Solution, problem 1, Homework 6: Lie groups. By the "intrinsic curve method" of computing derivatives

Parts (a) and (b) ask you to show that the identity I is a regular value of the map $F(A) = AA^T$ when viewed as a map from the real vector space $End(\mathbb{V})$ of all linear maps $\mathbb{V} \to \mathbb{V}$ onto its subspace $Sym(\mathbb{V})$ the space of of symmetric operators on $\mathbb{V}$. (Note we have $End(\mathbb{V}) = \mathbb{M}_n$, the space of n by n real matrices, where $n = dim(\mathbb{V})$. This identification requires choosing an orthonormal basis for the Euclidean vector space $\mathbb{V}$.)

a) Computing the derivative of F. Let $A(t)$ be a smooth curve passing through $A_0 \in End(\mathbb{V})$ at time $t = 0$. Write B for its time derivative at time $t = 0$, thus $B = \dot{A}(0) := \frac{d}{dt}|_{t=0}A(t)$. Then, on the one hand

$$dF_{A_0}(B) = \frac{d}{dt}|_{t=0}F(A(t))$$

While, on the other hand, because AA^t is homogeneous quadratic in A we have that

$$\frac{d}{dt}|_{t=0}A(t)A(t)^T = A(0)\dot{A}(0)^T + \dot{A}(0)A(0)^T = A_0B^T + BA_0^T$$

So that

$$dF_{A_0}(B) = A_0B^T + BA_0^T.$$

Remark. You could also do this calculation by setting $A(t) = A_0 + tB$, expanding out $F(A(t)) = A_0A_0^T + t(A_0B^T + BA_0^T) + t^2BB^T$, and selecting the coefficient of the part linear in t.

b) We are to show that dF_{A_0} is onto provided that $A_0A_0^T = Id$. To this end, let $S \in Sym(\mathbb{V})$ be an arbitrary symmetric operator on $\mathbb{V}$, so that $S = S^T$. We must show that there is a B for which $dF_{A_0}(B) = S$. Use that A_0 is invertible which comes from $A_0A_0^T = I$. As a first try, set $B = SA_0$. Then

$$dF_{A_0}(SA_0) = A_0(SA_0)^T + (SA_0)A_0^T = A_0A_0^TS + SA_0A_0^T = 2S.$$

Oops. Not quite. $B = \frac{1}{2}SA_0$ does the trick.

c) In general, the inverse image $F^{-1}(c)$ of a regular value c of a smooth map is a smooth submanifold whose tangent space is the kernel of F. This kernel is constant in dimension, that dimension being the dimension of F's domain minus the dimension of F's range.

In (b) we showed that I is a regular value for our F. We have that $End(\mathbb{V}) = Sym(\mathbb{V}) \oplus Skew(\mathbb{V})$ (direct sum) where $Skew(\mathbb{V})$ is the space of skew-symmetric operators. Thus, the dimension of $O(V)$ is the dimension of the space of skew symmetric operators which is $\binom{n}{2}$. Note: $\binom{n}{2} + \binom{n+1}{2} = n^2$.

d) Almost finally, $dF_I(B) = B + B^T$, so that we have that $T_IO(n) = kerdF_I = \{B : B + B^T = 0\}$ is $Skew(\mathbb{V})$, the space of skew symmetric operators on $\mathbb{V}$. This is the Lie algebra of the Lie group $O(\mathbb{V}) = O(n)$