, the three arcs into which they divide the equator, and the two hemispheres into which the equator splits the sphere $\mathbb{CP}^1$ define a triangulation of the sphere with three vertices, three edges and two faces.

Exercise 2.4. *Apply h_n^{-1} to each vertex, edge and face of this $3, 3, 2$ triangulation of $\mathbb{CP}^1 = S^2 =$ sphere. Show that this resulting collection of subsets of F_n defines a triangulation of F_n consisting of $2n^2$ faces, $3n^2$ edges and $3n$ vertices. Show this counting is consistent with the Euler characteristic of F_n computed by way of the degree-genus formula in an earlier exercise. Color the two hemispheres (faces) of the sphere black and white, the three vertices three colors '0', 1, and ∞ and the three edges by the pairs of vertices they join. Correspondingly color the faces of this triangulation of F_n with the colors to which their images are labelled. In this way the faces are are split into 2 types with n^2 of each type, the vertices are split into 3 types with n vertices of each type and the edges are split into 3 types with n^2 edges of each type. Show that exactly one edge connects any two vertices provided they are of different types.*

Cartographic groups. In the above exercise you built a specific Riemann surface X out of triangles with an associated "cartographic group" which permutes the faces, and hence the vertices and edges.

Exercise 2.5. *Show that $Aut(F_n)$ acts transitively on the black faces of the triangulation, transitively on the white faces, transitively on the edges and transitively on the vertices. By a 'flag' let us mean a triple consisting of a vertex, edge, and face of the triangulation where each one is adjacent to the others in the standard way. Define an orientation on flags and show that $Aut(F_n)$ acts freely transitively on the space of oriented flags.*

We saw that there are n vertices of type 0. Out of each, comes $2n$ edges. The symmetry group acts transitively on the edges. If we are to put a smooth Riemannian structure on F_n then the total angle at each vertex is 2π. There is an isometry fixing the vertex and taking each edge to each other. It follows that the $2n$ edges coming out of any single vertex must be equally spaced. **Hence we find that the angle between two consecutive edges meeting at the vertex must be $2\pi/2n = \pi/n$.** We have established that, upon uniformization, F_n is triangulated by $2n^2$ equilateral triangles of type $\pi/n, \pi/n, \pi/n$.

2.2. **Cartographic Groups, Dessigns D'Enfants, Belyi.** *This subsection is by way of a remark and pointer to related ideas.*

Consider a holomorphic map $h : X \to \mathbb{CP}^1$ branched over $0, 1, \infty$ and branched over no other points. Let N be the degree of the map. Then the inverse image of the triangulation of $\mathbb{CP}^1$ just described yields a triangulation of X consisting of $2N$ triangular faces, split equally into black and white, $3N$ edges and $m_1 + m_2 + m_\infty$ edges where $m_i = \#h^{-1}(i) < N$ is related to N and the total branching number at i a per the Riemann-Hurwitz type formulae. The vertices are labelled according to i and edges are labelled by pairs ij of distinct labels coming from $i = 0, 1, \infty$. according to the vertices they join. Each edge adjoins exactly one black and one white face.

See section 4 of Ityz for more details. This is the type of situation that inspired Grothendieck's program of "dessigns d'enfants'. See also "Belyi's theorem".

2.3. **Triangle groups.** The discussion of the triangulation of F_n induced by the branched cover $h_n : F_n \to F_1 = S^2$ shows the central nature of equilateral triangles with vertex π/n in understanding the uniformization of F_n. We have drawn some, along with the associated tiling of the universal cover, for $n = 2, 3, 4$.

Do it! -RM

We proceed generally. The complete connected simply connected Riemannian surfaces of constant curvature are the sphere $S^2 = \mathbb{CP}^1$, the plane $\mathbb{R}^2 = \mathbb{C}$ and the Poincare upper half plane H. For uniformity, we will refer to the space of each geometry as simply "the plane", or "its plane". In each geometry we have the notion of a triangle, and the edges of this triangle generate a subgroup of isometries as we now explain.

Each geometry has its notion of 'line'. Lines represent the shortest curves joining any two points. Any two points are joined by a line segment and that line segment is unique in all cases but one: that of antipodal points on the sphere. A line segment forms part of a line.

Associated to each line ℓ is a reflection R_ℓ of that geometry's plane: an isometry whose fixed point set is precisely that line. The product of two reflections whose lines intersect in a a point P forms a rotation about P. If the angle between the two lines is θ radians, directed from the first line to the second, then this rotation is by 2θ radians, in this same direction. Any isometry of the plane can be expressed as the product of three or fewer reflections.

Take a triangle in the plane: so three points, or 'vertices' and the three line segments - or edges - joining them. Label the vertices and edges as per usual. The reflections about the lines containing the three edges generates a subgroup of the isometry group of that plane. In order for this subgroup to be 'nice' we require that all the angles of the triangle are rational multiples of π. For suppose that the angle at vertex A is θ_A. By the above, the reflections about the two edges through A generate the rotation $\theta \mapsto \theta + 2\theta_A$, angles being taken mod 2π. If θ_A is an irrational multiple of π then this action on the circle is of "Kronecker type': every orbit is dense in the circle. Such groups - groups for which the orbits are not all a collection of isolated points but rather have limit points - yield bad quotient spaces - spaces that are not manifolds in particular. Thus we assume that the angles of the triangle are all integer. If that multiple is of the form $\pi p/q$ with p, q integers with no common factors, then by repeated iteration of this rotation we can achieve the rotation by $2\pi/q$.

Definition 2.1. *An n, m, ℓ triangle group is the group generated by reflections about the edges of a triangle whose vertices have angles $\pi/n, \pi/m, \pi/\ell$.*

The angle sum of a Euclidean triangle is always π: that is 180 degrees. The angle sum of a spherical triangle is always greater than π. The angle sum of a hyperbolic triangle is always less than π. Consequently the group is a subgroup of Euclidean, spherical or hyperbolic isometries depending upon whether $1/n + 1/m + 1/\ell$ is equal to, greater than, or less than 1.

For an isometry of the plane to be orientation preserving it is necessary and sufficient that it be generated by an even number of isometries.

Definition 2.2. *The von-Dyck group $D(n, m, \ell)$ is the subgroup of $\Delta(n, m, \ell)$ consisting of orientation preserving isometries*

Consequently, $D(n, m, \ell) \subset \Delta(n, m, \ell)$ is a normal subgroup of index 2.

Here is the theorem which leads to uniformization. To state it, recall that the "commutator" of a group G is the subgroup $[G, G]$ of G generated by the elements of the form

$ghg^{-1}h^{-1}$. It is a normal subgroup and the quotient $G/[G, G]$ is an Abelian group which is called the Abelianization of G.

Theorem 2.1 (main). *The Abelianization of $D(n, n, n)$ is $N_n := \mathbb{Z}_n \times \mathbb{Z}_n$. Consequently, the commutator $\Gamma_n := [D(n, n, n), D(n, n, n)]$ of $D(n, n, n)$ is an index n^2 normal subgroup of $D(n, n, n)$. Γ_n is the fundamental group of F_n and $H/\Gamma_n = F_n$ as Riemann surfaces.*

2.4. **Orbifolds.** The key to the proof of the main theorem seems to be the fact that $H/D(n, n, n)$ is isometric to the orbifold known as $S^2_{n,n,n}$. Topologically this space is the two-sphere S^2 endowed with three marked points called 'orbifold points". Now F_1, viewed as the image of h_n, also has three marked points, the three branch points. H induces a a Riemann surface structure on the orbifold and ,as a Riemann surface it must be the sphere. Thus there is a unique biholomorphism

$$b_n : S^2_{n,n,n} \to F_1 = \mathbb{CP}^1$$

once we choose a bijection from the three orbifold points to the three branch points of F_1.

The theorem then is achieved as a kind of 'unique lift' theorem for b_n, this lift being a map from $H \to \tilde{F}_n$ which takes the Γ_n action to the $\pi_1(F_n)$ action.

2.5. **Final notes.** Final note. By the work of Takeuchi, it seems that the only n for which we can have an explicit -technically an *arithmetic* uniformization are $n = 4, 5, 6, 7, 8, 9, 12, 15$. Why only these? Who knows?!

References

[1] Itzykson et al `http://archive.numdam.org/article/RCP25_1997__48__1_0.pdf`

[2] Bayer and Guardia. Hyperbolic Uniformization of Fermat curves. Ramanujan Journal/ 2006.

[3] Ganon et al. Automorphic forms for triangle groups. `https://arxiv.org/pdf/1307.4372.pdf`

[4] Grothendieck `https://webusers.imj-prg.fr/~leila.schneps/grothendieckcircle/EsquisseEng.pdf`

[5] Thurston. Orbifolds. `http://library.msri.org/books/gt3m/PDF/13.pdf`

[6] AUTHOR = Takeuchi, Kisao, TITLE = Arithmetic triangle groups, JOURNAL = J. Math. Soc. Japan, FJOURNAL = Journal of the Mathematical Society of Japan, VOLUME = 29, YEAR = 1977, NUMBER = 1, PAGES = 91–106,

[7] wiki Triangle Group `https://en.wikipedia.org/wiki/Triangle_group`